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INERTIAL NAVIGATION THEORY (AUTONOMOUS SYSTEMS)

V. D. Andreev

Foreign Technology Division
Wright-Patterson Air Force Base, Ohio

17 December 1974

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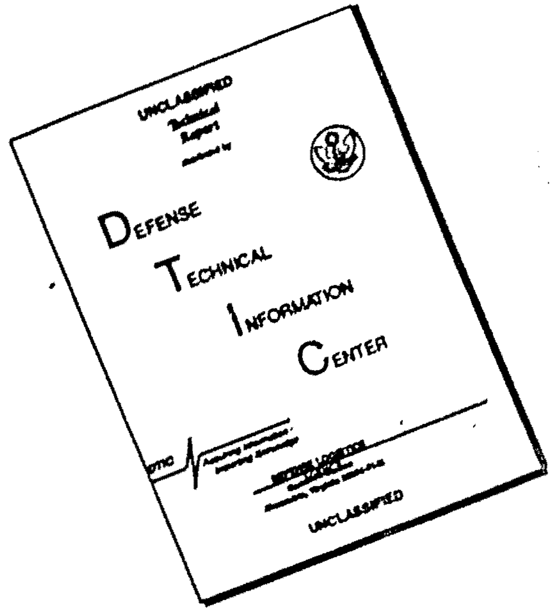

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В в	*В в*	V, v	Т т	*Т т*	T, t
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М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*<u>ye</u> initially, after vowels, and after ъ, ь; <u>e</u> elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

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RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}

rot	curl
lg	log

Inertial Navigation Theory
(Autonomous Systems)

V. D. Andreyev

Nauka Press
Moscow 1966

Foreword

The impetuous development of aviation, missile technology and the Naval fleet led to the necessity of fundamental improvement of the means of navigation and control of moving objects. Besides high accuracy, a number of such specific requirements as universality, reliability, short preparation time, electronic countermeasures, and sometimes concealment of operation are now placed on automatic navigation systems.

Along with development of other principles, special attention has been devoted in recent years to inertial navigation systems, in which the current position of a moving object is determined by integration of the on-board measured accelerations. Inertial systems have such important advantages as universality, autonomy and electronic countermeasures over other means of navigation. However, realization of these systems requires highly accurate and reliably operating elements: accelerometers, integrators, gyroscopes, tracking systems and computer devices. The interest displayed in inertial navigation systems is explained both by their principal advantages and also to a great extent by the fact that inertial systems, which provide the required navigation accuracy, can be developed on the basis of modern components.

The development of shipboard gyrocompasses by H. Anshutz-Kämpfe (1908) and Elmer A. Sperry (1911) can be considered the first use of inertial methods in navigation. The next important advance was the investigations of M. Schuller, who established the conditions of the unperturbability of the gyrocompass (1910) and of physical and gyroscopic pendulums (1923) by horizontal accelerations. Further stages in the development of the idea of inertial navigation are the principle of power-assisted gyroscope stabilization, proposed by S. A. Nozdrovskiy (1924), and also the principle of integral gyroscope correction, proposed in 1932 by Ye. B. Levental and in 1935 by I. M. Boykov.

For some time the development of inertial systems was related to gyropendulum systems and gyroscopic systems with integral correction, which simulate M. Schuller's physical pendulum and which permit plotting of an acceleration-proof vertical on a moving object. Significant results are related to the names of B. I. Kudrevich, I. V. Gekkeler, B. V. Bulgakov, Ya. N. Roytenberg and A. Yu. Ishlinskiy.

Another aspect of the inertial navigation method, namely, the circumstance that not only the vertical can be plotted, but the current coordinates of the object and its speed can be determined by using it, developed somewhat later. The first practical achievement in this direction was apparently the development of a control system for the FAU-2 rocket. Further development of this direction can be traced from data of American publications.[1] The beginning of development of inertial systems in their modern form in the United States dates from 1946-1947 and is related to development of control systems for ballistic (Atlas type) and winged (Navaho and Snark type) missiles. Practical realization of inertial systems was possible at that time because of development of flotation gyroscopes, proposed in 1946 by Draper (in the Soviet Union flotation gyroscopes were proposed in 1945 by L. I. Tkachev).

During the past few years considerable attention has been devoted in the non-Soviet literature, especially in American literature, to problems of inertial navigation. A large number of articles devoted to individual theoretical and engineering problems of inertial navigation have been published in various journals and several monographs have been issued. The most significant of these investigations have been translated into Russian. In 1958 the Foreign Languages Publishing House published a book by research associates W. Rigley, R. Woodberry, and J. Govorky of the Massachusetts Institute of Technology entitled "Inertial Navigation". In 1964 translations of K. L. MacClure's book "Inertial Navigation Theory" (Nauka Publishing House) and the collection "Inertial Control Systems", edited by D. Pittman (Voyenizdat) were also published.

During the past few years a number of articles, including several investigations of A. Yu. Ishlinskiy in which the fundamentals of a strict theory of inertial systems[2] have been outlined, have been published in the Soviet periodical literature on the problems of inertial navigation. In 1961 the Publishing House of Physicomathematical Literature published G. O. Fridlender's book "Inertial Navigation Systems" and in 1962 the Sovetskoyeradio Publishing House published I. A. Gorenshteyn, I. A. Schul'man, and A. S. Safaryan's book "Inertial Navigation".

It should be noted that the numerous investigations on the problems of the theory of inertial systems published in the periodical press are usually of an unrelated nature, and in the greater part of them there is lacking a clear statement of the problems and the required strictness of their solution. The monographs enumerated above are limited to consideration of individual classes of inertial systems. As a rule, various types of simplifications of the structure of inertial systems and the laws of motion of an object are introduced from the very beginning. Because of this, the exposition falls into separate and usually unrelated parts, the community of the basic principles of inertial navigation is obscured, and the theoretical results obtained are sometimes unsuitable for rough approximation. Introduction of a priori simplifications is usually explained by the insurmountable complexity of precise consideration.

At the same time the continuous increase in the demands on accuracy of inertial navigation systems forces consideration of the finer and finer circumstances of their operation, such as the asphericity of the earth's shape, the eccentricity of its gravitational field etc., and leads to the necessity of detailed analysis of the dynamics of their perturbed operation. The desire for universality leads, on the other hand, to rejection of the simplifications possible during development of a navigation system for a fully defined object.

In this book the author sets himself the task of systematic and strict exposition of the theoretical operational bases of inertial navigational systems from a common viewpoint without a priori simplifications and limitations, determined by the level of present technology. The methods of analyzing the operation of inertial navigation systems used by the author are the development of the ideas contained in the investigations of academician A. Yu. Ishlinskiy. The basis of the book were the author's articles, published during the past few years in journals of the USSR Academy of Sciences: Prikladnaya Matematika i Mekhanika and Izvestiya AN SSSR (serii Mekhanika and Tekhnicheskaya Kibernetika). The examples which concern schematic solutions and numerical evaluations are constructed on the basis of data from foreign publications.

Main attention is devoted in the book to the equations of ideal operations (unperturbed functioning) of inertial systems, which determine their structure, and to equations of inertial navigation system errors, an analysis of which permits evaluation of the operating stability of the system and establishment of the relationship between the errors of the elements and the accuracy of determining the navigational parameters of the object: the current coordinates of position and its orientation in space. Problems of autonomous preparation of inertial systems for operation are also considered. The book is devoted to the theory of autonomous inertial systems. The problems related to drawing up additional information and correction of inertial systems, are considered in another book of the author [Inertial Navigation Theory (Corrected Systems)] which is directly related to the present book and which was published immediately after it.

The book consists of seven chapters.

In the first chapter the theoretical and mechanical bases of inertial navigation are outlined, the equations of accelerometer operation are derived, the precession theory of gyroscopic devices for inertial systems is presented, the basic equation of inertial navigation is found and the general principles of constructing an inertial navigational system and the problems of the theory of these systems are discussed.

In the second chapter the necessary data on the shape, gravitational field and motion of the earth are presented. The main point in this chapter is the derivation of expressions from the solution of the Stokes problem for projections of the earth's gravitational field intensity onto its bodv axes.

The third chapter contains derivation of equations of the ideal operation of an arbitrary inertial system, first for calculation of Cartesian and then for calculation of curvilinear coordinates. The various special cases and examples for the more commonly used coordinates: geocentric, geographic and orthodromic, are also presented and an example of non-orthogonal curvilinear coordinates is also given. The theory of so-called gravimetric systems, which do not contain gyroscopes, is also outlined in this chapter.

The derivation and transformation of the equations of inertial navigation systems errors are presented in the fourth chapter. Both equations of coordinate errors and equations of orientation errors are considered. The problem of reducing the errors of the inertial system elements to equivalent instrumental errors of the main sensitive elements - accelerometers and gyroscopes - is given special consideration.

In the fifth chapter the common properties are considered, the stability and integration of error equations are investigated and the relationship of errors in calculating the location of an object and its orientation to the instrumental errors of the elements is considered. The case of Kepler motion of an object is given special consideration.

The sixth chapter is devoted to the theory of inertial navigation on the earth's surface. Both inertial systems with three arbitrarily oriented accelerometers and those with two horizontally positioned accelerometers are considered. The latter are compared to Schuller's pendulum - gyroscopic systems, the strict theory of which is also presented in this chapter.

Finally, in the last, the seventh chapter, the problems related to autonomous preparation of an inertial system for beginning of operation in the case of a fixed starting point with respect to the earth, are considered.

For purposes of compactness, the exposition is performed primarily in a vector form, and the elements of tensor calculus are employed when considering curvilinear coordinates. The final results are usually written in a scalar form. References to the literature are given in footnotes and, moreover, a bibliography is presented at the end of the book.

The author is aware that the book is not devoid of deficiencies. Some results could apparently be obtained by simpler means; improvements in the portion of selecting the sequence of outlining the individual problems are also probably possible. Critical comments and desires of the readers will be gratefully accepted.

The author feels it his pleasant duty to express deep gratitude to A. Yu. Ishlinskiy for unflagging attention and assistance in the work on the book. The author also thanks Ye. A. Devyanin, I. V. Novozhilov and N. A. Parusnikov for participating in the discussion of individual sections of the book.

1 The author did not set himself the task of presenting a complete survey of the history of development of the ideas of inertial navigation. This task is specific in itself and can be the subject of a separate investigation. There is apparently a need for such an investigation. Ths is especially indicated by publication of H. Helman's article "Development of Inertial Navigation" in the American journal Navigation (Vol. 9, No. 2, 1962). Problems of the history and priority are illuminated unilaterally and inaccurately in this article. References to a number of other well-known investigations of Soviet authors are lacking in it. The main references from these investigations are indicated in the bibliography at the end of this book. Of course, the list does not claim to be complete.

2 See, for example: Ishlinskiy, A. Yu. "On the Theory of the Gyrohorizon - Compass," Prikladnaya Matematika i Mekhanika Vol. 20, No. 4, 1956; "Equations of the Problem of Calculating the Location of a Moving Object by means of Gyroscopes and Accelerometers," Prikladnaya Matematika i Mekhanika Vol. 21, No. 6, 1957.

Chapter 1

Theoretical and Mechanical Bases of Inertial Navigation: Sensing Elements, the Fundamental Equation of Inertial Navigation and the Principle of Constructing Inertial Navigation Systems

§ 1.1. The Overall Characteristics of the Method of Inertial Navigation

The main task of any navigation method is to determine the location of the object, i.e., to determine the coordinates of some point, for example, of the center of mass, in a given system of reference. The problem of an inertial navigation system usually includes calculation of the rates of variation of these coordinates and also calculation of the parameters which characterize orientation of the object in a given system of reference and calculation of the variation of orientation parameters.

The principal characteristic of the inertial method of navigation includes the fact that the coordinates of the object are obtained essentially by integration of the equation of motion of its center of mass in the absolute (inertial) system of coordinates. The vector of the composite force, applied to the object, which is required for integration of this equation, is determined by the indications of special devices - accelerometers (specific force sensors) - in the form of projections onto the directions of their axes of sensitivity. The axes of sensitivity of accelerometers are oriented into the inertial system of coordinates by using gyroscopes or by the indications of the accelerometers themselves.

The inertial (Galilean) system of coordinates, in which Newton's laws of dynamics are valid, is the main system of reference in inertial navigation.

The indicated circumstances are more typical for the method of inertial navigation and it is associated with them by its name.

§ 1.2. The Operating Principle and the Equations of Operation of the Accelerometer (Specific Force Sensor)

The idealized scheme of a spatial accelerometer can be represented (Fig. 1.1) in the form of a mass point m, suspended in the housing of a device in a three-stage weightless elastic suspension.

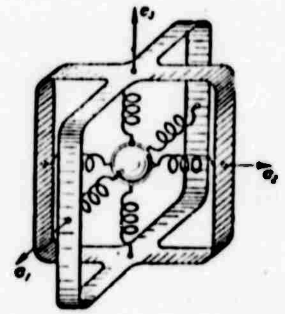

Fig. 1.1

To derive the equations of operation of the accelerometer let us introduce a right-hand orthogonal system of coordinates $0_2\xi_*\eta_*\zeta_*$ - some inertial (Galilean) system in which, by definition, Newton's laws are valid. Selection of the position of point 0_2 and orientation of the axis $\xi_*\eta_*\zeta_*$ are not subject to any other conditions.

Let the accelerometer housing move arbitrarily in this coordinate system. Let us consider the motion of point O, in which the sensitive mass of the accelerometer is concentrated. The sensitive mass of the accelerometer is obviously affected only by the sum F_Σ of the Newtonian forces of attraction of the sensitive mass by the entire aggregate of celestial bodies, including strictly speaking, the attraction by the masses of the object, in which the accelerometer is installed, and force $\vec{f}$, which is determined by elastic deformation of the suspension. Thus, if $\vec{r}_{O_2}$ is the radius vector of point O in the inertial system of coordinates, then the equation of motion of point O has the form:

$$m\frac{d^2 r_{02}}{dt^2} = F_\Sigma(r_{02}) + f. \tag{1.1}$$

The differentiation in equation (1.1) is absolute, i.e., $d^2\vec{r}_{O_2}/dt^2$ is the absolute acceleration of point O in the coordinate system $O_2\xi_*\eta_*\zeta_*$.

To an observer, bound to the housing of the accelerometer, the only effect on the sensitive mass m of the accelerometer is that of the elastic forces of the suspension, while the parameters which characterize this effect are the magnitudes of deformation of the suspension, whose function is elastic forces. Only the extent of deformation of the suspension can be measured and these deformations are the indications of the accelerometer.

By assuming that deformation is small and assuming that force $\vec{f}$ is proportional to the vector $\vec{n}$ of deformation of the suspension, we have:

$$f = kn. \tag{1.2}$$

The equality (1.2) assumes the isotropy of the elastic properties of the suspension. The three-dimensional suspension depicted in figure 1.1 satisfied this condition at small deformations.

Having taken for simplicity the ratio m/k equal to unity, we find from equation (1.1) the following expression for calculating the value measured by a three-dimensional accelerometer:

$$n = \frac{f}{m} = \frac{d^2 r_{02}}{dt^2} - F(r_{02}). \tag{1.3}$$

Here $\vec{F}=\vec{F}_{\Sigma}/m$, where $\vec{F}$ is the attractive force acting per unit of sensitive mass, i.e., the intensity of gravity at point O.

Thus, the specific force, i.e., the effective force of suspension per unit of sensitive mass, is measured by means of an accelerometer. It is equal to the difference of acceleration of the sensitive mass and of the intensity of gravity at the point of the current location of this mass.

Other names of the described device are often used in the literature - accelerometer and specific force sensor. The first name, and to a known degree the traditional one, does not accurately reflect the physics of operation of the device. The term specific force was introduced by Draper.[1] The name specific force sensor or the specific force meter accurately corresponds to the value measured by the device. We will usually employ the term newtonometer, introduced by A. Yu. Ishlinskiy.[2] This name correctly reflects the essence of operation of the device as a force meter (the name Newton has been given to the unit of force in the international system of units).

In the diagram shown in Figure 1.1, where the three-dimensional elastic suspension is realized by three pairs of springs, the indications of the newtonometer will be numerically equal to the values of projections n_{e_s} of vector $\vec{n}$ to unit vectors $\vec{e}_s$ of the spring axes

$$n_{e_s} = n \cdot e_s. \tag{1.4}$$

The actual designs of newtonometers are usually single-component. An idealized diagram of a one-component linear (axial) newtonometer is shown in Figure 1.2. The sensitive mass of this newtonometer has one degree of freedom with respect to the housing and can move only in a straight line, called the axis of sensitivity. It is along this axis that the reactive force of the spring of the suspension, deformation of which is being measured, acts on the sensitive mass. It is easy to see that in this case the reading of the newtonometer will also be numerically equal to the projection of vector $\vec{n}$ to the direction of the axis of sensitivity $\vec{e}$.

Fig. 1.2

Along with linear newtonometers, so-called pendulum newtonometers are used. An idealized diagram of a pendulum newtonometer is shown in Figure 1.3 and is a plane physical pendulum (its axis of suspension is perpendicular to the plane of the diagram), connected to the housing by springs whose direction of axes are normal to the axis of suspension and the axis of symmetry of the pendulum. It is obvious that with small deformations of the springs, i.e., at small deviations of the pendulum from the average position, this diagram of the device is equivalent to a linear newtonometer.

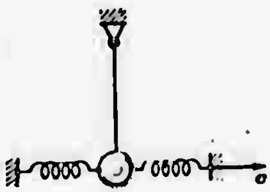

Fig. 1.3

Schemes of newtonometers, called integrating newtonometers or integrator-newtonometers are possible in which the readings of the newtonometers are proportional to the integrals or even to double integrals of n_{e_s} in time. These schemes are completely equivalent to that of a linear newtonometer: the first (or, accordingly, the second) time derivative of their readings is equal to n_{e_s} and is calculated by equations (1.3) and (1.4).

In the considered schemes of newtonometers (Figures 1.1, 1.2 and 1.3), the elastic suspension of the sensitive mass is provided by using mechanical springs. In real designs of newtonometers elastic (restoring) forces of a different nature, most often electromagnetic forces, are usually employed. However, this circumstance is unimportant to explain the principle of operation of the newtonometer and to derive equations (1.1) and (1.3). Therefore, henceforth only a mechanical elastic (spring-loaded) suspension will be considered. Let us note, incidentally, that the condition of smallness of deformation of the elastic suspension of the newtonometer is not the principal one and we can disregard it. Of course the presence of a linear dependence between deformation and the elastic force of the suspension is also not compulsory. This function should be only single-valued. However, henceforth for purposes of simplicity, the relationship of deformation and force will be assumed to be linear, which does not negate the essence of the consideration.

As already noted, real designs of newtonometers are one-component. Three one-component newtonometers whose axes of sensitivity are not coplanar, may be assumed equivalent to a single three-dimensional newtonometer.[3] Thus, in speaking of vector $\vec{n}$, we will henceforth have in mind equation (1.3). We will assume that the readings of the newtonometers are the projections n_{e_s} of vector $\vec{n}$ to unit vectors $\vec{e}_s$ of the axes of sensitivity.

The readings of the newtonometers are the main information which is used in inertial navigation systems. The accuracy of operation of inertial navigation systems is determined mainly by the accuracy of the specific force measured by the newtonometer. Therefore, it is very important to have a distinct concept of the principal sources of errors of newtonometers. The first of them is related to the inaccuracy of measuring the extent of deformation of the springs, which is the carrier of information about the magnitude of the elastic force. The second source of errors is determined by the fact that the actual dependence of the extent of deformation on the magnitude of the elastic force can be distinguished from the calculating relation used. The third source of errors may be the presence of unaccounted for forces, acting on the sensitive mass of the newtonometer, in addition to the force of elasticity of the suspension. These forces may be, for example, forces of dry and viscous friction, which occur in the device when the sensitive mass moves with respect to the housing. We note that the indicated categories of errors generally occur in any measuring device. Therefore, we can be concerned with them not only in the case of a mechanical spring-loaded suspension, which was discussed as an example, but also in the case of an elastic suspension of any nature. This in itself means that all the indicated errors can be both deterministic and random.

The essence of the method of inertial navigation reduces to integration of equation (1.3). Integration of this vector equation obviously requires conversion to three scalar equations, which can be obtained by projecting the vector equation to any three non-coplanar directions, Equation (1.3) is valid in the inertial system of coordinates $0_2\xi_*\eta_*\zeta_*$, while vector $\vec{n}$, contained in this equation, is known by its projections n_{e_s} to the axes of sensitivity $\vec{e}_s$ of the newtonometers. Thus, the most natural conversion to scalar equations is the projection of equation (1.3) either to the axes of the coordinate system $0_2\xi_*\eta_*\zeta_*$ or to the directions of the axes of sensitivity $\vec{e}_s$ of the newtonometers. It would be simplest if the directions of $\vec{e}_s$ were fixed in the coordinate system $0_2\xi_*\eta_*\zeta_*$, for example, if they coincided with the directions of the axes of this coordinate system.

If the directions of $\vec{e}_s$ vary their orientation in the coordinate system $0_2\xi_*\eta_*\zeta_*$, then one must know at each instant of time the position of the directions of $\vec{e}_s$ with respect to axes $\xi_*\eta_*\zeta_*$. One must also know the rates of change of the directions of $\vec{e}_s$ in the coordinate system $0_2\xi_*\eta_*\zeta_*$, because the right side of equation (1.3), which contains the second derivative $d^2\vec{r}_{o_2}/dt^2$, is projected to the movable direction of $\vec{e}_s$.

§ 1.3. The Precession Theory of Gyroscopic Devices of Inertial Systems

1.3.1. The Free Gyroscope

One of the possible methods of fixing the direction of the axes of sensitivity of newtonometers in the inertial system of coordinates or to obtain information about the position of these directions and the rates of their change is to use gyroscopic devices. The gyroscope, like the newtonometer, is the main sensing element of the inertial navigation system.

Let us consider the operating principle and the equations of operation of the main gyroscopic devices which can be used in inertial systems.

When deriving the equations of operation of gyroscopic devices, we will not go beyond the bounds of precession theory. This theory makes it possible to obtain the relations of interest to us simply and clearly. At the same time restriction to laws of the precession theory of gyroscopes only, as was indicated in A. Yu. Ishlinskiy's investigation,[4] does not lead to any appreciable errors or inaccuracies in the consideration of those aspects of the phenomena with which we must be concerned. The operating principles are usually selected and the circuits of gyroscopic devices are constructed usually on the basis of this theory. We resort to the complete equations of motion of the gyroscope in most cases only to provide stability of operation of the circuit (the stability of the operating conditions determined by precession equations) and the smallness of deviations of real from precession motion. In those cases when the motion of the gyroscope within the environs of precession motion is of a pre-oscillation nature, the complete equations are required to investigate the stability of the natural oscillations and to find their amplitudes, respectively.

Henceforth, when outlining the theory of gyroscopic devices of inertial navigation, we shall employ the methods of precession theory in the form developed by A. Yu. Ishlinskiy.[5] In this case we shall assume that the considered precession conditions are stable and we will not be concerned with the nature of the transient processes which provide this stability. Let us also note that precession theory in the problems which will be subsequently investigated yields high accuracy. This is a result of the circumstance that small and slow time-variable rates of precession are being considered.

Let us consider an ideal free gyroscope (Figure 1.4) that is a heavy disc rotating at constant angular velocity and installed without friction in a weightless gimbel suspension with three degrees of freedom.[6] The center of mass of the disc is located at the point of intersection of the suspension axes, which are assumed to be mutually perpendicular. The rotational axis of the disc coincides with its axis of symmetry.

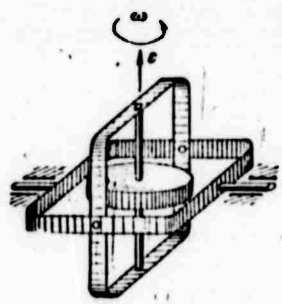

Fig. 1.4

The equation of motion (rotation) of a heavy solid with respect to a fixed point has the following form in the inertial coordinate system (the theorem of angular momentum):

(1.5)

$$\frac{dK}{dt} = M. \qquad (1.5)$$

where K is the vector of angular momentum and M is the vector of the total moment of external forces with respect to the point of the suspension.

It is assumed in precession theory that the angular momentum of a gyroscope is determined only by its natural rotation and is always directed along the axis of its figure. Therefore, by denoting the moment of inertia of the gyroscope with respect to the rotational axis by C, the angular velocity of natural rotation by $\vec{\omega}$ and the unit vector of the gyroscope axis (the axis of natural rotation) by $\vec{\varepsilon}$, we will have:

$$\frac{d}{dt}(C\omega\varepsilon) = M. \qquad (1.6) \tag{1.6}$$

Assuming that the kinetic moment C_ω of the gyroscope is constant and denoting it by H, we find the equation

$$\frac{d\varepsilon}{dt} = \frac{M}{H}, \tag{1.7}$$

which relates the rate of change of direction of vector $\vec{\varepsilon}$ to the external force moment.

If M equals zero, it follows from expression (1.7) that

$$\frac{d\varepsilon}{dt} = 0, \quad \varepsilon = \varepsilon_0. \tag{1.8}$$

Thus, a free gyroscope maintains a constant direction of its rotational axis (the axis of the kinetic moment) in the inertial coordinate system.

If three free gyroscopes are taken and the directions of the axes of sensitivity of the newtonometers are related to the directions of their kinetic moments, for example by aligning them identically ($\vec{e}_s=\vec{c}_s$), and if the directions of $\vec{e}_s$ are combined with the directions of the coordinate axes ξ_*,η_* and ζ_*, then the newtonometer readings of n_{e_s} will be projections of equations (1.3) to the axes of the inertial system of coordinates. It is easy to see that two free gyroscopes, with whose axes t e of ξ_* and η_*, for example, can be combined are sufficient. The equalities $\vec{e}_s=\vec{c}_s$, of course, do not have to be fulfilled. It is sufficient to have only two free gyroscopes with non-collinearly arranged kinetic moments and to be given the position of the directions of the axis of sensitivity of $\vec{e}_s$ with respect to the directions of their kinetic moments. The position of the directions of $\vec{e}_s$ is completely determined by this in inertial space.

In real designs the moment M is distinct from zero because of friction in the suspension axes, residual unbalance of the rotor, etc. Therefore,

$$\frac{de}{dt}=\frac{M_*}{H}. \tag{1.9}$$

where M_* is the perturbing moment. Consequently, the axis of the gyroscope rotor will be slowly processed (the so-called free deflection of the gyroscope) by varying its orientation in space with time.

We note that, along with the effect of the above perturbing moments, a number of effects determined by the characteristics of the dynamics of motion of a free gyroscope in a gimbal suspension and related primarily to the effect of equatorial moments of inertia of the gyroscope rotor and the moments of inertia of the suspension rings, is also added to the free deflection.[7]

1.3.2. A one-component absolute angular rate meter.

Let us consider a gyroscope (Figure 1.5), mounted on a platform in a suspension with two degrees of freedom. The center of mass of the gyroscope coincides with the center of the suspension. The gyroscope housing is connected to the platform by a spring, which creates an elastic moment around the axis of the housing as it rotates with respect to the platform.

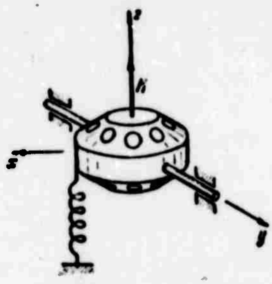

Fig. 1.5

For comparison of the equations of motion of the gyroscope, let us introduce a right-hand orthogonal system of coordinates Oxyz, bound to the platform. Let us locate its origin in the center of the gyroscope suspension, let us align the y axis along the axis of its housing and the z axis normal to the plane of the platform. Let point O be fixed in the inertial coordinate system and let the platform rotate arbitrarily with respect to this point, so that projections of its absolute angular velocity $\vec{\omega}$ to the x, y and z axes are ω_x, ω_y, and ω_z.

Let us connect the trihedron $Ox_1y_1z_1$ obtained from the trihedron Oxyz by rotation of it by angle δ around the y axis, to the gyroscope housing. Rotation is counter clockwise if we look from the end of the y axis (fig. 1.6), so that the vector of relative angular velocity $\vec{\dot\delta}$ is directed along this axis.

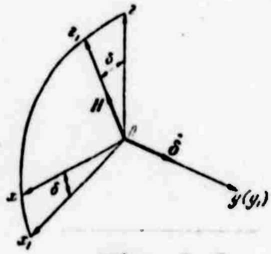

Fig. 1.6

Let us apply the theorem of the kinetic moment to the gyroscope housing with rotor. Let us project the vector equation (1.5), given in the inertial system of coordinates, to the mobile x, y and z axes.

Let

$$K = K_x x + K_y y + K_z z. \tag{1.10}$$

where K_x, K_y and K_z are projections of vector $\vec{K}$ to the x, y and z axes, and $\vec{x}$, $\vec{y}$ and $\vec{z}$ are the unit vectors of these axes. Then,

$$\frac{dK}{dt} = \dot{K}_x x + \dot{K}_y y + \dot{K}_z z + K_x \frac{dx}{dt} + K_y \frac{dy}{dt} + K_z \frac{dz}{dt} \tag{1.11}$$

(time differentiation is denoted by the dots). Since $d\vec{x}/dt$, $d\vec{y}/dt$ and $d\vec{z}/dt$ are the velocities of the ends of the unit vectors of the mobile coordinate system, we have

$$\frac{dx}{dt} = \omega \times x, \quad \frac{dy}{dt} = \omega \times y, \quad \frac{dz}{dt} = \omega \times z. \tag{1.12}$$

Consequently,

$$\frac{dK}{dt} = \dot{K}_x x + \dot{K}_y y + \dot{K}_z z + \omega \times (K_x x + K_y y + K_z z). \tag{1.13}$$

The vector $\dot{K}_x \vec{x} + \dot{K}_y \vec{y} + \dot{K}_z \vec{z}$ is the derivative of vector $\vec{K}$, if we assume that the coordinate system xyz is fixed with respect to the inertial system. This derivative is usually called the local derivative of the vector.

Thus

$$\frac{dK}{dt} = \dot{K} + \omega \times K \tag{1.14}$$

($\dot{K}$ is the local derivative) and from expressions (1.14) and (1.5), we find:

$$\left.\begin{aligned} \dot{K}_x + \omega_y K_z - \omega_z K_y &= M_x, \\ \dot{K}_y + \omega_z K_x - \omega_x K_z &= M_y, \\ \dot{K}_z + \omega_x K_y - \omega_y K_x &= M_z. \end{aligned}\right| \tag{1.15}$$

Limiting ourselves to within the scope of precession theory, let us take into account during calculation of $\vec{K}$ only the kinetic moment $\vec{M}$ of the gyroscope rotor.

It follows from Figures 1.5 and 1.6 that

$$K_x = H \sin\delta, \quad K_y = 0, \quad K_z = H \cos\delta. \tag{1.16}$$

It is obvious that in the considered case the moments M_x, M_y and M_z are made up of the elastic moment of the spring and of the moments of the normal reactions of the suspension pins of the axis of the housing. By noting that normal reactions do not create a moment with respect to the y axis of the suspension and by assuming that the elastic moment is proportunal to the deformation of the spring, i.e., to angle $\vec{\sigma}$, we find from relations (1.15) and (1.16):

$$H(\omega_z \sin\delta - \omega_x \cos\delta) = -k\delta, \tag{1.17}$$

where k is the proportionality constant

Thus,

$$\omega_z \sin\delta - \omega_x \cos\delta = -\omega_{x_1}, \tag{1.18}$$

then from equality (1.17), we find:

$$\delta = \frac{H}{k}\omega_{x_1}. \tag{1.19}$$

By assuming that angle δ is small and by assuming cos δ=1 and sin δ=δ, according to equality (1.18) we can write:

$$\delta = \frac{H}{k}\omega_x \frac{1}{1+\frac{H\omega_x}{k}}. \tag{1.20}$$

If now

$$\frac{H\omega_x}{k} \ll 1, \tag{1.21}$$

then

$$\delta = \frac{H}{k}\omega_x. \tag{1.22}$$

Thus, the value δ of elastic deformation of the spring is proportional to the projection of ω_{x_1} of the absolute angular velocity of the platform to the axis x_1, and if δ is small and if condition (1.21) is observed, then the value of δ is proportional to the projection of ω_x of the absolute angular velocity of the platform to its x axis. The value of the elastic deformation of the spring can obviously be measured. The considered device may be called a one-component absolute angular rate meter.

1.3.3. A two-component single-gyroscopic absolute angular rate meter. We can show that two components of absolute angular velocity of the platform can be measured with certain assumptions using a single gyroscope, i.e., the rate of variation of some direction in the inertial system of coordinates can be measured. This possibility is indicated by the circumstance that a free gyroscope maintains a direction of the vector of the kinetic moment, fixed in absolute space.

Let us consider a diagram (Figure 1.7) which differs from that presented in Figure 1.5 by the fact that the gyroscope is mounted on a platform in a suspension with three degrees of freedom. The gyroscope housing is connected to the frame of the gimbal suspension by a spring whose deformation leads to generation of a moment which acts on the housing and which is directed along its axis (as in a one-component meter). The frame of the device (the platform) is connected to the housing in the same fashion. Consequently, the gyroscope housing is mounted in a flexible suspension with two degrees of freedom. The total elastic moment of the suspension is the only external moment which acts on the gyroscope. The vector of the elastic moment, divided by the value of the kinetic moment of the gyroscope H, determines the rate of variation of the direction of the gyroscope axis in the inertial system of coordinates according to equation (1.7). Therefore, the projections of the absolute angular velocity to the axes of the housing and frame can be determined by measuring the values of the deformation of the springs.

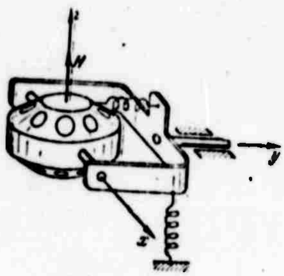

Fig. 1.7

Let us analyze in more detail the operation of the device. Let us connect to its housing a right-hand orthogonal system of coordinates Ox'y'z' (Figure 1.8), whose origin we locate in the center of mass of the gyroscope, we direct the y' axis along the axis of the frame, and we locate the x' axis in the plane in which the frame is located, when the spring of its suspension is not deformed. Let us connect to the frame the coordinate system Oxyz' obtained as a result of rotating system $Ox'y'z'$ by an angle δ around the y' axis. Let us also introduce the coordinate system $Ox_1y_1z_1$, rigidly bound to the gyroscope housing. The trihedron $Ox_1y_1z_1$ is obtained from the trihedron $Oxyz$ by rotating the latter by an angle δ_2 (Figure 1.8) around the x axis, which is coincident with the axis of suspension of the housing.

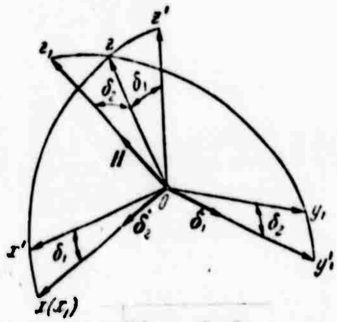

Fig. 1.8

Let us now make use of the theorem of the kinetic moment [equation (1.5)], having applied it to the two mechanical systems: to the housing of the gyroscope and to the frame with housing. If the values contained in equation (1.5) are denoted for the first system by $\vec{K}^1$ and $\vec{M}^1$ and those for the second system are denoted by $\vec{K}^2$ and $\vec{M}^2$, we find:

$$\frac{dK^1}{dt} = M^1, \quad \frac{dK^2}{dt} = M^2. \tag{1.23}$$

Equations (1.23) are equivalent to two systems of scalar equations of the type of (1.15). The six equations obviously permit calculation of the unknown values of δ_1, δ_2 and four moments of the normal reactions of the axial supports of the suspension of the housing and of the inner gimbal.

Since we are primarily interested in the relationship of the values of δ_1 and δ_2 to the elastic moments of the suspension, we can project the equations (1.23) to those axes with respect to which the normal reactions do not yield moments. For the gyroscope housing, this axis is the x axis of the housing suspension and for the housing-frame system, it is the y(y') axis of the frame suspension. Then, according to equations (1.15), we will have:

$$\left.\begin{aligned} \dot{K}^1_x + \omega_y K^1_z - \omega_z K^1_y &= M^1_x, \\ \dot{K}^2_y + \omega_z K^2_x - \omega_x K^2_z &= M^2_y. \end{aligned}\right\} \tag{1.24}$$

Since only the natural kinetic moment of the gyroscope is taken into account, we have

$$K^1_x = K^2_x = 0, \quad K^1_y = K^2_y = -H\sin\delta_2, \quad K^1_z = K^2_z = H\cos\delta_2. \tag{1.25}$$

By noting that moments M^1_x, M^2_y are created only by the springs of the suspension, and by assuming that they are proportional to the deformations of the latter,

$$M^1_x = -k\delta_2, \quad M^2_y = -k\delta_1, \tag{1.26}$$

we find from equations (1.24) and relations (1.25) the dependence of δ_1 and δ_2 on ω_x, ω_y and ω_z of interest to us:

$$\left.\begin{aligned} H(\omega_y\cos\delta_2 + \omega_z\sin\delta_2) &= -k\delta_2, \\ H\left(+\frac{d}{dt}\sin\delta_2 - \omega_x\cos\delta_2\right) &= -k\delta_1. \end{aligned}\right\} \tag{1.27}$$

The first equation of (1.27) is similar to equation (1.17), and since

$$\omega_y\cos\delta_2 + \omega_z\sin\delta_2 = \omega_{y_1}, \tag{1.28}$$

it can be written in the form of equation (1.19):

$$\delta_2 = -\frac{H}{k}\omega_{y_1}. \tag{1.29}$$

Since

$$\omega_x + \dot{\delta}_2 = \omega_{x_1}, \tag{1.30}$$

then it follows from the second equation of (1.27) that

$$\delta_1 = \frac{H \cos \delta_2}{k} \omega_{x_1}. \tag{1.31}$$

If we assume $\cos \delta_2 = \cos \delta_1 = 1$, $\sin \delta_1 = \delta_1$ and $\sin \delta_2 = \delta_2$, we have

$$\left.\begin{aligned} \delta_2 &= -\frac{H}{k} \frac{1}{1 + \frac{H\omega_x}{k}} \omega_y, \\ \delta_1 &= \frac{H}{k} \omega_x + \frac{H}{k} \dot{\delta}_2. \end{aligned}\right\} \tag{1.32}$$

If we required that the following equality be fulfilled

$$\frac{H}{k} |\omega_x| \ll 1, \quad \frac{H^2}{k^2} |\dot{\omega}_y| \ll 1, \tag{1.33}$$

we find

$$\delta_2 = -\frac{H}{k} \omega_y, \quad \delta_1 = \frac{H}{k} \omega_x. \tag{1.34}$$

Thus, we can find the projections of ω_{x_1} and ω_{y_1} of the absolute angular velocity of the gyroscope housing to its axes according to equalities (1.29) and (1.31) from the results of measuring the deformations δ_1 and δ_2 of the elastic suspension, these projections coinciding with those of ω_x and ω_y of the absolute angular velocity ω to the axis of the housing suspension and to the axis of the frame at small values of δ_1 and δ_2 and if the requirements of (1.33) are fulfilled.

The considered device can be called a two-component single-gyroscope absolute angular rate meter.[8]

1.3.4. A three-dimensional absolute angular rate meter. Three one-component meters, structurally connected into a single block so that their axes of sensitivity form an orthogonal trihedron, are employed more often than other schemes for measuring the absolute angular velocity of a rotating trihedron. This unit is a platform (Figure 1.9), on which three gyroscopes G_1, G_2, G_3 are installed in suspensions with two-degrees of freedom. A right-hand orthogonal coordinate system Oxyz, whose Oz axis is normal to the plane of the platform, is connected to the platform. The axes of the housings are parallel to the plane of the platform, where the x_1 and x_3 axes of the housings of gyroscopes G_1 and G_3 are parallel to the x axis of the platform, while the y_2 axis of the housing of gyroscope G_2 is parallel to the y axis of the platform. The gyroscope housings are connected to the platform by springs (they are not shown in Figure 1.9), which create moments around the axes of the housings similar to that which occurred in a one-component absolute angular-rate meter (figure 1.5). In the position when the springs are not deformed, vectors H_1 and H_2 of the kinetic moments of gyroscopes G_1 and G_2 are normal to the plane of the platform, and vector H_3 of the kinetic moment of gyroscope G_3 is parallel to the y axis.

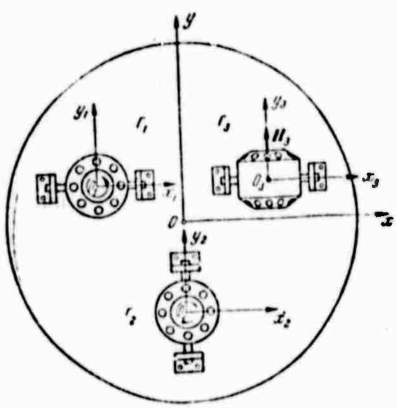

Fig. 1.9

The orientation of the gyroscope housings with respect to the direction of the x, y and z axes is determined by the position of the systems $O_1x_1y_1z_1$, $Ox_2y_2z_2$, and $O_3x_3y_3z_3$. The origin of each of these systems is shifted with the center of the suspension of the corresponding gyroscope, the z_1, z_2, and y_3 axes coincide with vectors H_1, H_2, and H_3 of the kinetic moments, while the x_1, x_3, and y_2 axes are directed along the axes of the housings of the corresponding gyroscopes. As was already noted, when the springs of the suspensions are not deformed, the z_1, z_2 and z_3 axes are normal to the plane of the platform (the xy plane). In the general case, these axes are deflected from the normal toward the platform by angles δ_1, δ_2 and δ_3, respectively, so that the table of the direction cosines between axes x_1, y_1, z_1; x_2, y_2 and z_2 and x_3, y_3 and z_3 and between axes x, y and z has the form:

(1.35)

	x_1	y_1	z_1	x_2	y_2	z_2	x_3	y_3	z_3
x	1	0	0	$\cos\delta_2$	0	$\sin\delta_2$	1	0	0
y	0	$\cos\delta_1$	$-\sin\delta_1$	0	1	0	0	$\cos\delta_3$	$-\sin\delta_3$
z	0	$\sin\delta_1$	$\cos\delta_1$	$-\sin\delta_2$	0	$\cos\delta_2$	0	$\sin\delta_3$	$\cos\delta_3$.

Let point O (the center of the platform) be fixed in the inertial coordinate system. Then the motion of the platform consists only of rotation around point O, so that the projections of the absolute angular velocity $\vec{\omega}$ of the platform to the x, y and z will be ω_x, ω_y and ω_z.

Let us compose the equations of motions of gyroscopes G_1, G_2 and G_3 in projections to the x, y and z axes, having applied the theorem of the kinetic moment to each of the three gyroscope housings.

Projections of the kinetic moments K^1, K^2, K^3 to the x, y and z axes are found by using the tables of the direction cosines (1.35), if we take into account that the vectors $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$ of the kinetic moments are directed along the z_1, z_2 and z_3 axes, respectively. These projections are equal to:

(1.36)

$$\left.\begin{array}{lll} K_x^1 = 0, & K_y^1 = -H_1 \sin\delta_1, & K_z^1 = H_1 \cos\delta_1, \\ K_x^2 = H_2 \sin\delta_2, & K_y^2 = 0, & K_z^2 = H_2 \cos\delta_2, \\ K_x^3 = 0, & K_y^3 = H_3 \cos\delta_1, & K_z^3 = H_3 \sin\delta_3. \end{array}\right\}$$

By projecting the equations of the angular momentum for gyroscopes G_1 and G_3 to the x axis and that for gyroscope G_2 to the y axis, we find according to equations (1.15):

(1.37)

$$\left.\begin{array}{l} H_1(\omega_y \cos\delta_1 + \omega_z \sin\delta_1) = M_x^1, \\ H_2(\omega_z \sin\delta_2 - \omega_x \cos\delta_2) = M_y^2, \\ H_3(\omega_y \sin\delta_3 - \omega_z \cos\delta_3) = M_x^3. \end{array}\right\}$$

We note that the moments of the normal reactions of the supports are not contained in the moments M_x^1, M_y^2 and M_x^3. Moreover, since the gyroscope housings are assumed to be balanced with respect to the axes of their own suspensions, the moments of gravitational forces may be assumed equal to zero. However, one should bear in mind that in the previously considered cases the origin of the rotating coordinate system Oxyz coincided with the center of the gyroscope suspension (and with its center of mass). In the case now being considered, the centers O_1, O_2, O_3 of the gyroscope suspensions do not coincide with the center of rotation of the platform O. Therefore, additional forces of transient motion inertia and Coriolis forces, which, generally speaking, may create moments around the axes of the housings, act on the gyroscope masses. However, because of the small distances of points O_1, O_2 and O_3 from the center of rotation of O and because of the limitation of values ω_x, ω_y and ω_z, these moments are negligible. Also taking into account that perturbing moments may be created by only that portion of the forces of inertia, which determines the inhomogeneity of the inertial force field within the gyroscope housing rather than by all the forces of inertia because of the balance of the gyroscopes, we disregard the indicated moments as is accepted.

Thus, the only moments applied to the gyroscopes along the axes of the housings are those of the flexible couplings of the suspensions. By assuming that they are proportional to the deformation of the suspensions, we have:

$$M_x^1 = -k_1\delta_1, \quad M_y^2 = -k_2\delta_2, \quad M_z^3 = -k_3\delta_3.$$

By substituting these expressions into equalities (1.37) we find

(1.38)

$$\left.\begin{aligned} \omega_y \cos\delta_1 + \omega_z \sin\delta_1 &= -\frac{k_1}{H_1}\delta_1, \\ \omega_z \sin\delta_2 - \omega_x \cos\delta_2 &= -\frac{k_2}{H_2}\delta_2, \\ \omega_y \sin\delta_3 - \omega_x \cos\delta_3 &= -\frac{k_3}{H_3}\delta_3. \end{aligned}\right\}$$

In the relations (1.38), as follows from the table of direction cosines (1.35),

(1.39)

$$\left.\begin{aligned} \omega_y \cos\delta_1 + \omega_z \sin\delta_1 &= \omega_{y_1}, \\ \omega_z \sin\delta_2 - \omega_x \cos\delta_2 &= -\omega_{z_2}, \\ \omega_y \sin\delta_3 - \omega_x \cos\delta_3 &= -\omega_{z_3}. \end{aligned}\right\}$$

so that

(1.40)

$$\omega_{y_1} = -\frac{k_1}{H_1}\delta_1, \quad \omega_{z_2} = \frac{k_2}{H_2}\delta_2, \quad \omega_{z_3} = \frac{k_3}{H_3}\delta_3. \qquad (1.40)$$

The system of equations (1.38) should be solved to find the values of ω_x, ω_y and ω_z from the known values of δ_1, δ_2 and δ_3.

The determinant of the system of algebraic equations (1.38) with respect to ω_x, ω_y and ω_z is

$$\Delta = \begin{vmatrix} 0 & \cos\delta_1 & \sin\delta_1 \\ -\cos\delta_2 & 0 & \sin\delta_2 \\ 0 & \sin\delta_3 & \cos\delta_3 \end{vmatrix} = -\cos\delta_2\cos(\delta_1-\delta_3). \tag{1.41}$$

This determinant is equal to zero when the following equalities occur separately or simultaneously

$$\delta_2 = \pm\frac{\pi}{2}, \quad \delta_1-\delta_3 = \pm\frac{\pi}{2}. \tag{1.42}$$

When fulfilling the first equality of (1.42), the vector $\vec{H}_1$ of the kinetic moment becomes parallel to the y axis, and when the second equality is fulfilled, the vectors of the kinetic moments $\vec{H}_1$ and $\vec{H}_2$ become parallel.

In our case angles δ_1, δ_2 and δ_3 are small, the determinant (1.41) is different from zero and the system of equations (1.38) has a single-valued solution:

$$\left.\begin{aligned} \omega_x &= \frac{k_2}{H_2}\delta_2 + \frac{\sin\delta_2}{\cos\delta_2\cos(\delta_1-\delta_3)}\times \\ &\qquad \times\left(\frac{k_3}{H_3}\delta_3\cos\delta_1 - \frac{k_1}{H_1}\delta_1\sin\delta_1\right), \\ \omega_y &= -\frac{1}{\cos(\delta_1-\delta_3)}\left(\frac{k_3}{H_3}\delta_3\sin\delta_1 + \frac{k_1}{H_1}\delta_1\cos\delta_1\right), \\ \omega_z &= -\frac{1}{\cos(\delta_1-\delta_3)}\left(\frac{k_3}{H_3}\delta_3\cos\delta_1 - \frac{k_1}{H_1}\delta_1\sin\delta_1\right). \end{aligned}\right| \tag{1.43}$$

We note that formulas (1.43) are accurate. Their derivation did not require restrictions of the type of (1.21) and (1.33), which were introduced in one- and two-component (simple gyroscope) absolute angular rate meters.

If the values of angles δ_1, δ_2 and δ_3 are small, then, by retaining terms of the second order of the smallness, we find, from formulas (1.43)

$$\left.\begin{aligned} \omega_x &= \frac{k_1}{H_1}\delta_2 + \frac{k_1}{H_1}\delta_1\delta_3, \\ \omega_y &= -\frac{k_1}{H_1}\delta_1 - \frac{k_2}{H_2}\delta_1\delta_3, \\ \omega_z &= -\frac{k_3}{H_3}\delta_3 + \frac{k_1}{H_1}\delta_1\delta_3. \end{aligned}\right| \tag{1.44}$$

Thus, the relations

$$\omega_x = \frac{k_1}{H_1}\delta_2, \quad \omega_y = -\frac{k_1}{H_1}\delta_1, \quad \omega_z = -\frac{k_3}{H_3}\delta_1 \tag{1.45}$$

determine the projections of the absolute angular velocity of the platform to its axes with an accuracy up to terms linear with respect to δ_1, δ_2 and δ_3.

We note that the arrangement of the gyroscopes presented in Figure 1.9 is not the only one. Other arrangements are possible which satisfy the condition that the vectors

$$H_1 \times s_1, \quad H_2 \times s_2, \quad H_3 \times s_3 \tag{1.46}$$

form an orthogonal set of three (here $\vec{s}_1$, $\vec{s}_2$ and $\vec{s}_3$ are the unit vectors of the directions of the axes of the gyroscope housings).

In the gyroscopic indicators of absolute angular velocity considered above, the elastic moments around the axes of the gyroscope suspension were created by using the springs. In real designs these moments can also be created by forces of different origin, for example, by electromagnetic forces. The nature of the restoring moments has no essential significance for derivation of the relations which determine the operation of gyroscopic velocity meters. As in the newtonometer circuit, the elastic moment in gyroscopic absolute angular rate meters does not have to be proportional the the angle of rotation (deformation of the spring). If the dependence is linear, the corresponding relations become especially simple and principally important only in order that the dependence of the elastic forces onto the corresponding angles be known and single-valued. As a measuring device the gyroscopic absolute angular rate sensor is similar in many ways to the newtonometer. The sources of errors of newtonometers and of absolute angular rate meters, in particular, are similar in many ways. The main errors of the latter are related to inaccurate sampling of the value of spring deformation, to an imprecise knowledge of the actually existing dependence of the value of the elastic moments onto the corresponding deformations (or the instability of this dependence from measurement to measurement) and to moments not taken into account.

These moments are caused by two main factors: non-coincidence of the center of mass of the gyroscope to the center of its suspension and to the moments of dry and viscous friction in the supports of the axes of the gyroscope housings. Besides the indicated factors, certain affects related to the dynamics of motion of the gyroscopic measuring device in the gimbal suspension with regard to the moments of inertia of the wheels of the latter,[9] also leads to errors of the measuring device.

All these errors can be represented in the form of certain perturbing moments M^4_{2x}, M^5_{2y} and M^6_{2x}, which act along the axes of the housings of gyroscopes G_1, G_2 and G_3. The instrument errors $\Delta\omega_x$, $\Delta\omega_y$ and $\Delta\omega_z$ of the absolute angular rate meter will then be equal to:

$$\Delta\omega_x = -\frac{M^5_{2y}}{H}, \quad \Delta\omega_y = \frac{M^4_{2x}}{H}, \quad \Delta\omega_z = \frac{M^6_{2x}}{H}. \tag{1.47}$$

It is also necessary to bear in mind another circumstance. When deriving all the relations for angular rate meters it was assumed that the natural kinetic moment of the gyroscope is constant. Moreover, in real gyroscopes the constancy of the rate of turning of the rotor with respect to the housing can of course be maintained only with some finite accuracy. The difference of the value of the kinetic moment of the gyroscope from the constant value also leads to errors in absolute angular rate meters. The nature of these errors is easily established by resorting to the initial equation of angular momentum (1.5). Since only the natural kinetic moment of the gyroscope was taken into account when deriving the equations of motion of angular rate meters, then by introducing the unit vector $\vec{\varepsilon}$ of the direction of the kinetic moment vector, we find:

$$\frac{d}{dt}(H\varepsilon) = M, \quad H = H_0 + \Delta H(t), \tag{1.48}$$

where $\Delta H(t)$ is variation of the value of the kinetic moment. Then,

$$(H + \Delta H)\frac{d\varepsilon}{dt} = M - \varepsilon\frac{d}{dt}\Delta H. \tag{1.49}$$

It follows from expression (1.49) that variation of the kinetic moment H by value $\Delta H(t)$ leads to the fact that only $H+\Delta H$ instead of H should be substituted in all the derived equations, because the perturbing moment $-\vec{\varepsilon}\frac{d}{dt}\Delta H$ is immaterial in view of the fact that it is directed along the gyroscope axis.

For a free gyroscope some (small) variation of the value of H of course has no significance whatever.

1.3.5. Free and controlled gyrostabilized platforms. In conclusion let us consider yet another type of gyroscopic device, used to maintain fixed orientation in an absolute space bound to the gyroscopes of a trihedron or to change this orientation by a given law. We have in mind devices which are called gyrostabilized platforms. These devices are employed extensively in view of a number of their inherent advantages. Without familiarization with them, exposition of the operating principles of gyroscopic orientation displays would be essentially incomplete.

A three-dimensional gyrostabilized platform (Figure 1.10) is a platform mounted in a suspension with three degrees of freedom. Three gyroscopes G_1, G_2 and G_3 are secured on the platform in suspensions with two degrees of freedom in the same manner as in the previously considered three-component absolute angular rate meter (Figure 1.9). Unlike the latter, there is no flexible coupling of the gyroscope housings to the platform. Sensors DU_1, DU_2 and DU_3 of angles δ_1, δ_2 and δ_3 of rotation of the axes of the housings with respect to the platform are installed along the axes of the housings. These attitude sensors control operation of engines En_1, En_2 and En_3, which create moments with respect to the axes of the gimbal suspension. In the case of a controlled platform, moment sensors DM_1, DM_2 and DM_3, by means of which given (control or correcting) moments are transmitted to the gyroscopes of the platform, are installed along the axes of the housings. The attitude and moment sensors are denoted only by gyroscope G_2 in Figure 1.10.

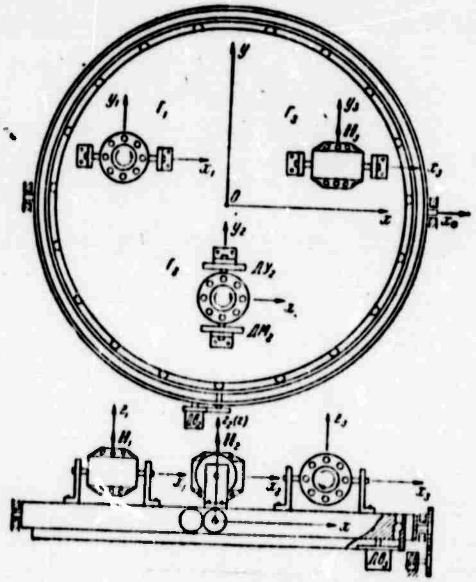

Fig. 1.10

Let us introduce the right-hand orthogonal coordinate systems

$$Ox_0y_0z_0,\ Ox'y'z',\ Ox''y''z'' \text{ and } Oxyz,$$

bound to the base on which the gimbal suspension of the platform is installed, to the outer ring of the gimbal suspension, to the inner ring of the gimbal suspension (to the outer ring of the ring mounting, Figure (1.10) and to the platform, respectively.

The x_0 axis is directed along the axis of the outer gimbal ring. The y_* and z_* axes form a right-hand orthogonal set of three with the x_* axis.

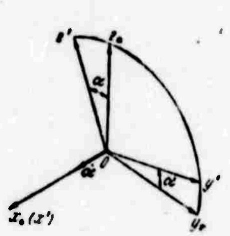

Fig. 1.11

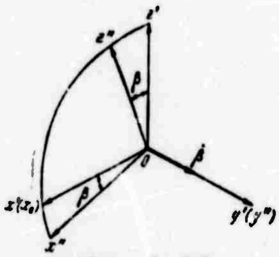

Fig. 1.12

The coordinate system Ox'y'z' (Figure 1.11) is obtained by rotating the coordinate system $Ox_*y_*z_*$ around the x_* axis by angle α. Counterclockwise rotation is assumed to be the forward direction of rotation if we look from the end of the x_* (x') axis. Thus, the relative angular velocity vector $\vec{\dot{\alpha}}$ coincides with the direction of the x_* (x') axis. The position of the y' axis determines the direction of the axis of the inner suspension ring. If α=o, the coordinate system Ox'y'z' accordingly coincides with the coordinate system $Ox_*y_*z_*$, bound to the base.

Trihedron Ox"y"z" (Figure 1.12) is obtained from trihedron Ox'y'z' by rotating it by angle β around the axis Oy' (the axis of the inner suspension ring). The vector $\vec{\dot{\beta}}$ of the relative angular rate of rotation is directed along axis y'(y"). Axis z" of trihedron Ox"y"z" coincides with the normal to the plane of the platform.

To convert to the coordinate system Oxyz (Figure 1.13), the trihedron Ox"y"z" should be rotated by angle γ around the z" axis, which obviously corresponds to rotation of the platform by angle γ with respect to the outer band of the ring mounting. Rotation counterclockwise is assumed to be positive if looking from the end of the z" axis. Vector $\vec{\dot{\gamma}}$ of the relative angular rate of rotation is directed along the z"(z) axis.

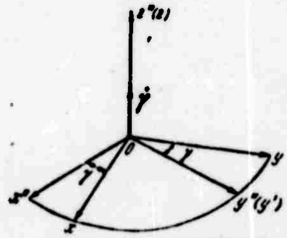

Fig. 1.13

The relative positions of the coordinate systems $Ox_*y_*z_*$, Ox'y'z', Ox"y"z" and Oxyz is determined by the following tables of direction cosines

(1.50)

	x'	y'	z'
x_*	1	0	0
y_*	0	$\cos\alpha$	$-\sin\alpha$
z_*	0	$\sin\alpha$	$\cos\alpha$

	x"	y"	z"
x'	$\cos\beta$	0	$\sin\beta$
y'	0	1	0
z'	$-\sin\beta$	0	$\cos\beta$

	x	y	z
x"	$\cos\gamma$	$-\sin\gamma$	0
y"	$\sin\gamma$	$\cos\gamma$	0
z"	0	0	1.

The vectors of the moments of the engines En_1, En_2 and En_3 are directed along the axes x_*(x'), y'(y") and z"(z) which are the axes of the platform suspension. The engine housings are installed on the base (object) (En_1), on the outer cardan ring (En_2) and on the platform (En_3), respectively.

This position of the gyroscopes on the platform (relative to the bound system of coordinates Oxyz) is the same as in the case of a three-component absolute angular rate meter (Figure 1.9). Therefore, to determine the position of the gyroscope housings relative to the x, y and z axes, the trihedrons $O_1x_1y_1z_1$, $_2x_2y_2z_2$ and $O_3x_3y_3z_3$, bound to them, whose orientation in the coordinate system Oxyz is given by the table of direction cosines (1.35), may be retained.

Let the center of the platform suspension - point O - be fixed in the inertial coordinate system and let the projections to the x, y and z axes of the absolute angular rate $\vec{\omega}$ of the platform in its motion with respect to point O be ω_x, ω_y and ω_z.

To construct the equations of motion of a gyrostabilized platform, six mechanical systems should be considered: 1) the device as a whole, 2) the inner gimbals that which is distributed on it, 3) the platform together with the gyroscopes mounted on it, 4) the housing of gyroscope G_1, 5) the housing of gyroscope G_2, and 6) the housing of gyroscope G_3. The motion of these systems completely determines the motion of all parts of the device both relative to the inertial system of coordinates and relative to each other.

The theorem of the kinetic moment [equation (1.5)] is used to compile the equations of motion. Having applied it to each of the systems being considered, we find:

$$\frac{dK^i}{dt} = M^i \quad (i = 1, 2, 3, 4, 5, 6) \tag{1.51}$$

The system of equations (1.51) is equivalent to 18 scalar equations, of which in the general case 18 unknowns can be determined: six angles $\alpha, \beta, \gamma, \delta_1, \delta_2$ and δ_3 and 12 moments of the normal reactions of the supports of six axes (three gimbal axes of the platform and three axes of suspension of the gyroscope housings on the platform).

However, on the basis of equations (1.51), we can find those six relations into which the moments of normal reactions do not enter. To do this, we should obviously project the i-th equation of (1.51) to the direction $\vec{\mu}^i$, so that the projection of the vector $\vec{N}_i$ of the moment of normal reactions in this direction is equal to zero.

According to relations (1.15) and (1.51), this type of equation will have the form:

$$\left(\frac{dK_x^i}{dt}+\omega_y K_z^i-\omega_z K_y^i\right)\cos(\widehat{x,\mu^i})+ \\ +\left(\frac{dK_y^i}{dt}+\omega_z K_x^i-\omega_x K_z^i\right)\cos(\widehat{y,\mu^i})+ \\ +\left(\frac{dK_z^i}{dt}+\omega_x K_y^i-\omega_y K_x^i\right)\cos(\widehat{z,\mu^i})= \\ = M_x^i\cos(\widehat{x,\mu^i})+M_y^i\cos(\widehat{y,\mu^i})+M_z^i\cos(\widehat{z,\mu^i}). \tag{1.52}$$

Since,

$$M_x^i\cos(\widehat{x,\mu^i})+M_y^i\cos(\widehat{y,\mu^i})+M_z^i\cos(\widehat{z,\mu^i})=M_{\mu^i}, \tag{1.53}$$

then we can select the directions of the suspension axes for the directions of $\vec{\mu}^i$.

As before, on the basis of precession theory, when calculating $\vec{K}^i$, we take into account only the natural kinetic gyroscopic moments. By noting that all three gyroscopes are contained in the first three systems into which we divided the considered device, we conclude that

$$K^1=K^2=K^3=K'.$$

As already noted, this position of the gyroscopes of the investigated device with respect to the x, y and z axes is similar to the disposition which occurred in the previously considered three-component absolute angular rate meter. Therefore, when looking for the projection of the vector $\vec{K}'$ to the x, y and z axes, we can use expressions (1.36) to project the kinetic moment of each gyroscope to these axes. By totalling the corresponding projections and by assuming for simplicity

$$H_1 = H_2 = H_3 = H,$$

we find

$$\left.\begin{array}{l} K'_x = H\sin\delta_2, \quad K'_y = H(-\sin\delta_1 + \cos\delta_3), \\ K'_z = H(\cos\delta_1 + \cos\delta_2 - \sin\delta_3). \end{array}\right\} \tag{1.54}$$

Let us take the direction of axis $0x_*(0x')$ of the outer gimbals of the device as the direction of $\vec{\mu}^i$ for the first system. The cosines of the angles of this axis with the x, y and z axes, according to the tables (1.50), are equal to:

$$\cos(\widehat{x', x}) = \cos\beta\cos\gamma, \quad \cos(\widehat{x', y}) = -\cos\beta\sin\gamma, \quad \cos(\widehat{x', z}) = \sin\beta. \tag{1.55}$$

By substituting expressions (1.53), (1.54) and (1.55) into equality (1.52), we find the equation of motion of the first system:

$$\begin{aligned} H\Big\{&\Big[\frac{d}{dt}\sin\delta_2 + \omega_y(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) - \\ &- \omega_z(-\sin\delta_1 + \cos\delta_3)\Big]\cos\beta\cos\gamma - \Big[\frac{d}{dt}(-\sin\delta_1 + \cos\delta_3) + \\ &+ \omega_z\sin\delta_2 - \omega_x(\cos\delta_1 + \cos\delta_2 + \sin\delta_3)\Big]\cos\beta\sin\gamma + \\ &+ \Big[\frac{d}{dt}(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) + \omega_x(-\sin\delta_1 + \cos\delta_3) - \\ &- \omega_y\sin\delta_2\Big]\sin\beta\Big\} = M'_{x'}. \end{aligned} \tag{1.56}$$

Let us take the direction of the y' axis of the inner ring as the direction of $\vec{\mu}^i$ for the second system. Taking into account that

$$\cos(\widehat{y', x}) = \sin\gamma, \quad \cos(\widehat{y', y}) = \cos\gamma, \quad \cos(\widehat{y', z}) = 0, \tag{1.57}$$

we find the equation of motion of the second system:

$$H\Big\{\Big[\frac{d}{dt}\sin\delta_2 + \omega_y(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) - \\ - \omega_z(-\sin\delta_1 + \cos\delta_3)\Big]\sin\gamma + \Big[\frac{d}{dt}(-\sin\delta_1 + \cos\delta_3) + \\ + \omega_x\sin\delta_2 - \omega_z(\cos\delta_1 + \cos\delta_2 + \sin\delta_3)\Big]\cos\gamma\Big\} = M_{y'}^2. \tag{1.58}$$

For the third system (the platform), the direction of μ^i is the direction of the z axis; therefore, its equation of motion is simpler than the two preceeding ones. It has the form:

$$H\Big[\frac{d}{dt}(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) + \omega_x(-\sin\delta_1 + \cos\delta_3) - \\ - \omega_y\sin\delta_2\Big] = M_z^3. \tag{1.59}$$

It remains for us to draw up the equation of motion of the gyroscopes G_1, G_2 and G_3. The directions of $\vec{\mu}^i$ for them will be the directions of the axes of the housings. Since disposition of the gyroscopes with respect to the platform is taken the same as in an angular rate meter with three degrees of freedom, then the equations of systems 4, 5 and 6 will coincide with equations (1.37), if we set:

$$H_1 = H_2 = H_3 = H; \quad M_x^1 = M_x^4, \quad M_y^2 = M_y^5, \quad M_z^3 = M_z^6. \tag{1.60}$$

By combining equations (1.37), (1.56), (1.58) and (1.59) we find a complete system of six first-order differential equations

/

which describe the motion of the gyrostabilized platform:

$$
\left.
\begin{aligned}
& H\Big\{\Big[\frac{d}{dt}\sin\delta_2 + \omega_y(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) - \\
& \qquad - \omega_z(-\sin\delta_1 + \cos\delta_3)\Big]\cos\beta\cos\gamma - \\
& \qquad - \Big[\frac{d}{dt}(-\sin\delta_1 + \cos\delta_3) + \omega_x\sin\delta_2 - \\
& \qquad - \omega_z(\cos\delta_1 + \cos\delta_2 + \sin\delta_3)\Big]\cos\beta\sin\gamma + \\
& + \Big[\frac{d}{dt}(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) + \omega_z(-\sin\delta_1 + \cos\delta_3) - \\
& \qquad\qquad - \omega_y\sin\delta_2\Big]\sin\beta\Big\} = M_x^1. \\
& H\Big\{\Big[\frac{d}{dt}\sin\delta_2 + \omega_y(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) - \\
& - \omega_z(-\sin\delta_1 + \cos\delta_3)\Big]\sin\gamma + \Big[\frac{d}{dt}(-\sin\delta_1 + \cos\delta_3) + \\
& + \omega_x\sin\delta_2 - \omega_z(\cos\delta_1 + \cos\delta_2 + \sin\delta_3)\Big]\cos\gamma\Big\} = M_y^2. \\
& H\Big[\frac{d}{dt}(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) + \\
& \qquad + \omega_z(-\sin\delta_1 + \cos\delta_3) - \omega_y\sin\delta_2\Big] = M_z^3. \\
& \qquad H(\omega_y\cos\delta_1 + \omega_z\sin\delta_1) = M_x^4. \\
& \qquad H(\omega_x\sin\delta_2 - \omega_z\cos\delta_2) = M_y^5. \\
& \qquad H(\omega_y\sin\delta_3 - \omega_x\cos\delta_3) = M_z^6.
\end{aligned}
\right| \tag{1.61}
$$

Let us consider the right sides of equations (1.61).

The moments M_x^1, M_y^2 and M_z^3 which act along the axes of the gimbals of the platform, can be represented in the following form:

$$
\left.
\begin{aligned}
M_x^1 &= M_{1x}^1 + M_{2x}^1, \\
M_y^2 &= M_{1y}^2 + M_{2y}^2, \\
M_z^3 &= M_{1z}^3 + M_{2z}^3.
\end{aligned}
\right| \tag{1.62}
$$

where $M^1_{1x'}$, $M^2_{1y'}$ and M^3_{1z} are the moments created by the relief engines En_1, En_2 and En_3 and which are dependent on the angles δ_1, δ_2 and δ_3 of the rotation of the gyroscope housings relative to the platform, and $M^1_{2x'}$, $M^2_{2y'}$ and M^3_{1z} are the destabilizing moments. The destabilizing moments are formed by the friction forces in the supports of the platform suspension axes and by attractive forces (within accurate balancing). The moments caused by errors of forming the unloading moments are also related to this.

The moments M^4_x, M^5_y and M^6_z, which act along the axes of the gyroscope housings, may be represented in the form:

(1.63)

$$\left.\begin{aligned} M^4_x &= M^4_{1x} + M^4_{2x}, \\ M^5_y &= M^5_{1y} + M^5_{2y}, \\ M^6_z &= M^6_{1z} + M^6_{2z}. \end{aligned}\right\}$$

Here M^4_{1x}, M^5_{1y} and M^6_{1x} are the controlling moments which orient the platform in the given manner. Moments M^4_{2x}, M^5_{2y} and M^6_{2x} occur because of friction in the supports of the axes of the gyroscope housings, unbalancing of the housings relative to their axes and because of errors of forming the controlling moments. The perturbing moments M^4_{2x}, M^5_{2y} and M^6_{2x} are the main cause of errors in orientation of the gyrostabilized platform.

We note that equations (1.61) are sufficient to describe the motion of the system (within the limits of precession theory) only on the assumption that the friction forces in the supports of the axes are not dependent on the magnitudes of the normal reactions. In the opposite case, it is of course necessary to retain all 18 equations of (1.51). Let us note those, where the left sides of equations (1.61) are dependent only on $\dot{\vec{\alpha}}$, but are not dependent on α. Angle α is thus a cyclic coordinate.

Together with relations (1.62) and (1.63), equations (1.61) describe the motion of both a free and controlled gyrostabilized platform. In the case of a free gyrostabilized platform

$$M_{1x}^{4} = M_{1y}^{5} = M_{1z}^{6} = 0. \tag{1.64}$$

In the case of a controlled gyrostabilized platform, these moments are distinct from zero.

Let us first consider the case of a free gyrostabilized platform. In this case the last three equations of (1.61) yield:

$$\left.\begin{array}{c}\omega_y \cos\delta_1 + \omega_z \sin\delta_1 = 0, \quad \omega_z \sin\delta_2 - \omega_x \cos\delta_2 = 0, \\ \omega_y \sin\delta_3 - \omega_z \cos\delta_3 = 0.\end{array}\right\} \tag{1.65}$$

Relations (1.65) are a homogeneous system of linear equations relative to ω_x, ω_y and ω_z. Its determinant Δ, according to expression (1.41), is equal to:

$$\Delta = -\cos\delta_2 \cos(\delta_1 - \delta_3). \tag{1.66}$$

When

$$|\delta_3| < \frac{\pi}{4}, \quad |\delta_1| < \frac{\pi}{4}, \quad |\delta_2| < \frac{\pi}{2} \tag{1.67}$$

the determinant is distinct from zero and system (1.65) permits only a zero solution:

$$\omega_x = \omega_y = \omega_z = 0. \tag{1.68}$$

This obviously means that the platform retains its fixed orientation in the inertial coordinate system.

If in addition to equality (1.64), we assume that

$$M^1_{2x'} = M^2_{2y'} = M^3_{2z} = 0, \tag{1.69}$$

then, by taking into account the solution of (1.68), from the first three equations of (1.61), we find:

$$\left.\begin{aligned} &H\left\{\left(\frac{d}{dt}\sin\delta_2\right)\cos\beta\cos\gamma - \left[\frac{d}{dt}(-\sin\delta_1 + \cos\delta_3)\right]\times\right. \\ &\left.\times\cos\beta\sin\gamma + \left[\frac{d}{dt}(\cos\delta_1 + \cos\delta_2 + \sin\delta_3)\right]\sin\beta\right\} = 0, \\ &H\left\{\left(\frac{d}{dt}\sin\delta_2\right)\sin\gamma + \left[\frac{d}{dt}(-\sin\delta_1 + \cos\delta_3)\right]\cos\gamma\right\} = 0, \\ &H\frac{d}{dt}(\cos\delta_1 + \cos\delta_2 + \sin\delta_3) = 0. \end{aligned}\right| \tag{1.70}$$

In the case where the destabilizing moments $M^1_{2x'}$, $M^2_{2y'}$, M^3_{2x} act along the x', y', z axes, values δ_1, δ_2, δ_3 will vary with time and can, in particular, take those values under which determinant (1.66) will become equal to zero. Then the existence condition for of solution (1.68) is broken and the orientation of the platform will no longer remain invariant. In order for this not to occur, that is, in order that angles δ_1, δ_2, δ_3 will be small and that inequalities (1.67) be trivially fulfilled, the engines En_1, En_2 and En_3 are introduced into the circuit of the device. These engines create unloading moments $M^1_{1x'}$, $M^2_{1y'}$, and M^3_{1x}, which counteract the affect of the perturbing moments. The unloading moments can be formulated in the following manner:

$$\left.\begin{aligned} M^1_{1x'} &= -k_2\delta_2\cos\beta\cos\gamma - k_1\delta_1\cos\beta\sin\gamma, \\ M^2_{1y'} &= -k_2\delta_2\sin\gamma + k_1\delta_1\cos\gamma, \\ M^3_{1z} &= -k_3\delta_3. \end{aligned}\right| \tag{1.71}$$

By taking into account inequalities (1.67), we note that the unloading of moments, calculated by relations (1.71), provides the existence of a trivial solution of the first three equations of (1.61), if of course there are no destabilizing moments. Having taken coefficients k_1, k_2 and k_3 sufficiently large, i.e., such that the values on the right sides of relations (1.71) exceed those corresponding to the destabilized moments at small values of δ_1, δ_2 and δ_3, we can provide trivial fulfillment of inequalities (1.67).

We can easily ascertain that the equilibrium position of the circuit

$$\delta_1 = \delta_2 = \delta_3 = 0 \tag{1.72}$$

is stable (within the limits of precession theory).

At small values of δ_1, δ_2 and δ_3, from relations (1.61), (1.68) and (1.71) we find:

$$\left.\begin{aligned} &H(\dot{\delta}_2 \cos\beta\cos\gamma + \dot{\delta}_1 \cos\beta\sin\gamma + \dot{\delta}_3 \sin\beta) = \\ &\qquad = -k_1\delta_1 \cos\beta\sin\gamma - k_2\delta_2 \cos\beta\cos\gamma, \\ &H(\dot{\delta}_2 \sin\gamma - \dot{\delta}_1 \cos\gamma) = k_1\delta_1 \cos\gamma - k_2\delta_2 \sin\gamma, \\ &H\dot{\delta}_3 = -k_3\delta_3. \end{aligned}\right| \tag{1.73}$$

It follows from the last equation of (1.73) that at $k_3 > 0$ the value of δ_3 approaches zero in time. Therefore, the stability of the equilibrium position of (1.72) is obviously determined by the properties of the solutions of the system of the two first equations of (1.73) at $\delta_3 = 0$, which in this case assume the form:

$$\left.\begin{aligned} &H\cos\beta(\dot{\delta}_2 \cos\gamma + \dot{\delta}_1 \sin\gamma) = \\ &\qquad = -\cos\beta(k_1\delta_1 \sin\gamma + k_2\delta_2 \cos\gamma), \\ &H(\dot{\delta}_2 \sin\gamma - \dot{\delta}_1 \cos\gamma) = k_1\delta_1 \cos\gamma - k_2\delta_2 \sin\gamma. \end{aligned}\right| \tag{1.74}$$

Having multiplied the first equation of (1.74) by cos γ and the second by cos β sin γ and having added the results obtained, we find

$$\cos\beta\,(H\dot{\delta}_2 + k_2\delta_2) = 0. \tag{1.75}$$

Having multiplied the first equation of (1.74) by sin γ and the second by cos β cos γ, and having added the results, we find:

$$\cos\beta\,(H\dot{\delta}_1 + k_1\delta_1) = 0. \tag{1.76}$$

The stability at $\beta \neq \pi/2, k_1 > 0$ and $k_2 > 0$ also follows from the form of equations (1.75) and (1.76).

The comment with respect to disposition [see (1.46)] of the gyroscopes on the platform, made during analysis of operation of the absolute angular rate meter, remains in force for the gyrostabilizer circuit.

It should be noted that consideration of the stability of the gyrostabilized platform within the limits of precession theory is usually insufficient. Final solution of the problem of the stability of the equilibrium position of (1.68) and (1.72) requires consideration of more complete equations than (1.61), in which the equatorial moments of inertia of the gyroscopes, the moments of inertia of the gyroscope housings and the gimbal , as well as the dynamic processes occurring in the formation circuits of the unloading moments, should be taken into account. Complete investigation of the stability of precession motion of gyroscopic devices is a special problem which is not considered here. A number of well-known investigations,[10] to which one should turn if necessary, is devoted to the solution of this problem.

Let us now consider the case of a controlled gyrostabilized platform. In this case orientation of the platform does not remain fixed in inertial space, as in the case of a free gyrostabilized platform, but varies by a given law. Moment sensors DM_1, DM_2 and DM_3 mounted on axes x_1, y_2 and x_3 of the gyroscope housings (Figure 1.10), are used to control rotation of the platform. The corresponding moments were denoted by M^4_{1x}, M^5_{1y} and M^6_{1x}.

It follows from the three last equations of (1.61), provided that the conditions of (1.67) are fulfilled, that

$$\left.\begin{aligned} \omega_x &= -\frac{M^5_{1y}}{H} + \frac{\sin\delta_2}{\cos\delta_2\cos(\delta_1-\delta_3)} \times \\ &\quad \times\left(-\frac{M^6_{1x}}{H}\cos\delta_1 + \frac{M^4_{1x}}{H}\sin\delta_3\right), \\ \omega_y &= \frac{1}{\cos(\delta_1-\delta_3)}\left(\frac{M^6_{1x}}{H}\sin\delta_1 + \frac{M^4_{1x}}{H}\cos\delta_3\right), \\ \omega_z &= \frac{1}{\cos(\delta_1-\delta_3)}\left(\frac{M^6_{1x}}{H}\cos\delta_1 - \frac{M^4_{1x}}{H}\sin\delta_3\right). \end{aligned}\right\} \qquad (1.77)$$

Expressions (1.77) are obtained in similar fashion to formulas (1.43), derived during analysis of operation of a three-component absolute angular rate meter. If δ_1, δ_2 and δ_3 are small, then, similar to (1.45), we obtain from expressions (1.77)

$$\omega_x = -\frac{M^5_{1y}}{H}, \quad \omega_y = \frac{M^4_{1x}}{H}, \quad \omega_z = \frac{M^6_{1x}}{H}. \qquad (1.78)$$

Thus, if moments M^4_{1x}, M^5_{1y} and M^6_{1x} are formed as the given time functions and if the value of H is assumed to be constant, then, according to the equalities of (1.78), the projections ω_x, ω_y and ω_z are also given time functions. The values of M^4_{1x}, M^5_{1y} and M^6_{1x} or any other values which uniquely determine these moments, may be used as the information source of the projections of ω_x, ω_y and ω_z of the absolute angular rate of the platform onto the axis of the coordinate system xyz, bound to it.

Equalities (1.68) for an uncontrolled gyrostabilized platform and relations (1.78) for a controlled gyrostabilized platform are valid if (1.69) is assumed. If this assumption is not fulfilled, i.e., if perturbing moments M^4_{2x}, M^5_{2y} and M^6_{2x} act along the axes of the gyroscope housings, then in both cases instrument errors $\Delta\omega_x$, $\Delta\omega_y$ and $\Delta\omega_z$, determined by the following equalities, occur:

(1.78a)

$$\Delta\omega_x = -\frac{M^5_{2y}}{H}, \quad \Delta\omega_y = \frac{M^4_{2x}}{H}, \quad \Delta\omega_z = \frac{M^6_{2z}}{H}.$$

The values of $\Delta\omega_x$, $\Delta\omega_y$ and $\Delta\omega_z$ are called "free deflections" of the gyrostabilized platform.

1.3.6. Free and controlled gyro frames. The gyroscopic platform may not be the load-bearing element but the friction in its suspension may be insignificant. For example, the platform may be surrounded by a spherical shell and suspended in a liquid with a low viscosity factor.

In this case, the angle of rotation sensors of the gyroscope housings with respect to the platform, the unloading engines and circuits of formation of the unloading moments may be eliminated from the circuit considered in section 1.3.5.

The corresponding gyroscopic devices are usually called gyro frames (in the given case this will be a three-dimensional three-gyroscopic gimbal). Like gyrostabilized platforms, gyro frames may be free or controlled, depending on whether the controlling moments are applied along the axes of the gyroscope housings or whether they are absent. In the first case the platform of the gyro frames retains its own fixed orientation and in the second case it rotates at an angular rate ω, whose projections onto the x, y and z axes of the gyroframe are bound to the controlling moments of relations (1.78). The perturbing moments along the axes of the gyroscope housings lead to deflections of the gyroframe according to the equalities of (1.78a).

1.3.7. Additional comments. In concluding consideration of gyroscopic devices of inertial navigation systems, it is useful to make several comments of a general nature.

We have considered several methods of constructing gyroscopic devices, by means of which information can be obtained about the orientation of some trihedron connected to the gyroscopes in an inertial coordinate system. All these devices can be combined by a single common name of absolute angular rate meters. This expansion of the concept "absolute angular rate meter" is useful because it permits consideration of almost all gyroscopic devices of inertial navigation systems from a single viewpoint. However, it should be noted immediately that there is a considerable difference between a free gyroscope and free gyrostabilized platform, on the one hand, and a controlled stabilized platform and essentially angular rate meters. Free gyroscopes and gyrostabilized platforms retain a given fixed orientation of the trihedron bound to them. Thus, the orientation of this trihedron in an inertial coordinate system is immediately known.

A strictly angular rate meter and a controlled platform permit only measurement of the value of projections of the absolute angular rate of a mobile trihedron to its axis. The orientation of the mobile trihedron in the inertial coordinate system can be determined by these values. This additional problem requires solution of a system of differential equations, which, as we shall see below, reduces to the well-known Poisson equations.[12]

One of the consequences of the noted difference is the circumstance that the inconstancy of the quantities of the natural kinetic moments of the gyroscopes in the circuit of the controlled gyrostabilized platform (or of the absolute angular rate meter) leads, as was already noted, to orientation errors (or to errors in determining the projections of the absolute angular velocity), and in the case of a free gyroscope and free gyrostabilized platform the inconstancy of the kinetic moments of the gyroscopes do not induce any of the indicated errors.

This is obvious from relations (1.65), (1.68) and (1.75)-(1.78). Relations (1.65) were obtained from the three last equations of (1.61) under the condition (1.64) and are not dependent on the value of H. The existence of solutions of (1.68) is also not dependent on H. The stability of this solution is retained according to (1.75) and (1.76) at any values of H distinct from zero. The value of H is essential in relations (1.77) and (1.78). When calculating ω_x, ω_y and ω_z from the known values of M^4_{1x}, M^5_{1y} and M^6_{1x} [according to formulas (1.77) and (1.78)], the difference of the real value of H from the calculated value by the quantity ΔH leads to errors:

$$\Delta\omega_x = \omega_x \frac{\Delta H}{H}, \quad \Delta\omega_y = -\omega_y \frac{\Delta H}{H}, \quad \Delta\omega_z = -\omega_z \frac{\Delta H}{H}.$$

In the circuits of a three-component absolute angular rate meter, free gyrostabilized platform and controlled platform considered above, the gyroscopes are installed so that their axes of sensitivity form an orthogonal set of three. The axis of sensitivity of a gyroscope is here understood as the direction perpendicular to the plane, containing the direction of the natural kinetic moment and axis of the gyroscope housing, and determined by equalities (1.46). The mutual orthogonality of the directions of the axes of sensitivity of the gyroscopic moments is of course not compulsory. The condition of orthogonality is usually observed in most real designs of devices, because this condition leads to simpler relations when calculating the values of the components of the absolute angular velocity, controlling and unloading moments etc.. As is well known, it is also suitable for a number of design and technological concepts. Construction of circuits in which the directions of the axes of sensitivity are not orthogonal is essentially possible. It is important only that the three directions of the axes of sensitivity not be coplanar.

The following comments, which it is necessary to make, concern the assumption made during derivation of the equations of the precession motion of the gyroscopic devices considered. The fact is that the angular momentum theorem [expression (1.5)] is generally valid only if the point, relative to which the angular momentum of the system and the external force moments are determined, is fixed in the inertial coordinate system. In all cases when equation (1.5) was used, the stipulation was made that the origin O of the trihedron Oxyz is fixed in the coordinate system $O_2\xi_*\eta_*\zeta_*$. Actually, the platform of the gyroscopic device is mounted on a moving object and, therefore, the origin of the coordinate system Oxyz moves in inertial space. However, the derived equations remain valid in this case as well. In order to prove this, let us consider the coordinate system $O\xi_*\eta_*\zeta_*$, whose origin is combined with the vertex of trihedron Oxyz, while the directions of the axes coincide with the directions of the corresponding axes of the inertial coordinate system $O_2\xi_*\eta_*\zeta_*$. The coordinate system $O\xi_*\eta_*\zeta_*$ moves in a forward direction with respect to the system $O_2\xi_*\eta_*\zeta_*$; therefore, the left sides of the equations of angular momentum, written in these coordinate systems, are coincident. The right sides differ by the value of the force moments of inertia of transient motion. Since the motion of the trihedron $O\xi_*\eta_*\zeta_*$ is forward, the inertial forces are parallel (there are no Coriolis forces) and they are determined by the acceleration of the translational motion of trihedron $O\xi_*\eta_*\zeta_*$, i.e., by the acceleration of its origin. If the gyroscopic elements of the circuits are balanced, the forces of inertia of translational motion, like the attractive forces, do not create additional moments, hence follows the validity of the equations of motion derived for a fixed point O, and also for a moving point. When considering unbalanced systems, the moments of inertial forces should be taken into account along with the moments of attractive forces. In particular, the inertial forces will create perturbing moments if balancing is incomplete. The given argument, strictly speaking, is exhaustive only if the origin O of the moving trihedron coincides with the center of mass (and simultaneously with the center of suspension) of the gyroscopes. If several gyroscopes are placed on the platform, this condition is not fulfilled and moments of centrifugal and Coriolis forces, which occur as the result of rotation of the coordinate system $Oxyz$ (of the platform) with respect to the system $O\xi_*\eta_*\zeta_*$, act on the gyroscopes. However, these moments are negligible in view of the limitation of the values of ω_x, ω_y and ω_z and the small dimensions of the platform, as a result of which these additional moments are usually disregarded.

Finally, it is also useful to note the following. In considering gyroscopic devices of inertial navigational systems, we assumed that the gyroscopes are mounted in an ordinary mechanical gimbal suspension. In modern gyroscopic devices, other principles of suspensions - floating, gas-dynamic, magnetohydrodynamic, magnetic, electrostatic etc. - are coming into use more and more.

However, the main relations which determine the operation of gyroscopic devices and those obtained above under the example of a mechanical gimbal suspension, retain their validity for any other type of suspension as well. Therefore (as in newtonometer circuits), there is no need to go into the details of the operating principle of this or that type of suspension. We will also not find this necessary during further consideration.

1.4. The Fundamental Equation of Inertial Navigation. General Principles of Constructing Inertial Systems.

<u>1.4.1. Conversion of the fundamental equation of inertial navigation and integration of it with respect to fixed orientation axes.</u> The fundamental equation of inertial navigation is equation (1.1) of motion of the sensitive mass of a three-dimensional newtonometer or relation (1.3), which relates the reading of the newtonometer as a measuring device to the acceleration of motion $d^2\vec{r}_{O_2}/dt^2$ of its sensitive mass and to the total attractive force of the unit sensitive mass by the aggregate of celestial bodies:

$$n = \frac{d^2 r_{O_2}}{dt^2} - F(r_{O_2}). \tag{1.79}$$

The essence of the inertial navigation method consists, as already noted, in integration of equation (1.79), which differs from equation (1.1) only in its notations.

Equation (1.79) can be integrated, for example, in the following manner. Let three one-component newtonometers be mounted on a gyrostabilized platform, considered in the preceding section, such that the directions of their axes of sensitivity form an orthogonal trihedron whose axes are directed parallel to the axes of the inertial coordinate system $O_2\xi_*\eta_*\zeta_*$. Let us assume that the system of three one-component newtonometers is equivalent to a single three-dimensional newtonometer. The readings of the newtonometers will then be projections of the vector $\vec{n}$ onto the directions of their axes of sensitivity: n_{ξ_*}, n_{η_*} and n_{ζ_*}. Let us denote the projections of vector $\vec{r}_{O_2}$ onto the axes of the inertial coordinate system by ξ_*, η_* and ζ_*. In inertial space these projections are obviously Cartesian coordinates of point O of the location of the sensitive masses of the newtonometers.[13] From relation (1.79), we have

$$
\left.\begin{aligned}
n_{\xi_*} &= \frac{d^2\xi_*}{dt^2} - F_{\xi_*}(\xi_*, \eta_*, \zeta_*),\\
n_{\eta_*} &= \frac{d^2\eta_*}{dt^2} - F_{\eta_*}(\xi_*, \eta_*, \zeta_*),\\
n_{\zeta_*} &= \frac{d^2\zeta_*}{dt^2} - F_{\zeta_*}(\xi_*, \eta_*, \zeta_*).
\end{aligned}\right\} \tag{1.80}
$$

By integrating equality (1.80) twice, we find:

$$
\left.\begin{aligned}
\xi_* &= \int_0^t\int_0^t [n_{\xi_*} + F_{\xi_*}(\xi_*, \eta_*, \zeta_*)]\,dt\,dt + \frac{d\xi_*(0)}{dt}t + \xi_*(0),\\
\eta_* &= \int_0^t\int_0^t [n_{\eta_*} + F_{\eta_*}(\xi_*, \eta_*, \zeta_*)]\,dt\,dt + \frac{d\eta_*(0)}{dt}t + \eta_*(0),\\
\zeta_* &= \int_0^t\int_0^t [n_{\zeta_*} + F_{\zeta_*}(\xi_*, \eta_*, \zeta_*)]\,dt\,dt + \frac{d\zeta_*(0)}{dt}t + \zeta_*(0).
\end{aligned}\right| \tag{1.81}
$$

Integration of equations (1.80) requires that the corresponding computer and also the clocks, from which the absolute (world or Newtonian) time signals enter the computer, are contained in the apparatus of the inertial navigation system. It is obvious that the form of functions F_{ξ_*}, F_{η_*} and F_{ζ_*} should be known and that the initial values of coordinates $\xi_*(0)$, $\eta_*(0)$ and $\zeta_*(0)$ and their time derivatives be $d\xi_*(0)/dt$, $d\eta_*(0)/dt$ and $d\zeta_*(0)/dt$ should also be known.

The Cartesian coordinates ξ_*, η_* and ζ_* of the point at which are located the sensitive masses of the newtonometers are obtained as a result of double integration. The position of this point on the object on which the inertial system is mounted is generally arbitrary. In particular, it may not coincide with the center of mass of the moving object. It is not essential to determine the coordinates of the object, because the resulting error obviously does not exceed the linear dimensions of the object. However, determination of the velocity and acceleration of the object along with the coordinates may also be contained in the task of the inertial system. The velocity and acceleration of the center of mass of the object may differ considerably from those of the sensitive mass of the newtonometer if the latter is not located in the center of mass of the object. The resulting problems will be discussed in the following section of this section.

Relations (1.80) and (1.81) and the concepts expressed in regard to them fully determine the essence of the operating principle of inertial navigation systems. However, they do not yet provide a practical method of realizing this type of system. In fact, the inertial coordinate system $O_2\xi_*\eta_*\zeta_*$, to which are related all the arguments, have not yet been determined in practice. The form of functions F_{ξ_*}, F_{η_*} and F_{ζ_*} is also still unknown. The fundamental relations of inertial navigation in the coordinate system specifically bound to those celestial bodies (or body) in whose neighborhood and relative to which the navigation problem should be solved, must first be obtained for practical realization of the considered principle. A system whose origin is combined with the center of mass of some celestial body may be taken as this coordinate system. Henceforth, we shall consider this celestial body to be the earth.

Let us introduce a right-hand orthogonal coordinate system $O_1\xi_*\eta_*\zeta_*$, the origin O_1 of which coincides with the earth's center of mass. Let the orientation of trihedron $O_1\xi_*\eta_*\zeta_*$ be unchanged in the inertial coordinate system. Without loss of generality, we can obviously assume that the directions of the coordinate axes $O_2\xi_*\eta_*\zeta_*$ and $O_1\xi_*\eta_*\zeta_*$ coincide and retain their fixed position relative to the directions from the earth's center of mass O_1 to moving stars.

Let us denote radius vector of point O of the location of the sensitive mass of the newtonometer relative to the earth's center of mass O_1 by $\vec{r}$ and the radius vector of point O_1 relative to the origin O_2 of the inertial coordinate system by $\vec{r}_{O_1}$ (Figure 1.14). It is obvious that

$$r_{O2} = r_{O1} + r. \tag{1.82}$$

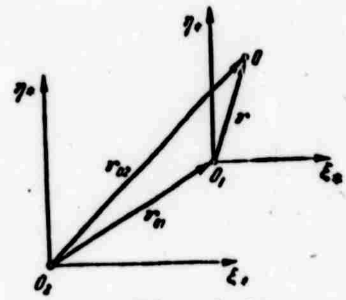

Fig. 1.14

By substituting equality (1.82) into relation (1.79), we find:

$$n = \frac{d^2 r_{O1}}{dt^2} + \frac{d^2 r}{dt^2} - F(r_{O2}). \tag{1.83}$$

Force $\vec{F}(\vec{r}_{O_2})$, which acts on the sensitive mass of the newtonometer, is the total attractive force of this mass by the earth and by the remaining celestial bodies. According to the law of Newton's gravitational force, the value of the attractive force by the earth of the unit sensitive mass of the newtonometer is dependent only on $\vec{r}$. Let us denote this force by $\vec{g}(\vec{r})$. Let us denote the attractive force of the unit sensitive mass of the newtonometer by the remaining celestial bodies by $\vec{F}_1(\vec{r})$. Expression (1.83) may then be rewritten in the form

$$n = \frac{d^2 r_{O1}}{dt^2} + \frac{d^2 r}{dt^2} - g(r) - F_1(r). \tag{1.84}$$

It is easy to see that

$$\frac{d^2 r_{01}}{dt^2} - F_1(0) = 0 \tag{1.85}$$

In fact, $\vec{F}_1(0)$ is the attractive force of the unit mass placed at point O_1 by the celestial bodies, with the exception of the earth.[14]

Therefore, equation (1.85) is nothing more than the equation of motion of the earth's center of mass within the gravitational field of the remaining celestial bodies.

Taking into account equation (1.85), equality (1.84) assumes the form:

$$n = \frac{d^2 r}{dt^2} - g(r) + F_1(0) - F_1(n). \tag{1.86}$$

If the motion of the object (and consequently, of point O) occurs at a small distance from the earth's center, commensurate, for example, to its radius, then the difference

$$\Delta F_1(r) = F_1(0) - F_1(r) \tag{1.87}$$

of the attractive forces at points O and O_1 become negligible compared to the force $\vec{g}(\vec{r})$ even for nearby celestial bodies, including that for the moon and sun.[15]

Thus, deflection of the vertical, induced by the difference of the sun's attractive forces at the center of the earth and at some point on its surface, does not exceed a value of 0.008". Accordingly, this deviation does not exceed a value of 0.017" for the moon. At the same time, deflection of the vertical, induced by the non-uniformity of the earth's distribution of mass, has, as was noted in § 2.1 an order of several angular seconds. Therefore, we may assume with a sufficient degree of accuracy that

$$n = \frac{d^2 r}{dt^2} - g(r). \tag{1.88}$$

The coordinate system $O_1\xi_*\eta_*\zeta_*$ moves in a forward direction relative to the inertial coordinate system $O_2\xi_*\eta_*\zeta_*$; therefore, we can obviously assume that differentiation in equation (1.88) is carried out in the coordinate system $O_1\xi_*\eta_*\zeta_*$.

This equation (1.88) is valid in the coordinate system $O_1\xi_*\eta_*\zeta_*$ and has the same form as equation (1.79), obtained for the inertial coordinate system. Consequently, with respect to Newton's laws, the coordinate system $O_1\xi_*\eta_*\zeta_*$ near its origin is practically indistinguishable from the inertial system. At the origin itself they are completely indistinguishable. The principle of the equivalence of the general theory of relativity, which, as is well known, is of a local nature, is essentially included in this.[16] The coordinate system $O_1\xi_*\eta_*\zeta_*$ is distinguished near its origin from the inertial system only to the extent to which the gravitational field in which the earth moves is inhomogeneous. The difference (1.87) also characterizes this inhomogeneity.

From equation (1.88), similar to equations (1.81), we find:

$$\left.\begin{aligned} \xi_* &= \int_0^t\int_0^t (n_{\xi_*}+g_{\xi_*})\,dt\,dt + \frac{d\xi_*(0)}{dt}t + \xi_*(0), \\ \eta_* &= \int_0^t\int_0^t (n_{\eta_*}+g_{\eta_*})\,dt\,dt + \frac{d\eta_*(0)}{dt}t + \eta_*(0), \\ \zeta_* &= \int_0^t\int_0^t (n_{\zeta_*}+g_{\zeta_*})\,dt\,dt + \frac{d\zeta_*(0)}{dt}t + \zeta_*(0). \end{aligned}\right| \tag{1.89}$$

If we assume that the earth's gravitational field is central (or rather spherical), we have

$$g(r) = -\frac{\mu r}{r^3}, \tag{1.90}$$

where μ is the product of the earth's mass by the gravitational constant. The equations (1.89) assume the form:

$$
\left.
\begin{aligned}
\xi_* &= \int_0^t \int_0^t \left(n_{\xi_*} - \frac{\mu \xi_*}{r^3} \right) dt\, dt + \frac{d\xi_*(0)}{dt} t + \xi_*(0), \\
\eta_* &= \int_0^t \int_0^t \left(n_{\eta_*} - \frac{\mu \eta_*}{r^3} \right) dt\, dt + \frac{d\eta_*(0)}{dt} t + \eta_*(0), \\
\zeta_* &= \int_0^t \int_0^t \left(n_{\zeta_*} - \frac{\mu \zeta_*}{r^3} \right) dt\, dt + \frac{d\zeta_*(0)}{dt} t + \zeta_*(0).
\end{aligned}
\right| \tag{1.91}
$$

If the sphericity of the earth's gravitational field is taken into account, then the projections of g_{ξ_*}, g_{η_*} and g_{ζ_*} in equations (1.89) may also be assumed unknown functions of coordinates ξ_*, η_* and ζ_* and time functions. In fact, if the earth's body axis system $O_1 \xi \eta \zeta$, (rotating together with it), is introduced, then the projections of g_ξ, g_η and g_ζ of vector $\vec{g}$ to the axes of this system will be known functions of coordinates ξ, η and ζ of point O. The time motion of the coordinate systems $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$ relative to each other is known. It is defined by the law of the earth's rotation with respect to its center. Therefore, the projections of g_ξ, g_η and g_ζ may be calculated as functions of coordinates ξ_*, η_* and ζ_* and as time functions.

The problem of determining the coordinates of the object during its motion near the earth's surface is essentially solved by equations (1.89) or (1.91). In fact, since the earth's motion in the coordinate system $O_1\xi_*\eta_*\zeta_*$ is known, we can transform from Cartesian coordinates ξ_*, η_* and ζ_* by appropriate calculation to any other coordinates, including the earth's body axis system. The orientation parameters of the object in any coordinate system may also be found by using the required calculations. In order to ascertain this, it is sufficient to recall that the angles of rotation of the gimbal rings of the gyrostabilized platform, which can be measured, determine the orientation of the object with respect to the coordinate system $O_1\xi_*\eta_*\zeta_*$, because in the considered case the orientation of the gyrostabilized platform relative to the coordinate system $O_1\xi_*\eta_*\zeta_*$ is fixed. By knowing the orientation of the object in the coordinate system $O_1'\xi_*\eta_*\zeta_*$, we can convert to the parameters which characterize its orientation in any other coordinate system, whose motion relative to the system $O_1\xi_*\eta_*\zeta_*$ is defined, of course including that in the earth's body axis system. A similar case holds for the rates of variation of the orientation parameters.

Let us consider in more detail the problem as to what extent, disregarding the inhomogeneity of the gravitational field, i.e., the difference of the attractive forces determined by equality (1.87), is essential. In other words, is this disregard essentially required or can we get along without it.

We can show that the latter case is valid, i.e., that difference (1.87) may be taken into account, and that the exact equality (1.86) rather than the simplified relation (1.88) may be taken as the equation of inertial navigation.

Let there be k celestial bodies whose gravitational difference at point O_1 and at point O of the position of the sensitive mass of the newtonometer should be taken into account. Let us denote the radius vector of the center of mass of the i-th of the celestial bodies relative to point O_1 by $\vec{r}_i$. The radius vector $\vec{r}_i'$ of the point O relative to the center of mass of the i-th body is then equal to:

$$r_i' = r - r_i. \tag{1.92}$$

Let us assume that the masses of the celestial bodies taken into account and their motion in the coordinate system $O_1\xi_*\eta_*\zeta_*$ are known, so that

$$r_i = r_i(t).$$

If we assume that the gravitational field of each of the celestial bodies is spherical, then on the basis of Newton's law of universal gravitation, we can write:

$$F_1(0) - F_1(r) = \sum_{i=1}^{k} \mu_i \left(\frac{r_i}{r_i} - \frac{r - r_i}{|r - r_i|^3} \right). \tag{1.93}$$

The right sides of the projections of the vector equality (1.93) onto the axes of the coordinate system $O_1\xi_*\eta_*\zeta_*$ depend only on $\xi_{*i}(t)$, $\eta_{*i}(t)$ and $\zeta_{*i}(t)$ and on ξ_*, η_* and ζ_*. Introduction of them into the integrands (1.89) or (1.91), although it complicates these expressions, essentially does not change the methods of solving equations (1.89), (1.91) and, consequently, equation (1.86).

Essentially, nothing changes if we reject the assumption of the sphericity of the gravitational fields of the celestial bodies taken into account. In this case it would be necessary to introduce k additional coordinates systems, rigidly linking them to the considered celestial bodies. We may assume that the gravitational fields in the body axis systems are defined, while the motions (rotations) of the latter relative to the coordinate system $O_1\xi_*\eta_*\zeta_*$ are known in time. Projections of the difference (1.87) to axes ξ_*, η_* and ζ_* will then be dependent on the time and parameters which characterize the disposition of the considered celestial body axis systems with respect to trihedron $O_1\xi_*\eta_*\zeta_*$ at the initial instant of time. Taking into account the non-sphericity of the gravitational fields of each of the k bodies is therefore quite similar to taking into account non-sphericity of the earth's gravitational field.

It follows from the foregoing that a knowledge of the required parameters of the gravitational fields in the coordinate function is a necessary condition for realization of the principle of inertial navigation. It is true that we shall subsequently see that the schemes which operate under specific conditions and with incomplete information about the gravitational field can be constructed for solution of some special problems of navigation.

1.4.2. Determining the velocity and acceleration of the center of mass of an object. The radius vector of point O of the position of its sensitive mass in the coordinate system $O_1\xi_*\eta_*\zeta_*$ is denoted by $\vec{r}$ in equation (1.88), which determines the readings of a three-dimensional newtonometer.

If we assume that point O always coincides with the center of mass of the object, then equation (1.88) will determine the acceleration of the object, and as the result of integration of this equation, the velocity and coordinates of the location of the center of mass of the object will be obtained.

Actually, the position of the sensitive mass of the newtonometer does not coincide with the center of mass of the object. This is explained by the followin~ factors. First, even if the center of mass of the object occupies a fixed position in its body and if the center of suspension of the sensitive mass of the newtonometer (the position in which the suspension is not deformed) coincides with the center of mass of the object, the sensitive mass completes some motion relative to the center of the suspension as the result of deformation. The velocities and accelerations of this motion may be significant.

Second, the center of mass of an object usually does not occupy a fixed position within the body of the object. Its position varies because of motion of the mass on the object, combustion of fuel etc.. Therefore, even if the center of suspension of the sensitive mass of the newtonometer and the center of mass of the object initially coincided, they would subsequently diverge.

Furthermore, a newtonometer can be established at some distance from the center of mass at the very beginning. Finally, additional variation of their mutual disposition is possible because of deformations (or elastic oscillations) of the object.

Because of the non-coincidence of the center of mass of the object and of the sensitive mass of the newtonometer, the acceleration, velocity and coordinates of the center of mass of the object, strictly speaking, may not be obtained directly from equation (1.88). Moreover, equation (1.88) is the equation of a three-dimensional (three-component) newtonometer, whereas three one-component newtonometers with three sensitive masses are actually used.

Let us consider the posed problems in more detail. This is even more necessary since exposition of the operating principle of the newtonometer and interpretation of the objective content of its readings are not always accurate and rigorous in the literature on inertial navigation.

Let us link trihedron $O'xyz$ to the housing of a three-dimensional newtonometer. Its origin will coincide with the center of suspension of the sensitive mass, i.e., with the position which it occupies when the suspension is not deformed and the readings of the newtonometer are equal to zero. For the diagram presented in Figure 1.1, the x, y and z axes may be directed along the axes of the springs.

The position of point O' relative to the earth's center O_1 is determined by the radius vector $\vec{r}'$, and the position of point O relative to O' is determined by radius vector $\vec{\rho}$. Vector $\vec{\rho}$ characterizes the motion of the sensitive mass relative to the housing of the device and, consequently, the deformation of the suspension. Obviously,

$$r = r' + \rho. \tag{1.94}$$

Let us find the equation for $\vec{\rho}$. It follows from that outlined in § 1.2 and section 1.4.1 that the equation of motion of the sensitive mass of the newtonometer in the coordinate system $O_1\xi_*\eta_*\zeta_*$ may be represented in the form

$$m\frac{d^2r}{dt^2} = mg(r) + f, \tag{1.95}$$

where $\vec{f}$ is the total force acting on the sensitive mass on the side of the suspension. By substituting the value of $\vec{r}$ from (1.94) and noting that the inhomogeneity of the gravitational field in the mass of the device may be disregarded, we find

$$m\frac{d^2\rho}{dt^2} = -m\left[\frac{d^2r'}{dt^2} - g(r')\right] + f. \tag{1.96}$$

If the force $\vec{f}$ is only the result of elastic deformation of the suspension, then $\vec{f}=k\vec{\rho}$ and equation (1.96) is represented in the following form:

$$\frac{d^2\rho}{dt^2} + \nu^2\rho = -\left[\frac{d^2r'}{dt^2} - g(r')\right], \quad \nu^2 = \frac{k}{m}. \tag{1.97}$$

Differentiation is carried out in the coordinate system $O_1\xi_*\eta_*\zeta_*$. By integrating in this same coordinate system, we find the expression for $\vec{\rho}$:

$$\rho = -\frac{1}{\nu}\int_0^t \left[\frac{d^2r'}{dt^2} - g(r')\right]\sin\nu(t-\tau)\,d\tau + \\ + \rho^0\cos\nu t + \frac{1}{\nu}\frac{d\rho^0}{dt}\sin\nu t, \tag{1.98}$$

where $\vec{\rho}^{\,0}$ and $d\vec{\rho}^{\,0}/dt$ are the corresponding initial values.

In order to maintain the analogy with relation (1.88), let us take as the readings of the three-dimensional newtonometer the vector

$$n = -\nu^2\rho. \tag{1.99}$$

It follows from relations (1.98) and (1.99) that the instantaneous values of the velocity and acceleration of the point of the object in which the center of suspension of the newtonometer is located, may not be found from the readings of the newtonometer. However, there is the following possibility here. Let the natural oscillation frequency ν of the sensitive mass be taken so large that the variation of function

$$q = \frac{d^2r'}{dt^2} - g(r') \tag{1.100}$$

over a period of $T=2\pi/\nu$ oscillations may be disregarded. This means that the range of essential frequencies of the function $\vec{q}$ are considerably below the frequency of ν. Solution of the problem then provides calculation of the average value of $\tilde{\vec{n}}$ of vector $\vec{n}$ within the period of natural oscillations.

According to equalities (1.98), (1.99) and (1.100) we have:

$$\tilde{n} = -\nu^2 \cdot \frac{\nu}{2\pi} \int_t^{t+\frac{2\pi}{\nu}} \rho \, dt^* = \frac{\nu^3}{2\pi} \int_t^{t+\frac{2\pi}{\nu}} dt^* \int_0^{t^*} q(\tau) \sin\nu(t^* - \tau) \, d\tau. \tag{1.101}$$

The range of integration in variables t*, τ is depicted in Figure 1.15. By changing the order of integration in equation (1.101), we find:

$$\tilde{n} = \frac{\nu^3}{2\pi}\left[\int_0^t d\tau \int_t^{t+\frac{2\pi}{\nu}} q(\tau)\sin\nu(t^* - \tau)\,dt^* + \int_t^{t+\frac{2\pi}{\nu}} d\tau \int_\tau^{t+\frac{2\pi}{\nu}} q(\tau)\sin\nu(t^* - \tau)\,dt^*\right]. \tag{1.102}$$

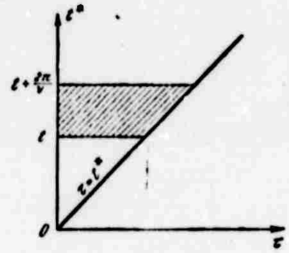

Fig. 1.15

The first integral in the square brackets is obviously equal to zero. From the second integral, we find

$$\tilde{n} = \frac{\nu}{2\pi} \int_t^{t+\frac{2\pi}{\nu}} q(\tau)[1 - \cos\nu(t - \tau)]\,d\tau. \tag{1.103}$$

Since the square bracket in the integrand (1.103) does not change sign, then, according to the well-known mean value theorem, we will have

$$\tilde{n} = \frac{\nu}{2\pi} q(\bar{t}) \int_t^{t+\frac{2\pi}{\nu}} [1 - \cos\nu(t - \tau)]\,d\tau. \tag{1.104}$$

Hence,

$$\tilde{n} = q(\tilde{t}), \quad \tilde{t} = t + \frac{2\pi}{\nu}\theta, \quad 0 < \theta < 1. \tag{1.105}$$

Thus, we found that the mean value $\vec{\tilde{n}}$ of vector $\vec{n}$ during the period of natural oscillations of the sensitive mass is equal to

$$\tilde{n} = \frac{d^2r'}{dt^2} - g(r'), \tag{1.106}$$

where the right side corresponds to some instant within the averaging interval. Expression (1.106) coincides with equality (1.88) with the only difference that the right side of expression (1.106) does not depend on the radius vector $\vec{r}$ of the current position of the sensitive mass of the newtonometer, but depends on the radius vector $\vec{r}'$ of the current position of the center of its suspension. Consequently, one can determine the value of the velocity and acceleration of the point of the object corresponding to the center of suspension of the newtonometer with a lag not exceeding $T=2\pi/\nu$ from the newtonometer readings. This lag is insignificant at large values of ν.

In practice the newtonometer readings are averaged due to damping of the natural oscillations of the sensitive mass, which is introduced to provide stability of the newtonometer operation. Damping is accomplished by forces proportional to the rate of displacement of the sensitive mass with respect to the newtonometer housing. If the newtonometer is mounted on a gyrostabilized platform, then obviously the damping forces will be proportional to the absolute derivative $d\vec{\rho}/dt$. Then in equation (1.96), one should set

$$f = -k\rho - k_1\frac{d\rho}{dt} \tag{1.107}$$

and, instead of equation (1.97), we find

(1.108)

$$\frac{d^2\rho}{dt^2} + 2h\frac{d\rho}{dt} + \nu^2\rho = -\left[\frac{d^2r'}{dt^2} - g(r')\right],$$
$$h = \frac{k_1}{2m}, \quad \nu^2 = \frac{k}{m}, \quad \nu^2 > h^2.$$

By integrating equation (1.108) in the coordinate system $O_1\xi_*\eta_*\zeta_*$, we find the forced solution in the following form (the solution of the homogeneous equation vanishes rapidly and it can be discarded immediately):

(1.109)

$$\rho = -\frac{1}{\sqrt{\nu^2 - h^2}} \int_0^t \left[\frac{d^2r'}{dt^2} - g(r')\right] \times$$
$$\times e^{-h(t-\tau)} \sin\sqrt{\nu^2 - h^2}\,(t-\tau)\,d\tau.$$

The rigidity of the suspension k and the damping coefficient k_1 are selected so that the values of ν, h, $(\nu^2-h^2)^{1/2}$ are considerably greater than the maximum value of the frequencies taken into account in the range of vector function $d^2\vec{r}'/dt^2-g(\vec{r}')$. This function may then be assumed constant in the subintegral expression of the right side of solution (1.109). In this case, after integration, we find the established value

(1.110)

$$\tilde{n} = -\nu^2\rho = \frac{d^2r'}{dt^2} - g(r').$$

i.e., we again arrive at relation (1.106).

Thus, we can find the velocity and acceleration of the sensitive mass of the center of suspension, i.e., the velocity and acceleration of the corresponding point of the object, from the readings of a three-dimensional newtonometer.

Let us return to the problem of calculating the velocity and acceleration of the center of mass of an object. Let us denote the radius of the center of mass C of the object with respect to the center of mass O_1 of the earth by $\vec{r}_C$ and the radius vector of the center of mass C of the object with respect to the center O' of the newtonometer suspension by $\vec{\rho}_C$. Obviously,

$$r_C = r' + \rho_C. \tag{1.111}$$

If vectors $d\vec{r}'/dt$ and $d^2\vec{r}'/dt^2$ are taken instead of $d\vec{r}_C/dt$ and $d^2\vec{r}_C/dt^2$, the resulting errors of calculating the velocity and acceleration are vectors $d\vec{\rho}_C/dt$ and $d^2\vec{\rho}_C/dt^2$.

If the object is assumed to be a rigid body, then the vectors $d\vec{\rho}_C/dt$ and $d^2\vec{\rho}_C/dt^2$ can be calculated as soon as the position of the center of mass C and the body of the object is known. In fact, the projections of vector $\vec{\rho}_C$ onto the axes of the platform can then be found by the angles of rotation of the gyrostabilized platform in a gimbal suspension, whose values can be measured, and the projection of vectors $d\vec{\rho}_C/dt$ and $d^2\vec{\rho}_C/dt^2$ can be found by differentiating these projections.

It is more difficult to calculate the elastic oscillations of the object, because this requires knowledge of the time of its deformation at each instant.

It follows from the foregoing that if the problem of the inertial system is calculation of only the coordinates of the object, then it makes no difference where the newtonometers are located on the object.

But if it is necessary to calculate rather accurately the velocity of an object (for example, during control of a ballistic missile on the active leg of its flight), and even more so acceleration, the newtonometers should be located near the center of mass of the object. In any case one should keep in mind that disposition of them far from the center of mass may lead to considerable errors in calculating the velocity and acceleration of the center of mass of the object, mainly because of its elastic deformations.

Let us turn to the problem of the correctness of replacing three linear newtonometers with a single three-dimensional device. To do this, let us find the precise equation of operation of the linear newtonometer.

Let trihedron O'xyz again be rigidly bound to the newtonometer housing. Let its x axis be the axis of sensitivity of the newtonometer, i.e., the axis along which the sensitive mass may move and along which the elastic force of the suspension is applied to it. Let point O' correspond to the position of the sensitive mass in which its suspension is not deformed. For generality, let us assume that the newtonometer housing, i.e., trihedron O'xyz, rotates at an absolute angular velocity $\vec{\omega}$. Let us compile the equation of motion of the sensitive mass of the newtonometer along the x axis.

Let us use equation (1.95). Instead of $\vec{r}$, let us substitute in it the value

$$r = r' + \rho, \quad \rho = \rho_x x. \tag{1.112}$$

and, instead of $\vec{f}$, the value

$$f = -(k\rho_x + k_1\dot{\rho}_x)x \tag{1.113}$$

[the unit vector of the O'x axis in relations (1.112) and (1.113) is denoted by $\vec{x}$]. We find

$$\frac{d^2}{dt^2}(\rho_x x) + 2h\dot{\rho}_x x + \nu^2\rho_x x = -\left[\frac{d^2r'}{dt^2} - g(r')\right],$$
$$h = \frac{k_1}{2m}, \quad \nu^2 = \frac{k}{m}. \tag{1.114}$$

Projection to the O'x axis yields:

$$x \cdot \frac{d^2}{dt^2}(\rho_x x) + 2h\dot{\rho}_x + \nu^2\rho_x = -x \cdot \left[\frac{d^2 r'}{dt^2} - g(r')\right]. \tag{1.115}$$

Let us find the value of the first term in the left side of equality (1.115). Obviously,

$$x \cdot \frac{d^2}{dt^2}(\rho_x x) = \ddot{\rho}_x + 2\dot{\rho}_x x \cdot \frac{dx}{dt} + \rho_x x \cdot \frac{d^2 x}{dt^2}. \tag{1.116}$$

Since vector $\vec{x}$ is the unit vector,

$$\left.\begin{aligned} x \cdot \frac{dx}{dt} &= \frac{1}{2}\frac{d}{dt}(x \cdot x) = 0, \\ x \cdot \frac{d^2 x}{dt^2} &= -\frac{dx}{dt} \cdot \frac{dx}{dt}. \end{aligned}\right| \tag{1.117}$$

But

$$\frac{dx}{dt} = \omega \times x. \tag{1.118}$$

therefore,

$$x \cdot \frac{d^2 x}{dt^2} = -(\omega \times x) \cdot (\omega \times x). \tag{1.119}$$

According to the well-known Lagrange identity, the right side of equality of (1.119) is expanded in the following manner:

$$(\omega \times x) \cdot (\omega \times x) = \omega^2 - \omega_x^2 = \omega_y^2 + \omega_z^2. \tag{1.120}$$

where ω_x, ω_y and ω_z are the projections of the absolute angular velocity of trihedron O'xyz on its axes.

Taking into account equalities (1.116), (1.119) and (1.120), equation (1.115) assumes the form:

$$\ddot{\rho}_x + 2h\dot{\rho}_x + (\nu^2 - \omega_y^2 - \omega_z^2)\rho_x = -\varkappa \cdot \left[\frac{d^2 r'}{dt^2} - g(r')\right]. \tag{1.121}$$

When

$$\nu^2 \gg \omega_y^2 + \omega_z^2 \tag{1.122}$$

the established value of deformation of the suspension spring is

$$\tilde{\rho}_x = -\frac{1}{\nu^2}\varkappa \cdot \left[\frac{d^2 r'}{dt^2} - g(r')\right]. \tag{1.123}$$

Having taken as the newtonometer readings the value

$$n_x = -\nu^2\rho_x = \varkappa \cdot \left[\frac{d^2 r'}{dt^2} - g(r')\right]. \tag{1.124}$$

we arrive at the relation similar to relation (1.4), by which we earlier determined the readings of a linear newtonometer. We can find the same relation by considering the equation of motion of the sensitive mass of a one-component pendulum newtonometer.

Now let three one-component newtonometers n_x, n_y and n_z be mounted (Figure 1.16) on a gyrostabilized platform or on the platform of a gyroscopic absolute angular rate meter. Let the axes of newtonometer sensitivity coincide with the axes of the trihedron Oxyz associated with the platform, and let the centers of the suspensions of their sensitive masses be separated from the vertex of the indicated trihedron by distance ℓ_x, ℓ_y and ℓ_z. Then the readings of newtonometer n_x will be calculated according to relation (1.121) by the equality

$$n_x = \left[\frac{d^2 r_1}{dt^2} - g(r_1)\right] \cdot x. \tag{1.125}$$

where $\vec{r}_1$ is the radius vector of the center of suspension of newtonometer n_x relative to the center of the earth O_1'.

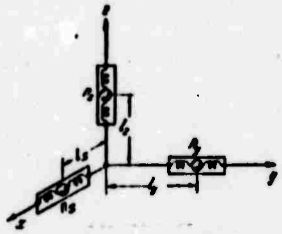

Fig. 1.16

If the radius vector of point O relative to the earth's center of mass is denoted by $\vec{r}$, then

$$r_1 = r + l_x x. \tag{1.126}$$

By substituting equality (1.124) into relation (1.123) and by noting that the difference $\vec{g}(\vec{r}) - \vec{g}(\vec{r}_1)$ at a small value of ℓ_x is negligible, we find (at ℓ_x = const):

$$n_x = \left[\frac{d^2 r}{dt^2} - g(r)\right] \cdot x + l_x \frac{d^2 x}{dt^2} \cdot x. \tag{1.127}$$

Similar expressions are also obtained for n_y and n_z, so that the projections of n_x, n_y and n_z determine the vector

$$n = n_x x + n_y y + n_z z = \frac{d^2 r}{dt^2} - g(r) + \Delta n. \tag{1.128}$$

where

$$\Delta n_x = l_x \frac{d^2x}{dt^2} \cdot x, \quad \Delta n_y = l_y \frac{d^2y}{dt^2} \cdot y,$$
$$\Delta n_z = l_z \frac{d^2z}{dt^2} \cdot z.$$

Thus, three linear newtonometers, mounted near point O', are equivalent to a single three-dimensional device mounted at this point, with an accuracy up to the error determined by the vector $\Delta\vec{n}$.

If orientation of the x, y and z axes is fixed (a gyrostabilized platform), then $d^2\vec{x}/dt^2=d^2\vec{y}/dt^2=d^2\vec{z}/dt^2=0$ and this means that the vector $\Delta\vec{n}$ is also equal to zero. If the newtonometers are mounted on a platform rotating at angular velocity $\vec{\omega}$ (for example, on the platform of an absolute angular rate meter), then, according to relations (1.119) and (1.120),

$$\left.\begin{aligned} &\Delta n_x = -l_x(\omega_y^2 + \omega_z^2), \\ &\Delta n_y = -l_y(\omega_z^2 + \omega_x^2), \quad \Delta n_z = -l_z(\omega_x^2 + \omega_y^2) \end{aligned}\right\} \tag{1.129}$$

At small values of ℓ_x, ℓ_y and ℓ_z (usually of the order of several centimeters) and at limited values of ω_x, ω_y and ω_z, the modulus of vector $\Delta\vec{n}$ is negligible. It should be noted that since ℓ_x, ℓ_y and ℓ_z are known, while projections ω_x, ω_y and ω_z are measured by a gyroscopic meter, then in principle the error of $\Delta\vec{n}$ can be completely eliminated.

In the above consideration, the axes of sensitivity of the three linear newtonometers formed a rigid orthogonal trihedron. Obviously, this does not change if this trihedron is not orthogonal or if it is not even rigid, but orientation of the axes of sensitivity of all three newtonometers is independent. If the axes of sensitivity are non-coplanar, then the newtonometer readings still determine the vector

$$n = \frac{d^2r}{dt^2} - g(r). \tag{1.130}$$

where the radius vector of the suitably selected point, near which the newtonometers are located, may be taken as $\vec{r}$.

1.4.3. General principles of constructing inertial navigational systems. A typical block diagram. The method of integrating a fundamental equation of inertial navigation, considered above (section 1.4.1), when the directions of the axes of sensitivity of three newtonometers form an orthogonal trihedron, invariant oriented in absolute space, as already noted, completely solves the problem of calculating the navigation parameters. This method, is, in any case, from the formal viewpoint, the more natural one and a direct method of solving the problem. However, the formal simplicity and naturalness of constructing the diagram is not always, as is well known, related to the simplicity and even the possibility of its technical and engineering realization.

Therefore, in real designs the directions of the axes of sensitivity of newtonometers may vary their orientation in inertial space during operation of the inertial system, where variation of the orientation of the newtonometers is usually a function of the coordinates determined by the inertial system itself. The orientation of newtonometers may be varied with respect to the inertial coordinate system, for example, by linking them rigidly to the controlled gyrostabilized platform, considered in the preceding section, and by forming in the required manner the controlling moments M^4_{1x}, M^5_{1y} and M^6_{1x}. A free gyrostabilized platform may be taken as the base of the diagram and the required orientation of the newtonometers relative to the platform and, consequently, relative to the axes of the inertial coordinate system can be provided by using a special kinematic diagram. Finally, orientation of the directions of the axes of sensitivity does not have to be a previously known coordinate function. For example, newtonometers can be linked to the platform of a three-component absolute angular rate meter and integration of the fundamental equation in the coordinate system bound to the platform[17] can be accomplished by taking advantage of the fact that rotation of the platform in inertial space is known from the readings of the angular rate meter.

A number of circumstances must be taken into account in each con-rete case in order to dwell on various schemes for constructing an inertial navigation system. One of the problems which must be solved here is to select the reference grid in which it is more convenient for some reason than in others to navigate a specific object (or class of objects). The coordinates which determine the position of point O with respect to trihedron $O_1'\xi_*\eta_*\zeta_*$ may be in the general case some curvilinear and non-orthogonal coordinates x^1, x^2 and x^3. They may obviously be transient as well, i.e., the coordinate surfaces x^i= const may alter its position in time with respect to trihedron $O_1\xi_*\eta_*\zeta_*$. This, for example, will occur if coordinates x^1, x^2 and x^3 determine the position of the object in the earth body axes system.

If the readings of three newtonometers are denoted by n_1, n_2 and n_3, the values of n_1, n_2 and n_3 with arbitrary orientation of the axes of sensitivity will be some time functions, functions of the three coordinates of x^i and of their first and second time derivatives:

$$n_i = f_i(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3, \ddot{x}^1, \ddot{x}^2, \ddot{x}^3, t) \quad (i = 1, 2, 3). \tag{1.131}$$

Equalities (1.131) are nothing more than projections of equation (1.88) on the directions of the axes of sensitivity of the newtonometers. The essence of the principle of inertial navigation, as already noted, reduces to integration of equation (1.88). In the considered case this reduces to integrating the system of three differential equations (1.131), which (if the directions of the axes of sensitivity of newtonometers n_1, n_2 and n_3 are not co-planar) are equivalent to the vector equation (1.88). In order to integrate equations (1.131), we could, for example, proceed in the following manner: reproject equalities (1.131) to axes $\xi_*\eta_*$ and ζ_*, use equations (1.89) and find the values of x^1, x^2 and x^3 from the obtained values of ξ_*, η_* and ζ_*. This method presumes computer operations on the readings of the newtonometers until integration of these readings.

However, it is well known that the signals taken from the newtonometers are rather rapidly variable time functions. Performing the computer operations directly on these signals with the required accuracy is related to considerable difficulties and usually leads to significant errors, adding to the errors of the newtonometers themselves. Therefore, the solution of equations (1.131), in which the first operation completed on the newtonometer readings is the integration operation, is more feasible. The condition of integrating the newtonometer readings until the computer operations on them have been executed is obviously a practically required condition which must be satisfied in constructing the diagram of an inertial navigation system.

At least two variants are possible. The highest derivatives $\ddot{x}^1$, $\ddot{x}^2$ and $\ddot{x}^3$ occur in the functions of f_i, which are on the right sides of equations (1.131), as the result of projection of the acceleration $d^2\vec{r}/dt^2$ on the directions of the axes of sensitivity of the newtonometers. Because of this, the functions f_i are linear in $\ddot{x}^1$, $\ddot{x}^2$ and $\ddot{x}^3$, i.e., equalities (1.131) may be represented in the form

$$n_i = a_{i1}\ddot{x}^1 + a_{i2}\ddot{x}^2 + a_{i3}\ddot{x}^3 + f_i'(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3, t). \qquad (1.132)$$

Coefficients a_{ij}, as we shall subsequently see, are functions of the coordinates x^1, x^2 and x^3, time t and of the parameters which determine orientation of the directions of the axes of sensitivity of the newtonometers. The latter can be either known time functions or time functions of coordinates x^1, x^2 and x^3. Consequently, equalities (1.132) can be written in the form

$$n_i = \frac{d}{dt}\left(\sum_{k=1}^{3} a_{ik}\dot{x}^k\right) - \sum_{k=1}^{3}\sum_{l=1}^{3}\left(\frac{\partial a_{ik}}{\partial t} + \frac{\partial a_{ik}}{\partial x^l}\dot{x}^l\right)\dot{x}^k + f'_i(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3, t). \tag{1.133}$$

The first variant of integrating equations (1.131) therefore involves the solution of equations (1.133) with respect to the sums $\Sigma a_{ik}\dot{x}^k$ according to the relations

$$\sum_{k=1}^{3} a_{ik}\dot{x}^k = \int_0^t\left[n_i + \sum_{k=1}^{3}\sum_{l=1}^{3}\left(\frac{\partial a_{ik}}{\partial t} + \frac{\partial a_{ik}}{\partial x^l}\dot{x}^l\right)\dot{x}^k - f'_i\right]dt + \sum_{k=1}^{3} a_{ik}(0)\dot{x}^k(0). \tag{1.134}$$

The values of $\dot{x}^k$ and x^k are then found, which are then used to form the subintegral expressions (1.134). This variant places no restrictions of any kind on disposition of the axes of sensitivity of the newtonometers. Orientation of the axes of sensitivity with respect to the axes $\xi_* \eta_* \zeta_*$ should be only a known time function and a function of coordinates x^1, x^2 and x^3.

The second variant presumes a completely specific dependence of the directions of the axes of sensitivity of the newtonometers on the coordinates and time: orientation of the axes of sensitivity should be selected such that only a_{11}, a_{22} and a_{33} of all values of a_{ij} be distinct from zero in equalities (1.132). Then, instead of relations (1.134) we find the following:

$$a_{kk}\dot{x}^k = \int_0^t \left[n_k + \sum_{l=1}^{3} \left(\frac{\partial a_{kk}}{\partial t} + \frac{\partial a_{kk}}{\partial x^l} \dot{x}^l \right) \dot{x}^k - f_k' \right] dt + a_{kk}(0)\dot{x}^k(0) \tag{1.135}$$

Of course, variants which are intermediate between relations (1.134) and (1.135) are also possible, when orientation of one (or two) of the three newtonometers is subject to deriving equations of type (1.134), while the readings of the remaining two (or one) are integrated according to relations (1.135).

To provide the required dependence of the directions of the axes of sensitivity of the newtonometers on coordinates x^1, x^2 and x^3 and time t, it is obviously necessary to form some controlling effects that depend in the general case on the coordinates x^i, their time derivatives and clearly reentrant time. The number of controlling effects may vary from zero, when orientation of the newtonometers relative to axes ξ_*, η_* and ζ_* is fixed, to six, when the directions of the axes of sensitivity of all three newtonometers vary independently of each other.

Selection of both the reference grid x^1, x^2 and x^3 and also the directions of the axes of sensitivity of the newtonometers, and, consequently, of the kinematics of the diagram and of the form of the controlling effects, should of course provide the greatest simplicity possible of the latter. One naturally strives in this case toward simultaneous simplification of functions f_i, and not only of their parts which occurred due to projection of the acceleration $d^2\vec{r}/dt^2$ on the directions of the newtonometer axes, but also of those which are obtained from projecting vector $\vec{q}$. This usually leads to the necessity of specifically orienting the axes of sensitivity of the newtonometers relative to the gravitational field.

Selection of the diagram is greatly affected by the possibility of using one or another means of correcting the operation of the inertial system, the requirements placed on the process of preparing the system for operation and on the process of operating it, the general characteristics of the object for which the system is designed, its velocity, range etc.

Finally, the given accuracy with which the navigational parameters must be determined and navigation must be accomplished, both in selecting the structure and operating algorithm of the initial system and in selecting its elements, is of decisive importance.

Selection of the elements of the diagram and selection of its structure and algorithm (equations of ideal operation) are of course unrelated to each other. The typical properties of the elements and devices selected for constructior of the diagrams usually place quite specific requirements on the structure of the diagram and on its operating algorithm. And, on the other hand, elements with quite specific properties are required to realize the given structure.

Of course, the method of constructing the diagram is primarily determined by the characteristics of the sensing elements and mainly by the accuracy and the possible range of measurement. But the properties of the remaining elements and devices -- computers, altitude and moment sensors, tracking systems etc. -- may also be no less determining.

Thus, if we assume rather large and accurate computer facilities (for example, a digital computer with sufficiently high speed and sufficiently large storage capacity in combination with analog-to-digital converters with the required accuracy), it is obviously possible to facilitate the task performed by the sensing elements, especially by the gyroscopic elements. In particular, in combination with accurate tracking systems this makes it possible to use free gyroscopes rather than load-bearing gyroscopic components (of the stabilized platform type).

It should be noted that the properties of elements which can be used in the system naturally affect not only the structure of the inertial navigation system, but its structural performance as well, as far as arrangement of the system on the object.

Thus, when using accurate tracking system and high-speed computers, the inertial sensing elements (gyroscopes and newtonoometers) can be linked to each other by the tracking systems without forming a common rigid unit. In the opposite case, the sensing elements should obviously comprise a monounit, in which the arrangement of individual sensing elements is rigidly fixed relative to each other.

The concepts presented above about the common principles of constructing the diagrams of inertial navigation systems have a common nature and of course do not contain a number of important details, whose significance can be discussed only after detailed analysis of them. However, these common concepts permit rather good representation of the typical block diagram of the inertial navigation system. It may be represented as consisting of four functional blocks (Figure 1.17): the block of sensing (inertial) elements 1, computer block 2, time block 3, and initial data input block 4. Of course (which follows from the foregoing), these functional blocks do not have to be common blocks in the design sense and in the configuration.

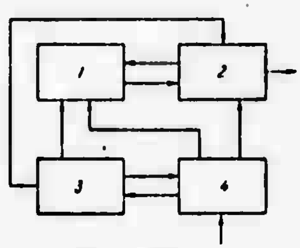

Fig. 1.17

The sensitive element block contains the newtonometers and the absolute angular rate meters. The block accomplishes given orientation of the axes of sensitivity of the newtonometers and of the absolute angular rate meters. Data is fed from the sensing elements into the computer block.

The initial orientation of the sensing elements and input into the computer block of the initial conditions required to integrate the fundamental equation of inertial navigation are accomplished by the initial data input block. World (absolute) time signals are cleared from the time block to the computer.

The main purpose of the computer block is to integrate the fundamental equation of inertial navigation and to calculate the required navigational parameters. Therefore, the operational program of the computer block should contain double integration. If the block provides variation of orientation of the newtonometers and of the gyroscopes of the inertial element block, the task of the computer includes formation of the corresponding controlling affects. Finally, if automatic navigation is assumed, the task of forming the programmed trajectory of motion of the object is also placed on the computer block and the number of output parameters will contain the instructions which control the steering gear of the object to maintain it on the programmed trajectory with the required accuracy.

1.4.4. The main problems of the theory of autonomous inertial navigation. Data on the principles of operation and on the equations of operation of inertial sensing elements were outlined in the preceding sections and the fundamental equation of inertial navigation was also derived. An example was given for constructing the diagram of an inertial navigation system with directions of the axes of sensitivity of the newtonometers and of the gyroscopic absolute angular rate meters, invariantly fixed in inertial space. Some common concepts were also presented on the possible methods of constructing the structural diagrams and operating algorithms of inertial navigation systems, and the more essential circumstances were enumerated which should be taken into account when selecting the method of constructing the diagram in various specific cases.

We can now formulate the essence of the problems which occur during theoretical analysis of operation of inertial systems.

The first problem which occurred here may be called the problem of construction and analysis of the equations of ideal operations of an inertial navigation system, i.e., the mathematical algorithm of its operation with ideal elements and correctly given initial conditions. This problem obviously includes determination of the form of projections of the fundamental equation of inertial navigation onto the directions of the axes of sensitivity of the newtonometers with different selection of coordinates x^1, x^2 and x^3, which characterize the current position of the object in space, and which characterize it as a function of the orientations of the directions of the axes of sensitivity of the newtonometers. The indicated problem also contains a search for (with the given reference grid x^1, x^2 and x^3) of the newtonometer orientation which permits rather simple integration of the fundamental equation of inertial navigation directly in projections onto the axes of sensitivity of the newtonometers and which permits rather simple construction of the algorithm of integration itself. This also includes mathematical formulation of the problems of forming the instructions for controlling variation of orientation of the axes of sensitivity of the newtonometers and gyroscopes with consideration of the kinematics of the gyroscopic devices described in §1.2, and also calculation, if possible, of the parameters which characterize the orientation of the object.

Consideration of the equations of ideal operation and of inertial navigation systems should be preceded by the derivation of the required functions which characterize the earth's gravitational field, its shape and motion.

The second important problem is derivation and analysis of the equations of perturbed functioning (motion) of the inertial navigation system, i.e., study of its operation with regard to the instrument errors of the elements, inaccuracies of the initial arrangement of the directions of the axes of sensitivity of the newtonometers and gyroscopes, and also with regard to errors of introducing the initial conditions. It is obvious that perturbed functioning of the system is different from that which is attributed to it by the equations of ideal operation, and the navigational parameters are determined in this case inaccurately and with errors.

The equations which describe time variation of deviations of perturbed motion of the system from unperturbed and ideal motion, are therefore naturally called error equations. Error equations are consequently equations in variations. The importance of studying the properties of these equations is obvious, because they determine the operational stability of the inertial system and they relate the errors of the elements and of the initial conditions to the errors of calculating the navigational parameters. The main purpose of analyzing the error equations is to establish a direct relationship between the accuracy of the system and the instrument errors of its elements.

The next problem is theoretical analysis of the phenomena and affects which occur during correction of inertial systems due to additional data sources.

The use of outside information sources to correct an inertial system has as its purpose an increase in the accuracy of calculation by the system of navigational parameters, i.e., reduction of the magnitude of errors. Accurate data on the coordinates of the object at some known time instants or on the rate of variation of the coordinates or, finally, the possibility of "tying in" to some direction whose orientation relative to the inertial coordinate system is known, for example, to the direction of some celestial body (astrocorrection), can be used as the data for making the correction. Correction can be accomplished by different methods. The simplest method is obviously introduction of corrections into the output parameters of the inertial navigation system. The second method is to bring the system at the point of correction to a state similar to that in which it was located at the moment of beginning operation at the starting point, with simultaneous introduction of corrections into the output parameters. The first method generally has no essential effect on the inertial system. The second method essentially differs in no way from preparation of the system for beginning of operation. Both methods affect neither the error sources or the dynamics of their time variation. Correction methods are possible which alter the structure of the error equations; they can be used to improve the stability of the inertial system, for example, a system unstable without correction can be made stable. These methods make it possible to avoid error accumulations. Analysis of this type of methods of correction is closely related to study of the properties of the error equations of autonomous inertial systems.

We can further isolate the group of problems related to simplification of the equations of ideal operation. Simplifications are possible not only by selecting the reference grid and special orientation of the axes of sensitivity of the newtonometers and gyroscopes relative to this reference grid and gravitational field.

The extent and nature of time variation of various terms of the equations of ideal operation are determined to a great extent by the motion of the object. For a given class of the programmed trajectories of motion of an object, some terms of the equation of ideal operation may be small and may be disregarded. Other terms may be close to their values on the programmed trajectory, and, therefore, it may be possible to form them as functions of their programmed values, i.e., as time functions, rather than as functions of the current coordinates of the object. The equations of ideal operation may also be simplified if the time variation of one or another navigational parameters are known from the outside information sources or from the specifics of motion of the object or if the functional relations which link some of these parameters are known.

Introduction of simplifications into the algorithm of ideal operation of the system usually leads to the occurrence of additional errors in calculating the navigational parameters. Simplifications are permissable if the value of the errors caused by them are small compared to other errors, for example, to those which occur as the result of instrument errors of the elements. The possibility of simplifying the equations of ideal operation of the system can best be determined only as the result of analyzing these equations together with the corresponding error equations.

The given list of problems whose solution is required when investigating operation of inertial navigation systems is of course not exhaustive. Only the main groups of problems and only those in the most common form were touched on here. These problems may, of course, be more detailed only during their solution.

1. Draper C. C., Wrigley W., Hovorka J., Inertial guidance, Pergamon Press, New York, 1960.

2. Ishlinskiy, A. Yu., Equations of the problem of calculating the location of a moving object by means of gyroscopes and accelerometers, <u>Prikladnaya Matematika i Mekhanika</u> Vol. 21, No. 6, 1957.

3. Strict proof of this statement will be given in the next section of this chapter.

4. Ishlinskiy, A. Yu., On the Theory of Gyroscopic Stabilization of Complex Systems, <u>Prikladnaya Matematika i Mekhanika</u> Vol.22, No. 3, 1958.

5. Ishlinskiy, A. Yu., On the Theory of Gyroscopic Stabilization of Complex Systems, <u>Prikladnaya Matematika i Mekhanika</u> Vol. 22, No. 3, 1958.

6. A suspension which provides three degrees of freedom of the gyroscope rotor is called a suspension with three degrees of freedom. This name is generally accepted in the Soviet literature on gyroscopic devices. In the non-Soviet literature (for example, in American literature) this suspension is often called one with two degrees of freedom, bearing in mind the number of degrees of freedom of the gyroscope housing.

7. Magnus, K., On the stability of motion of a heacy symmetrical gimbaled gyroscope. <u>Prikladnaya Matematika i Mekhanika</u>, Vol. 22 No. 3, 1958; Klimov, D.M. Appendices 1 and 2 to: Nikolai, Ye. L., Giroskop v vardanovom podvese (Gimbaled Gyroscopes), Fitzmatgiz, 1964.

8. Such devices are sometimes called specific moment sensors in the literature (for example, Gorenshteyn, I. A., I. A. Schul'man and A. S. Safaryan, Inertsial'naya navigatsiya (Inertial navigation), Sovetskaya Radio, 1962), bearing in mind the rotational moment relative to the kinetic moment of the gyroscope. This name may lead to a misunderstanding, because the specific moment is most often called the rotational moment relative to the moment of inertia. Therefore, we will use the generally accepted name absolute angular rate meter (sensor).

9. See the remark on page 19 (Footnote 3).

10. Bylgakov, B. V. Prikladnaya teoriya giroskcpov (Applied theory of Gyroscopes), Gostekhizdat, 1955; Roytenberg, Ya. N. Free oscillations of gyroscopic stabilizers, Prikladnaya Matematika i Mekhanika, Vol. II, No. 2, 1947.

. Deflections of a gyrostabilized platform, like deflections of a free gyroscope, may be caused not only by the perturbing moments along the axes of the housings but also by certain dynamic affects of the motion of the platform (see the literature indicated in footnote 7).

12. Appel'P., Teoreticheskaya mekhanika (Theoretical mechanics), Vol 2, Fizmatgiz, 1960.

13. The assumption of the equivalence of a single three-dimensional newtonometer to three linear ones, as already indicated, assumed the fact that the sensitive masses of all three newtonometers are located at the same point of space.

14. Duboshin G. H., Teoriya prityazheniya (Theory of Attraction), Fizmatiz, 1961.

15. Blazhko S. N., Kurs sfericheskoy astronomii (A Course in Spherical Astronomy), Gostekhizdat, 1954.

16. Einstein A., Infeld L., The Evolution of Physics, Gostekhizdat, 1948.

17. Andreyev V. D., "On the General Equations of Inertial Navigation", Prikladnaya Matematika i Mekhanika, Vol. 28, No. 2, 1964.

Chapter 2

THE SHAPE, GRAVITATIONAL FIELD AND MOTION OF THE EARTH.

§ 2.1. The shape of the earth. The fundamental earth body axis systems.

The earth's surface is usually assumed to be a fluid surface of oceans and seas which is thought of as continuing inside the continents along thin canals which do not change the distribution of masses.

The shape of this surface is the result of the total effects of the gravitational forces of the earth's mass and of the centrifugal force caused by rotation of the earth about its own axis. The normal to the quiet surface of the ocean thus coincides with the direction of the resulting gravitational forces of the earth and centrifugal force, i.e., to the direction of gravity. This direction is called the perpendicular direction or the true vertical.

The level surface of the earth is very complex and may not be accurately represented by any true geometric figure. A special term - geoid, proposed in 1873 by the German scientist I. Listing, was used to define it.

A geoid can be approximated with a sufficient degree of accuracy by a surface formed by rotation of an ellipse around its small axis, coinciding with the earth's rotational axis. The ellipsoid of rotation obtained in this case, usually called Clairaut's ellipsoid will obviously be determined if its semiaxes a and b are given. The ellipsoid of rotation may also be defined by being given one of the semiaxes, for example, the major semiaxis a and the compression

$$\alpha = \frac{a-b}{a}. \qquad (2.1)$$

or eccentricity e, whose square is equal to

$$e^2 = \frac{a^2 - b^2}{a^2}, \tag{2.2}$$

In view of the smallness of α and e^2 and, consequently, due to the proximity of Clairaut's ellipsoid to the sphere, another name of the level surface is also used - the terrestrial spheroid.

The ends of the minor semiaxis b of the terrestrial ellipsoid are called poles: one north and the other south. The cross sections of the ellipsoid surface, normal to the minor semiaxis, are circles called parallels. The largest of them is called the equator. The plane of the equator passes through the center of the earth. The cross sections of the surface of Clairaut's ellipsoid by the planes which pass through the minor axis are called meridians. These are obviously ellipses with semiaxes a and b.

The parameters of the terrestrial ellipsoid (the reference ellipsoid) are obtained by geodetic measurements carried out especially for this purpose. In different countries the parameters of the reference ellipsoid are taken as somewhat different from each other.

The parameters obtained in 1940 by the Soviet geodesist F. N. Krasovskiy are used for geodetic and cartographic work in the Soviet Union. The parameters of F. N. Krasovskiy's ellipsoid are the following [1]

(2.3)

major semiaxis $a = 6{,}378{,}245$ m,
minor semiaxis $b = 6{,}356{,}863$ m,
primary compression

$$\alpha = \frac{a-b}{a} = \frac{1}{298.3} = 0.00335233,$$

square of first eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2} = 0.0066934,$$

square of second eccentricity

$$l^2 = \frac{a^2 - b^2}{b^2} = 0.0067386,$$

polar radius of curvature of ellipsoid

$$c = \frac{a^2}{b^2} = 6{,}399{,}699 \text{ m},$$

radius of the sphere of an identical volume with the terrestrial ellipsoid

$$R' = 6{,}371{,}110 \text{ m},$$

radius of the sphere of an identical surface with the terrestrial ellipsoid

$$R'' = 6{,}371{,}116 \text{ m}.$$

We note that deviation of the normal to the geoid, i.e., the true vertical, from the direction of the normal to Clairaut's ellipsoid, does not exceed several angular seconds (2-3") with appropriate selection of its parameters, while the deviation of the geoid surface from the ellipsoid surface along the normal is of the order of tens of meters (100-150)[2].

For further exposition of the properties of the terrestrial ellipsoid, let us associate with it the right-hand orthogonal coordinate system $O_1\xi\eta\zeta$ (Figure 2.1). Let us locate the origin of this coordinate system at the center of the earth O_1, and let us direct the axis $O_1\zeta$ along the minor axis of the terrestrial ellipsoid in the direction of the north pole. The axes $O_1\xi$ and $O_1\eta$ will then be located in the equatorial plane. In order to finally determine this coordinate system, let us locate the axis $O_1\xi$ along the line of intersection of the equatorial plane with the plane of the Greenwich meridian.

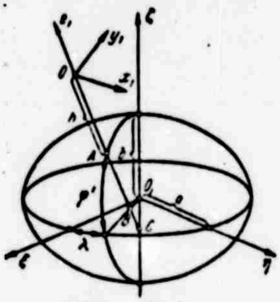

Fig. 2.1

The equation of Clerot's ellipsoid in the given coordinate system has the form:

$$\frac{\xi^2+\eta^2}{a^2}+\frac{\zeta^2}{b^2}=1.$$

Let point O be some arbitrarily selected point in the coordinate system $O_1\xi\eta\zeta$. Let us draw the normal to Clairaut's ellipsoid through this point. It will obviously be located in the meridional plane O containing point O, and will intersect the ellipsoid at point A, the equatorial plane at point B and axis ζ at point C (Figure 2.1). The location of point O in the coordinate system $O_1\xi\eta\zeta$ may be determined by angle φ', formed by the normal to the ellipsoid with the equatorial plane, angle λ between the meridianal planes of point O and the Greenwich meridian and by the segment of the normal h from point A to point O. Angles φ' and λ are called geographic latitudes and longitudes, and the value of h coincides with the greatest accuracy to the height of point O above sea level.

Let us link the right-hand orthogonal trihedron $Ox_1y_1z_1$ point O. Let us direct the z_1 axis along the positive normal to the terrestrial ellipsoid, let us locate the y_1 axis in the meridianal plane, containing point O, and let us direct it in the direction of the north pole. The position of the x_1 is now clearly determined. It is easy to see that trihedron $x_1y_1z_1$ will be oriented along the cardinal points by the accompanying trihedron (Darboux's trihedron) on to the surface h = const, surrounding the earth. Orientation of this trihedron relative to the earth's body axes ξ, η and ζ is characterized by the table of direction cosines:

(2.4)

	x_1	y_1	z_1
ξ	$-\sin\lambda$	$-\sin\varphi'\cos\lambda$	$\cos\varphi'\cos\lambda$
η	$\cos\lambda$	$-\sin\varphi'\sin\lambda$	$\cos\varphi'\sin\lambda$
ζ	0	$\cos\varphi'$	$\sin\varphi'$.

Having considered the meridianal cross section of the ellipsoid (Figure 2.2), which passes through point O and whose equation obviously has the form

$$\frac{x^2}{a^2}+\frac{\zeta^2}{b^2}=1, \tag{2.5}$$

we find the following expression for calculating φ'

$$\cot\varphi'=-\frac{d\zeta}{dx}. \tag{2.6}$$

It is easy to find from relations (2.2), (2.5) and (2.6) the expressions to calculate x and ζ by φ', h and the parameters of the ellipsoid:

$$\left.\begin{aligned} x&=\left[\frac{a}{(1-e^2\sin^2\varphi')^{1/2}}+h\right]\cos\varphi', \\ \zeta&=\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{1/2}}+h\right]\sin\varphi'. \end{aligned}\right\} \tag{2.7}$$

The coordinates ξ and η are expressed in turn by x and λ:

(2.8)

$$\xi = x\cos\lambda, \quad \eta = x\sin\lambda.$$

At $h = 0$ formulas (2.7) and (2.8) yield the expressions for coordinates ξ, η and ζ of the point of the ellipsoid surface to φ' and λ.

Let us determine the radii of curvature r_2 and r_3 of two mutually perpendicular main normal cross sections of the surface $h = \text{const}$ which pass through axes Ox_1 and Oy_1. Having turned to the first formula of (2.7), we note that it yields an expression for the radius of the parallel of surface $h = \text{const}$ at latitude φ'. According to Menier's theorem[3], it follows directly from this formula that the radius of curvature of a normal cross section tangent to the parallel is

(2.9)

$$r_2 = \frac{a}{(1 - e^2\sin^2\varphi')^{1/2}} + h,$$

i. e., it is equal to segment OC (Figure 2.2).

The radius of curvature of the meridional cross section is calculated by the well-known formula of differential geometry

(2.10)

$$r_3 = \frac{(x'^2 + \zeta'^2)^{3/2}}{x'\zeta'' + \zeta' x''}$$

(differentiated with respect to φ is denoted by prime). By using (2.7), we find:

(2.11)

$$r_3 = \frac{a(1 - e^2)}{(1 - e^2\sin^2\varphi')^{3/2}} + h.$$

If we now draw some arbitrary normal cross section such that it forms an angle ψ with the meridianal plane, then the radius of curvature of this cross section is calculated from Euler's formula

$$\frac{1}{r_\psi}=\frac{\sin^2\psi}{r_2}+\frac{\cos^2\psi}{r_3}. \tag{2.12}$$

It follows from formulas (2.9), (2.11) and (2.12) that the radius of curvature of the normal cross section of the surface h = const (and of the surface of the terrestrial ellipsoid h=0) with the angle ψ varying from 0 to $\pi/2$ increases continuously from its minimum value r_3 to the maximum value r_2. It is also easy to see from formulas (2.9), (2.11) and (2.12) that the meridianal cross section at the equator has the minimum radius of curvature, when

$$r_3=a(1-e^2)+h, \tag{2.13}$$

while the maximum value of the radius of curvature corresponds to latitude $\varphi'=\pi/2$, when

$$r_3=r_2=\frac{a}{\sqrt{1-e^2}}+h. \tag{2.14}$$

In view of the smallness of eccentricity of the terrestrial ellipsoid, formulas (2.7), (2.9) and (2.11) can be simplified. By decomposing the right sides of these formulas into series of powers of e^2 and by retaining only values of the first order of smallness with respect to the square of eccentricity, instead of (2.7) and (2.8), we find:

$$\left.\begin{aligned}\xi&=\left[a\left(1+\frac{e^2}{2}\sin^2\varphi'\right)+h\right]\cos\varphi'\cos\lambda,\\ \eta&=\left[a\left(1+\frac{e^2}{2}\sin^2\varphi'\right)+h\right]\cos\varphi'\sin\lambda,\\ \zeta&=\left[a\left(1-e^2+\frac{e^2}{2}\sin^2\varphi'\right)+h\right]\sin\varphi'.\end{aligned}\right| \tag{2.15}$$

Accordingly, instead of formulas (2.9) and (2.11), we will have:

$$\left.\begin{aligned} r_2 &= a\left(1 + \frac{1}{2}e^2\sin^2\varphi'\right) + h, \\ r_1 &= a\left(1 - e^2 + \frac{3}{2}e^2\sin^2\varphi'\right) + h. \end{aligned}\right\} \tag{2.16}$$

Along with the geographic coordinates φ', λ and h, let us introduce the additional geocentric coordinates of the point O. In order to calculate them, let us combine point O with the center of the earth O_1 by the segment of a straight line (Figure 2.3). The direction toward the center of the earth may be called the geocentric vertical. The geocentric coordinates of point O will be length r of segment O_1O, angle φ between the meridianal plane and direction O_1O, and angle λ between the plane containing axis $O\zeta$ and point O, and the plane $O_1\xi\zeta$.

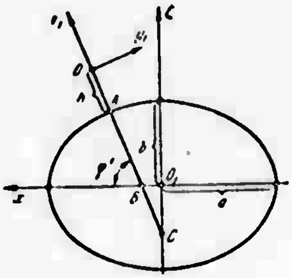

Fig. 2.2

It is obvious that the geocentric longitude is equal to the geographic longitude. Let us establish the relationship of φ and r to φ' and h. To do this, we note that coordinates ξ, η and ζ of point O are expressed by r, φ and λ by means of the equalities

$$\xi = r\cos\varphi\cos\lambda, \quad \eta = r\cos\varphi\sin\lambda, \quad \zeta = r\sin\varphi. \tag{2.17}$$

hence, it follows that

$$\tan\varphi = \frac{\zeta}{\eta}\sin\lambda. \tag{2.18}$$

But, from formulas (2.7) and (2.8),

$$\frac{\zeta}{\eta}\sin\lambda=\left[1-\frac{ae^2}{a+h(1-e^2\sin^2\varphi')^{1/2}}\right]\tan\varphi'. \tag{2.19}$$

By substituting relation (2.19) into formula (2.18), we find:

$$\tan\varphi=\left[1-\frac{ae^2}{a+h(1-e^2\sin^2\varphi')^{1/2}}\right]\tan\varphi'. \tag{2.20}$$

It now follows from formulas (2.7) and (2.17) that

$$r=\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{1/2}}+h\right]\frac{\sin\varphi'}{\sin\varphi}. \tag{2.21}$$

Having substituted instead of sin φ its expression by h and φ', easily obtained from relation (2.20), we find the dependence of r on φ' and h. The dependence of r on φ' and h may also be found directly from formulas (2.7) and (2.8). In fact,

$$r=\sqrt{\xi^2+\eta^2+\zeta^2}. \tag{2.22}$$

By substituting the values of ξ, η and ζ from formulas (2.7) and (2.8), we find

$$r=\left\{\left[\frac{a}{(1-e^2\sin^2\varphi')^{1/2}}+h\right]^2\cos^2\varphi'+\right.$$
$$\left.+\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{1/2}}+h\right]^2\sin^2\varphi'\right\}^{1/2}. \tag{2.23}$$

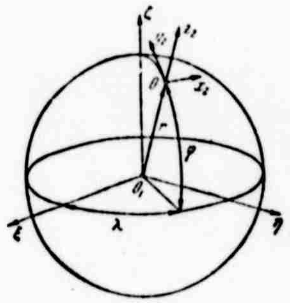

Figure 2.3

Let us introduce the moving trihedron $Ox_2y_2z_2$ (Figure 2.3), associated with the point O on the sphere of radius r concentric to the earth, similar to the manner in which the moving trihedron $Ox_1y_1z_1$ on the surface $h = \text{const}$ was introduced. Let us direct the z_2 axis along the geocentric vertical from the center of the earth. Let us locate the y_2 axis in the meridianal plane of point O and let us direct it toward the north pole. Let us select the direction of the x_2 axis so that the y_2 and z_2 axes are completed to form a right-hand orthogonal set of three.

Orientation of the trihedron $Ox_2y_2z_2$ to the coordinate system $O_1\xi\eta\zeta$ is calculated by the table of direction cosines, similar to table (2.4). The difference will be only in that, instead of geographic latitude φ', the expressions for the geocentric cosines will contain the geocentric latitude φ.

It is easy to see that the x_2 and x_1 axes of trihedrons $Ox_2y_2z_2$ and $Ox_1y_1z_1$ coincide. These trihedrons are expanded with respect to each other by angle $(\varphi' - \varphi)$, i.e., by the value of the difference of the geographic and geocentric latitudes. The mutual disposition of these trihedrons is characterized by the table of direction cosines:

(2.24)

	x_2	y_2	z_2
x_1	1	0	0
y_1	0	$\cos(\varphi' - \varphi)$	$-\sin(\varphi' - \varphi)$
z_1	0	$\sin(\varphi' - \varphi)$	$\cos(\varphi' - \varphi)$.

The difference $(\varphi' - \varphi)$ is calculated from formula (2.20), hence,

(2.25)

$$\tan(\varphi' - \varphi) = \frac{ae^2 \sin\varphi' \cos\varphi'}{(1 - e^2\sin^2\varphi')^{1/2}\,[a(1 - e^2\sin^2\varphi')^{1/2} + h]}.$$

In view of the smallness of the values of e^2 and $(\varphi' - \varphi)$, formulas (2.20), (2.23) and (2.25) can be simplified. By assuming that the value h/a is also small and by decomposing the right sides of the indicated formulas into series in powers of e^2 and h/a, we find

$$\tan\varphi = \left[1 - e^2\left(1 - \frac{h}{a}\right)\right]\tan\varphi'. \tag{2.26}$$

$$r = a\left(1 - \frac{1}{2}e^2\sin^2\varphi'\right) + h = a\left(1 - \frac{1}{2}e^2\sin^2\varphi\right) + h. \tag{2.27}$$

$$\varphi' - \varphi = \frac{e^2}{2}\left(1 - \frac{h}{a}\right)\sin 2\varphi' = \frac{e^2}{2}\left(1 - \frac{h}{a}\right)\sin 2\varphi. \tag{2.28}$$

The smallness of the values of e^2 and $(\varphi' - \varphi)$ also simplify the table of directions cosines (2.24). Assuming that

$$\cos(\varphi' - \varphi) = 1, \quad \sin(\varphi' - \varphi) = \varphi' - \varphi,$$

we find

$$\begin{array}{cccc} & x_2 & y_2 & z_2 \\ x_1 & 1 & 0 & 0 \\ y_1 & 0 & 1 & -\frac{e^2}{2}\left(1 - \frac{h}{a}\right)\sin 2\varphi \\ z_1 & 0 & \frac{e^2}{2}\left(1 - \frac{h}{a}\right)\sin 2\varphi & 1. \end{array} \tag{2.29}$$

Substituting the value $e^2=0.0067$ into formula (2.28), we find that the maximum deviation of the true vertical from the geocentric vertical is equal to

$$|\varphi' - \varphi|_{max} \approx 0.00335, \tag{2.30}$$

which corresponds to $\approx 11.6'$ and is achieved at latitude $\varphi=45^0$ (or $\varphi'=45^0$) on the earth's surface. As the distance from the earth's surface increases, the value of this difference decreases. However, this decrease is very slow in direct proximity to the earth's surface. Thus, at h=100 km, from formula (2.28) we obtain

$$|\varphi' - \varphi|_{max} \approx 0.00355\left(1 - \frac{1}{64}\right) \approx 0.0033,$$

which corresponds to $\approx 11'$.

Therefore, at small values of h, we may assume:

(2.31)

$$\varphi' - \varphi = \frac{1}{2} e^2 \sin 2\varphi' \cong \frac{1}{2} e^2 \sin 2\varphi.$$

and ℓ is the second eccentricity of Clairaut's ellipsoid, calculated by the fifth equality of (2.3).

The values of P_0 and Q_0 are constants which do not depend on coordinates x, y and z. They are calculated only by the value of $D\mu$ and by the second eccentricity.

For the potential [5] V of the gravitational field inside the spheroid, the following expression holds:

$$V = -\frac{1}{2} P_0 x^2 - \frac{1}{2} P_0 y^2 - \frac{1}{2} Q_0 z^2 + K_0, \tag{2.35}$$

where

$$K_0 = 2\pi D\mu \frac{a^2}{l} \tan^{-1} l \tag{2.36}$$

is the potential of the spheroid to its center, which is also a constant value.

The projections F_x, F_y and F_z can be expressed by derivatives the potential V of coordinates x, y and z:

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}. \tag{2.37}$$

Formulas (2.33)-(2.36) are valid for the interior points of a homogeneous ellipsoid, also including the points on its surface which are maximum points.

We are primarily interested in the gravitational field of a homogeneous spheroid outside its volume. In this case expressions (2.32) for F_x, F_y and F_z remain valid, but direct calculation of the integrals on the right sides is cumbersome. The resulting difficulties can be avoided here by using Maclaurin's theorem that two confocal homogeneous spheroids of equal mass produce an identical effect in the entire space external to both spheroids. This theorem permits easy distribution of formulas (2.33)-(2.36) to the case of the extrinsic point by altering them somewhat.

The semiaxes of the ellipsoid, confocal to the given ellipsoid and passing through point A(x,y,z), as is well known, are equal to:

$$a' = \sqrt{a^2+\nu}, \quad b' = \sqrt{b^2+\nu}, \tag{2.38}$$

where ν is the positive root of the equation

$$\frac{x^2+y^2}{a^2+\nu} + \frac{z^2}{b^2+\nu} = 1; \tag{2.39}$$

the second eccentricity of the confocal ellipsoid is equal to

$$e' = \sqrt{\frac{a^2-b^2}{b^2+\nu}}; \tag{2.40}$$

and finally, the density of the confocal spheroid, having the same mass as the given sphereoid, is found from the equality

$$D' = \frac{a^2 b}{(a^2+\nu)\sqrt{b^2+\nu}} D. \tag{2.41}$$

On the basis of Maclaurin's theorem, projections F_x, F_y and F_z for point A(x,y,z), extrinsic with respect to Clerot's ellipsoid, are calculated by formulas (2.33)-(2.36), if a', b', ℓ' and D', respectively, from formulas (2.38)-(2.41) are substituted in them instead of a, b, ℓ, and D. By carrying out this substitution and denoting

$$D' \frac{1+l'^2}{l'^3} = D \frac{1+l^2}{l^3} = D \frac{a^2 b}{(a^2-b^2)^{3/2}} = \text{const}, \tag{2.42}$$

we arrive at the following formulas for F_x, F_y and F_z:

$$F_x = -Px, \quad F_y = -Py, \quad F_z = -Qz. \tag{2.43}$$

where

$$\left.\begin{aligned} P &= 2\pi D\mu \frac{a^2 b}{(a^2-b^2)^{3/2}} \left(\tan^{-1} l' - \frac{l'}{1+l'^2}\right), \\ Q &= 4\pi D\mu \frac{a^2 b}{(a^2-b^2)^{3/2}} (l' - \tan^{-1} l'). \end{aligned}\right| \tag{2.44}$$

For potential V, we find the expression

$$V = -\frac{1}{2} Px^2 - \frac{1}{2} Py^2 - \frac{1}{2} Qz^2 + K, \tag{2.45}$$

where

$$K = 2\pi D\mu \frac{a^2}{l'} \tan^{-1} l' \tag{2.46}$$

Unlike formulas (2.33)-(2.36), the values of P, Q and K in formulas (2.43)-(2.46) are variables, because they are function of l', and consequently, in view of relations (2.40) and (2.39), they are functions of x, y and z. However, formulas (2.37) remain valid for F_x, F_y and F_z, because it is easily established, for example, that

$$-\frac{1}{2} \frac{\partial P}{\partial x} x^2 - \frac{1}{2} \frac{\partial P}{\partial y} y^2 - \frac{1}{2} \frac{\partial Q}{\partial x} z^2 + \frac{\partial K}{\partial x} = 0.$$

Similar equalities are obtained upon differentiation of V with respect to y and z.

Formulas (2.43) would yield accurate expressions for F_x, F_y and F_z if the terrestrial spheroid were homogeneous and its density D and gravitational constant μ were known with some accuracy. However, in fact the distribution of masses in the terrestrial spheroid is non-uniform, the value of μ is known from direct measurements with accuracy only up to 0.1%, while the average value of D (or, which is the same thing, the earth's mass M) is calculated only indirectly and also with an accuracy of the order of 0.1%. Their values are equal to:

$$\mu = 6.67 \cdot 10^{-8} \frac{cm^3}{g \cdot s^2}, \quad D = 5.52\, g \cdot cm^{-3}, \quad M = 5.98 \cdot 10^{27}\, g \tag{2.47}$$

Therefore, formulas (2.43) yield only some approximate values for projections of the earth's gravitational field intensity on its body axes.

2.2.2. Solution of the Stokes problem for a level surface given in the form of a spheroid. More accurate calculation of the earth's gravitational field can be had by solving the Stokes problem for the terrestrial ellipsoid. However, before going into exposition of the solution of the Stokes problem, let us show that the spheroid can be a figure of equilibrium of a homogeneous heavy rotating liquid.

It is known from hydrostatics that, for the equilibrium of a liquid, on which a force having the components f_x, f_y and f_z and arbitrary point (x,y,z), is acting, it is necessary that the following equalities be fulfilled

$$\frac{\partial p}{\partial x} = Df_x, \quad \frac{\partial p}{\partial y} = Df_y, \quad \frac{\partial p}{\partial z} = Df_z, \tag{2.48}$$

where p is pressure and D is the density of the liquid at the given point. It follows from these equalities that

$$dp = D(f_x dx + f_y dy + f_z dz). \tag{2.49}$$

Let the pressure on the external surface of the liquid be equal to zero: dp=0. Then,

$$f_x dx + f_y dy + f_z dz = 0. \tag{2.50}$$

Further, let the shape of the surface of the figure of equilibrium of the liquid be calculated by the equation

$$S(x, y, z) = 0.$$

Differentiation of this equation yields

$$\frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial y} dy + \frac{\partial S}{\partial z} dz = 0. \tag{2.51}$$

By comparing this equality with equality (2.50), we conclude that the partial derivatives of S should be proportional to the force components:

$$\frac{f_x}{\frac{\partial S}{\partial x}} = \frac{f_y}{\frac{\partial S}{\partial y}} = \frac{f_z}{\frac{\partial S}{\partial z}}. \tag{2.52}$$

But $\partial S/\partial x$, $\partial S/\partial y$ and $\partial S/\partial z$ are proportional to the direction cosines of the normal to the surface S. Consequently, surface S is a level surface.

Let forces f_x, f_y and f_z admit the force function $\tilde{W}$. Then, instead of relation (2.50), we find

$$\frac{\partial \tilde{W}}{\partial x} dx + \frac{\partial \tilde{W}}{\partial y} dy + \frac{\partial \tilde{W}}{\partial z} dz = d\tilde{W} = 0. \tag{2.53}$$

Hence, it follows that $\tilde{W} =$ const and this means that the surface S is again level.

For a spheroid, function S has the form

$$S = \frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0. \tag{2.54}$$

Therefore,

$$\frac{\partial S}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial S}{\partial y} = \frac{2y}{a^2}, \quad \frac{\partial S}{\partial z} = \frac{2z}{b^2}. \tag{2.55}$$

The components of gravity $\vec{F}$ by a homogeneous spheroid of unit mass, located inside or on the surface of a spheroid, are given by formulas (2.33). By adding centrifugal forces to them we find

$$f_x = u^2 x - P_0 x, \quad f_y = u^2 y - P_0 y, \quad F_z = -Q_0 z. \quad (2.56) \tag{2.56}$$

where u is the earth rate.

By introducing expressions (2.56) and (2.55) into the condition of equilibrium (2.52), we arrive at the equalities

$$a^2 (u^2 - P_0) = a^2 (u^2 - P_0) = -bQ_0. \tag{2.57}$$

The first of them is satisfied identically, and from the second one and from (2.34) follows the relation

$$\frac{\omega^2}{2\pi D\mu} = \frac{3+l^2}{l^3}\tan^{-1} l - \frac{3}{l^2}. \tag{2.58}$$

The function on the right side of this relation initially increases as l increases from zero, reaching a maximum equal to 0.22467 at $l=2.5293$, and then decreases, asymptotically approaching zero at when l increases without bounds.

Thus, at

$$\frac{\omega^2}{2\pi D\mu} < 0.22467 \tag{2.59}$$

there are two (one at the maximum point) solutions of equation (2.58), one of which corresponds to a slightly compressed spheroid. Lyapunov's and Poincare's investigations showed that stable figures of equilibrium of a rotating liquid are obtained only upon fulfillment of the condition

$$\frac{\omega^2}{2\pi D\mu} < 0{,}18712. \tag{2.60}$$

The conditions of (2.59) and (2.60) are fulfilled for the earth's parameters. In the first approximation equation (2.58) yields for compression of the terrestrial ellipsoid α the following value found by Newton

$$\alpha = \frac{5}{4} q. \tag{2.61}$$

where $q = \frac{u^2 a}{g_e}$, u is the earth rate and g_e is the value of the acceleration of gravity at the equator.

Equation (2.58) relates the second eccentricity of Clairaut's ellipsoid to the earth rate and its density. Therefore, if the shape of the spheroid and the angular velocity of its rotation are assumed to be given, the completely specific value of density D is obtained from equality (2.58). In this case the mass of the spheroid will not coincide with the earth's mass.

Let us turn to the Stokes problem. The Stokes theorem is valid: "Let there be a fixed body uniformly rotating about a fixed axis at a constant angular velocity u. Let there be known some level surface of gravity, which completely envelopes the body. The potential function of gravity and its first derivatives (i.e., the force components) will be clearly determined both on the level surface itself and in the entire external space if the total mass of the body is known regardless of the law of distribution of this mass."[6]

The principal possibility of determining the potential of gravity and of gravity itself follows from the Stokes theorem if the shape of the level surface and the total mass of the body are known. The Stokes problem also comprises the search for the potential function W of gravity by the given conditions. The potential of gravity consists of the gravitational potential V_0 and the potential of centrifugal forces U:

$$W = V_0 + U. \tag{2.62}$$

The potential of centrifugal forces does not depend on the shape of the level surface and is expressed by the obvious formula

$$U = \frac{u^2}{2}(x^2 + y^2). \tag{2.63}$$

Thus, the Stokes problem reduces to finding the potential function V_0 of gravitation.

Function V_0 should satisfy the general properties of the potential function of gravitation:[7]

1. Externally, it should satisfy the Laplace equation with relative to the level surface of the space

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + \frac{\partial^2 V_0}{\partial z^2} = 0. \tag{2.64}$$

2. It should be continuous and finite and it should have continuous and finite first derivatives at any finite values of coordinates x, y and z.

3. It should be subject to the limiting condition

$$\lim_{r \to \infty} r V_0 = \mu M, \tag{2.65}$$

where $r = \sqrt{x^2+y^2+z^2}$, and M is the earth's mass.

Moreover, the following equality should be fulfilled on the given level surface

$$V_0 = \mathrm{const} - \frac{\omega^2}{2}(x^2 + y^2) \tag{2.66}$$

Let us take Clairaut's ellipsoid as the reference surface and let us assume that the condition of Stokes theorem is fulfilled for it, i.e., let us disregard the circumstance that the masses of continents are not enveloped by the surface of Clairaut's ellipsoid Then, solution of the Stokes problem will be the function

$$W = CK + V + U. \tag{2.67}$$

where C is some arbitrary function; K is a function calculated by equality (2.46); V is the potential function of gravitation of a homogeneous spheroid limited by Clairaut's ellipsoid, taken at the reference surface; and U is the potential function of the centrifugal forces calculated by equality (2.63).

Then, for the potential function of gravitation V_0, from equalities (2.62) and (2.67), we find the expression

$$V_0 = CK + V. \tag{2.68}$$

The function V_0, given by equality (2.68), satisfies the first of the conditions formulated above, because each of the functions of V and K is individually a solution of the Laplace equation, which is easy to ascertain by taking the second derivatives in coordinates from the functions of V and K, calculated by formulas (2.45) and (2.46) and by taking into account equation (2.39) and relation (2.40). Function (2.68) also satisfies the second of the conditions indicated above.

The remaining two conditions, i.e., the conditions placed on equalities (2.65) and (2.66) can be fulfilled by selecting in the appropriate manner constant C and density D of the homogeneous spheroid contained in expression (2.45) for potential V according to relations (2.44).

In fact, let us take for D the value resulting from (2.68) The surface of Clairaut's ellipsoid will then be the reference surface of function V + U, i.e., we will have on this surface:

$$V + U = \text{const.} \tag{2.69}$$

But since function K on the given Clairaut's ellipsoid is also a constant, then condition (2.66) is fulfilled.

Now, by forming the product rV_0 and passing to the limit as $r \to \infty$, we find

$$\lim rV_0 = C 2\pi D\mu a^2 b + \frac{4}{3}\pi D\mu a^2 b. \tag{2.70}$$

By comparing the right sides of equalities (2.70) and (2.65), we conclude that, in order to satisfy condition (2.65), we must take

$$C = \frac{M}{2\pi D a^2 b} - \frac{2}{3}. \tag{2.71}$$

Thus, for the components of the earth's gravitational field intensity, we find the expressions:

$$\left.\begin{aligned} F_x = -Px + C\frac{\partial K}{\partial x}, \quad F_y = -Py + C\frac{\partial K}{\partial y}, \\ F_z = -Qz + C\frac{\partial K}{\partial z}, \end{aligned}\right| \tag{2.72}$$

where

$$\left.\begin{aligned} P &= 2\pi D\mu \frac{a^2 b}{(a^2-b^2)^{3/2}}\left(\tan^{-1} l' - \frac{l'}{1+l'^2}\right), \\ Q &= 4\pi D\mu \frac{a^2 b}{(a^2-b^2)^{3/2}}(l' - \tan^{-1} l'), \\ K &= 2\pi D\mu \frac{a^2 b}{\sqrt{a^2-b^2}} \tan^{-1} l'. \end{aligned}\right| \tag{2.73}$$

To calculate the constants D and C, contained in expressions (2.72) and (2.73), the following equalities are used

$$\frac{u^2}{2\pi D\mu} = \frac{3+l^2}{l^3}\tan^{-1} l - \frac{3}{l^2}, \quad C = \frac{M}{2\pi D a^2 b} - \frac{2}{3}. \tag{2.74}$$

Equation (2.39) and relation (2.40), by means of which the value of the second eccentricity of the confocal ellipsoid at the current point is found, and also equality (2.3), which determines the second eccentricity l of a level ellipsoid, must also be added to expressions (2.72), (2.73) and (2.74).

Thus, formulas (2.72), (2.73), (2.74), (2.39), (2.40) and (2.3) make it possible to find the values of F_x, F_y and F_z if the following are known: the semiaxes a and b of the level ellipsoid, the earth rate u, the earth's mass M and the gravitational constant μ.

However, as already indicated above, the accuracy with which the gravitational constant and earth's mass are known is estimated by a value of the order of 0.1%. Therefore, the constants contained in the formulas (2.72) and (2.73) are best calculated in the following manner. Find the value of Dμ from the first relation of (2.74) and obtain the value of C by comparing the acceleration of gravity, obtained from formulas (2.72), (2.73) and (2.74) to the measured value of the acceleration of gravity at any point on the earth's surface, for example, at the equator.

From the first equality of (2.72), for the component of acceleration of gravity g_x on the surface of the terrestrial ellipsoid, we find the value

$$g_x = P_0 x - C\left(\frac{\partial K}{\partial x}\right)_0 - u^2 x. \tag{2.75}$$

where $(\partial k/\partial x)_0$ is the value of the derivative of K with respect to x at the point taken on the surface of the reference ellipsoid.

It follows from equalities (2.46), (2.40) and (2.39) that

(2.76)

$$\left.\begin{aligned}\frac{\partial K}{\partial x} &= -2\pi D\mu\frac{a^2 b}{a^2+\nu}\frac{x}{T_1},\\ \frac{\partial K}{\partial y} &= -2\pi D\mu\frac{a^2 b}{a^2+\nu}\frac{y}{T_1},\\ \frac{\partial K}{\partial z} &= -2\pi D\mu\frac{a^2 b}{b^2+\nu}\frac{z}{T_1},\end{aligned}\right\}$$

where

$$T_1=\left[\frac{x^2+y^2}{(a^2+\nu)^2}+\frac{z^2}{(b^2+\nu)^2}\right](a^2+\nu)\sqrt{b^2+\nu}. \tag{2.77}$$

Having taken the point with coordinates $y=z=0$ and $x=a$ at the equator and taking into account that $\nu=0$ on a level ellipsoid, we find from (2.75)-(2.77):

(2.78)

$$g_e=C2\pi D\mu a+P_0 a-\omega^2 a,$$

where g_e is the measured value of acceleration of gravity at the equator. Hence,

(2.79)

$$C=\frac{g_e+\omega^2 a-P_0 a}{2\pi D\mu a}.$$

2.2.3. Calculating the projections of the earth's regularized gravitational field intensity onto the axes of the geocentric and geographic moving trihedrons. Let us find the explicit expressions for projections F_{x_2}, F_{y_2} and F_{z_2} of the earth's gravitational field intensity onto the axes of the geocentric moving trihedron $Ox_2y_2z_2$, introduced in the preceding section.

From relations (2.72), (2.73), (2.76) and (2.77), having taken into account the table of direction cosines (2.), we find

$$
\left.
\begin{aligned}
&F_{x_2} = 0,\\
&F_{y_2} = 2\pi D\mu \sin\varphi\cos\varphi\left[\frac{a^2br}{(a^2-b^2)^{3/2}}\left(3\tan^{-1}l' - \right.\right.\\
&\qquad\left.\left. - \frac{l'}{1+l'^2} - 2l'\right) - \frac{Cra^2b\,(a^2-b^2)}{T\,(a^2+\nu)(b^2+\nu)}\right],\\
&F_{z_2} = -2\pi D\mu\left\{\frac{a^2br}{(a^2-b^2)^{3/2}}\left[2\,(l' - \tan^{-1}l') + \right.\right.\\
&\qquad\left. + \cos^2\varphi\left(3\tan^{-1}l' - \frac{l'}{1+l'^2} - 2l'\right)\right] +\\
&\qquad\left. + \frac{a^2bCr}{T}\left(\frac{\cos^2\varphi}{a^2+\nu} + \frac{\sin^2\varphi}{b^2+\nu}\right)\right\}.
\end{aligned}
\right\}
\tag{2.80}
$$

By using equalities (2.17), (2.2) and (2.77), we find the following expressions for the terms of the right sides of formulas (2.80), containing T:

$$
\left.
\begin{aligned}
&\frac{Cra^2b}{T}\,\frac{(a^2-b^2)}{(a^2+\nu)(b^2+\nu)} =\\
&\qquad = Ca^2b\left(\frac{a}{r}\right)r^2\,\frac{(b^2+\nu)^{1/2}}{(b^2+\nu)\cos^2\varphi + (a^2+\nu)\sin^2\varphi},\\
&\frac{Cra^2b}{T}\left(\frac{\cos^2\varphi}{a^2+\nu} + \frac{\sin^2\varphi}{b^2+\nu}\right) =\\
&= Cab\left(\frac{a}{r}\right)\frac{[(b^2+\nu)\cos^2\varphi + (a^2+\nu)\sin^2\varphi]\,(b^2+\nu)^{1/2}}{(b^2+\nu)^2\cos^2\varphi + (a^2+\nu)^2\sin^2\varphi}.
\end{aligned}
\right\}
\tag{2.81}
$$

The right sides of the second and third formulas of (2.80) can be expanded into rather rapidly convergent series by powers of e. Let us find this expansion with an accuracy up to the terms containing the factor e^4.

From equation (2.39) of a confocal ellipsoid, calculation of (2.40) of its second eccentricity and from the equalities for x, y and z, similar to equalities (2.17) for ξ, η and ζ, we find:

(2.82)

$$l' = \sqrt{\frac{2(a^2-b^2)}{r^2-(a^2-b^2)+\sqrt{r^4+(a^2-b^2)^2-2r^2(a^2-b^2)\cos 2\varphi}}}.$$

The expression for l' is easily represented in the form

(2.83)

$$l' = \frac{ae}{r}s,$$

where

(2.84)

$$s = 1 + \frac{e^2}{2}\left(\frac{a}{r}\right)^2\cos^2\varphi + \frac{e^4}{16}\left(\frac{a}{r}\right)^4\left(\frac{3}{8}\cos^2\varphi - \frac{7}{32}\sin^2 2\varphi\right) + \ldots$$

We now find:

(2.85)

$$\left.\begin{aligned} &\frac{a^3br}{(a^2-b^2)^{3/2}}\left(3\tan^{-1}l' - \frac{l'}{1+l'^2} - 2l'\right) = \\ &\qquad = b\left(\frac{a}{r}\right)^4 e^2 s^5\left(-\frac{2}{5} + \frac{4}{7}l'^2 + \ldots\right). \\ &\frac{2a^3br}{(a^2-b^2)^{3/2}}\left(l' - \tan^{-1}l'\right) = \\ &\qquad = b\left(\frac{a}{r}\right)^2 s^3\left(\frac{2}{3} - \frac{2}{5}l'^2 + \frac{1}{7}l'^4 + \ldots\right). \end{aligned}\right\}$$

By substituting expressions (2.83) and (2.84) for l' and s, we find:

(2.86)

$$
\left.\begin{aligned}
&\frac{a^2br}{(a^2-b^2)^{3/2}}\left(3\tan^{-1}l'-\frac{l'}{1+l'^2}-2l'\right)=\\
&\qquad=b\left(\frac{a}{r}\right)^4e^2\left[-\frac{2}{5}+e^2\left(\frac{4}{7}-\cos^2\varphi\right)\left(\frac{a}{r}\right)^2\right]+\ldots,\\
&\frac{2a^2br}{(a^2-b^2)^{3/2}}\left(l'-\tan^{-1}l'\right)=\\
&\qquad=b\left(\frac{a}{r}\right)^3\left[\frac{2}{3}+e^2\left(\cos^2\varphi-\frac{2}{5}\right)\left(\frac{a}{r}\right)^2+\right.\\
&\qquad\left.+e^4\left(\frac{2}{7}+\frac{1}{4}\cos^2\varphi-\frac{9}{32}\sin^2 2\varphi\right)\left(\frac{a}{r}\right)^4\right]+\ldots
\end{aligned}\right|
$$

Now let us find the expansions in powers of e of expressions (2.81), also contained in the formulas for calculating Fy_2 and Fz_2. From relations (2.83), (2.84) and (2.40), we find

(2.87)

$$
v=r^2\left[1-(1-e^2\sin^2\varphi)\left(\frac{a}{r}\right)^2+\frac{e^4}{4}\sin^2 2\varphi\left(\frac{a}{r}\right)^4+\ldots\right].
$$

Taking this into account, we find:

(2.88)

$$
\left.\begin{aligned}
&(b^2+v)^{1/2}=r\left[1-\frac{e^2}{2}\cos^2\varphi\left(\frac{a}{r}\right)^2+\right.\\
&\qquad\left.+\frac{e^4}{8}\left(\frac{5}{4}\sin^2 2\varphi-\cos^2\varphi\right)\left(\frac{a}{r}\right)^4+\ldots\right].\\
&(b^2+v)\cos^2\varphi+(a^2+v)\sin^2\varphi=\\
&\qquad=r^2\left[1-e^2\cos 2\varphi\left(\frac{a}{r}\right)^2+\frac{e^4}{4}\sin^2 2\varphi\left(\frac{a}{r}\right)^4+\ldots\right].\\
&(b^2+v)^2\cos^2\varphi+(a^2+v)^2\sin^2\varphi=\\
&\qquad=r^4\left[1-2e^2\cos 2\varphi\left(\frac{a}{r}\right)^2+\right.\\
&\qquad\left.+e^4\left(1-\frac{1}{4}\sin^2 2\varphi\right)\left(\frac{a}{r}\right)^4+\ldots\right].
\end{aligned}\right|
$$

By substituting expressions (2.88) into formulas (2.81) and by performing termwise multiplication and division of the series, we find:

(2.89)

$$
\begin{aligned}
&\frac{Ca^2br}{T}\frac{(a^2-b^2)}{(a^2+\nu)(b^2+\nu)}=\\
&\quad=Cbe^2\left(\frac{a}{r}\right)^4\left[1+e^2\left(\frac{7}{2}\cos^2\varphi-2\right)\left(\frac{a}{r}\right)^2+\ldots\right],\\
&\frac{Ca^2br}{T}\left(\frac{\cos^2\varphi}{a^2+\nu}+\frac{\sin^2\varphi}{b^2+\nu}\right)=\\
&\quad=Cb\left(\frac{a}{r}\right)^2\left[1+e^2\left(\frac{3}{2}\cos^2\varphi-1\right)\left(\frac{a}{r}\right)^2+\right.\\
&\quad\left.+e^4\left(1-\frac{5}{8}\cos^2\varphi-\frac{35}{32}\sin^2 2\varphi\right)\left(\frac{a}{r}\right)^4+\ldots\right].
\end{aligned}
$$

Let us now substitute expressions (2.89) and (2.86) into formulas (2.80). Converting everywhere to trigonometric functions of sin φ and sin 2φ, we find the following equalities:

(2.90)

$$
\begin{aligned}
&F_{x_1}=0,\\
&F_{y_1}=\pi D\mu be^2\sin 2\varphi\left(\frac{a}{r}\right)^4\left\{-\frac{2}{5}-C+\right.\\
&\quad\left.+\left[-3\left(\frac{1}{7}+\frac{C}{2}\right)+\sin^2\varphi\left(1+\frac{7C}{2}\right)\right]e^2\left(\frac{a}{r}\right)^2\right\},\\
&F_{z_1}=-2\pi D\mu b\left(\frac{a}{r}\right)^2\left\{\frac{2}{3}+C+\left[\frac{1}{5}+\frac{C}{2}-\right.\right.\\
&\quad\left.-\sin^2\varphi\left(\frac{3}{5}+\frac{3C}{2}\right)\right]e^2\left(\frac{a}{r}\right)^2+\\
&\quad+\left[\frac{3}{28}+\frac{3C}{8}+\sin^2\varphi\left(\frac{5}{28}+\frac{5C}{8}\right)+\right.\\
&\quad\left.\left.+\sin^2 2\varphi\left(-\frac{5}{16}-\frac{35C}{32}\right)\right]e^4\left(\frac{a}{r}\right)^4\right\}.
\end{aligned}
$$

These equalities are also the desired expansions in powers of e of the projections of the earth's gravitational field intensity onto axes x_2, y_2 and z_2. The rapid convergence of the series obtained is provided by the smallness of the earth's eccentricity e.

The right sides of formulas (2.90) contain the constants $D\mu$ and C, calculated by the first equality of (2.74) and by equality (2.79), respectively. Let us find the explicit expressions for constants $D\mu$ and C.

From the first equality of (2.74)

(2.91)

$$D\mu = \frac{u^2 l^3}{2\pi\left[(3 + l^2)\tan^{-1} l - 3l\right]},$$

hence, by expanding the function on the right side by powers of l, we find:

(2.92)

$$D\mu = \frac{u^2}{2\pi l^5} \frac{15}{4}\left(1 + \frac{6}{7} l^2 + \frac{1}{49} l^4 + \ldots\right).$$

By converting in relation (2.92) to the first eccentricity and by using for this the relation

$$l^2 = \frac{e^2}{1 - e^2},$$

we find $D\mu$ in the following form:

(2.93)

$$D\mu = \frac{u^2}{2\pi e^5} \frac{15}{4}\left(1 - \frac{1}{7} e^2 + \frac{1}{49} e^4 + \ldots\right).$$

It remains to find C. However, it is more convenient to find directly the value of $\pi D\mu C$, because constant C is contained in formulas (2.90) in this combination.

From formula (2.79)

(2.94)

$$\pi D\mu C = \frac{1}{2a}(g_e + u^2 a - P_0 a).$$

According to (2.34),

$$P_0 = 2\pi D\mu \frac{1+l^2}{l^3}\left(\tan^{-1} l - \frac{l}{1+l^2}\right). \tag{2.95}$$

Expansion into a series in l and conversion afterwards to the first eccentricity yield the following value for P_0

$$P_0 = 2\pi D\mu\left(\frac{2}{3} - \frac{2}{15}e^2 - \frac{8}{105}e^4\right). \tag{2.96}$$

By substituting expression (2.96) into formula (2.94) and taking into account relation (2.93), we arrive at the equality

$$\pi D\mu C = \frac{g_e}{2a} + \frac{a^3}{2e^3}\left(-\frac{5}{2} + \frac{13}{7}e^2 + \frac{8}{49}e^4\right). \tag{2.97}$$

which also yields the explicit representation of the constant $\pi D\mu C$.

Substitution of expressions (2.93) and (2.97) into formulas (2.90) leads to the following expressions for Fy_2 and Fz_2:

$$\begin{aligned}
F_{y_2} &= \frac{g_e}{2}(q - e^2)\left(\frac{a}{r}\right)^4 \sin 2\varphi\left[1 + e^2\,\frac{7e^2 - 30q}{14(q - e^2)}\right]\times \\
&\quad\times\left\{1 + \left[\frac{30q - 21e^2}{14(q - e^2)} + \sin^2\varphi\,\frac{7e^2 - 10q}{2(q - e^2)}\right]e^2\left(\frac{a}{r}\right)^2\right\}. \\
F_{z_2} &= -g_e\left(\frac{a}{r}\right)^2\left\{1 - \frac{e^2}{2} - \frac{e^4}{8} + q\left(\frac{3}{2} - \frac{15}{28}e^2\right) + \right. \\
&\quad + \left[\frac{e^2}{2} - \frac{e^4}{4} + q\left(-\frac{1}{2} + \frac{15}{14}e^2\right) - \right. \\
&\quad - \sin^2\varphi\left(\frac{3e^2}{2} - \frac{3e^4}{4} + q\left(-\frac{3}{2} + \frac{45}{14}e^2\right)\right)\bigg]\left(\frac{a}{r}\right)^2 + \\
&\quad + \left[\frac{3e^2}{8} - \frac{15}{28}q + \sin^2\varphi\left(\frac{5e^2}{8} - \frac{25}{28}q\right) - \right. \\
&\quad \left.\left. - \sin^2 2\varphi\left(\frac{35}{32}e^2 - \frac{25}{16}q\right)\right]e^2\left(\frac{a}{r}\right)^4\right\}.
\end{aligned} \tag{2.98}$$

where the ratio of the centrifugal force occurring because of the earth's rotation to gravity at the equator is denoted by q, i.e.,

$$q = \frac{\omega^2 a}{g_e}. \qquad (2.99)$$

If we take[10]

$$\left.\begin{aligned} \omega &= \frac{2\pi}{86\,164.0} = 7.292116 \cdot 10^{-5}\ 1/\text{сек}, \\ a &= 6\,378\,245\ \text{м}, \quad e^2 = 0.0066934, \\ g_e &= 978.049\ \text{cm/sec}^2, \end{aligned}\right| \qquad (2.100)$$

then

$$q = 0.00346775. \qquad (2.101)$$

The numerical values of the coefficients contained in formulas (2.98) will then be equal to

$$\left.\begin{aligned} \frac{g_e}{2}(q - e^2) &= -1.577, \quad 1 + e^2\frac{7e^2 - 30q}{14(q - e^2)} = 1.008, \\ e^2\frac{30q - 21e^2}{14(q - e^2)} &= 0.005, \quad e^2\frac{7e^2 - 10q}{2(q - e^2)} = -0.013, \\ 1 - \frac{e^2}{2} - \frac{e^4}{8} + q\left(\frac{3}{2} - \frac{15}{28}e^2\right) &= 1.001837, \\ \frac{1}{2}e^2 - \frac{1}{4}e^4 + q\left(-\frac{1}{2} + \frac{15}{14}e^2\right) &= 0.001627, \\ \frac{3}{2}e^2 - \frac{3}{4}e^4 + q\left(-\frac{3}{2} + \frac{45}{14}e^2\right) &= 0.004878, \\ e^2\left(\frac{3}{8}e^2 - \frac{15}{28}q\right) &= 0.000005, \\ e^2\left(\frac{5}{8}e^2 - \frac{25}{28}q\right) &= 0.000008, \\ e^2\left(\frac{35}{32}e^2 - \frac{25}{16}q\right) &= 0.000009. \end{aligned}\right| \qquad (2.102)$$

The numerical values of the coefficients indicate that formulas (2.98) can be written with an accuracy of the order of 0.02 cm/sec^2 in the following manner:

(2.103)

$$\left.\begin{aligned} F_{y_2} &= \frac{g_e(q-e^2)}{2}\left(\frac{a}{r}\right)^4 \sin 2\varphi, \\ F_{z_2} &= -g_e\left(\frac{a}{r}\right)^2\left[1-\frac{e^2}{2}+\frac{3}{2}q+\right. \\ &\quad \left.+\frac{q-e^2}{2}(-1+3\sin^2\varphi)\left(\frac{a}{r}\right)^2\right]. \end{aligned}\right\}$$

where

(2.104)

$$\left.\begin{aligned} \frac{g_e(q-e^2)}{2} &= -1.58, \quad \frac{q-e^2}{2} = 0.0016, \\ 1-\frac{e^2}{2}+\frac{3}{2}q &= 0.0018. \end{aligned}\right\}$$

Formulas (2.98) or the similar formulas (2.103) yield expressions for Fy_2 and Fz_2 as a function of the geocentric coordinates φ and r. At the same time it may happen that the directly known values will be the geocentric latitude φ and the height above sea level. The latter, as was noted in §2.1, coincides with great accuracy to the distance h along the normal to Clairaut's level ellipsoid.

Formulas (2.23), which yields the expression r in φ', h and the parameters of Clairaut's ellipsoid, was obtained in §2.1. The following value is obtained from this formula for r^2:

(2.105)

$$r^2 = a^2 + h^2 - \frac{a^2e^2(1-e^2)\sin^2\varphi'}{1-e^2\sin^2\varphi'} + 2ah\sqrt{1-e^2\sin^2\varphi'}.$$

Let us consider the case of small values of h, when the ratio h/r has the same order of smallness as the square of the first eccentricity e^2. Then with an accuracy up to values of the order of e^4 inclusively, we have:

(2.106)

$$r^2 = a^2\left[\left(1+\frac{h}{a}\right)^2 - e^2\left(1+\frac{h}{a}\right)\sin^2\varphi' + e^4\sin^2\varphi'\cos^2\varphi'\right].$$

With an accuracy up to values of the order of e^2, the geocentric latitude φ' is related to the geocentric relation

(2.107)

$$\tan\varphi' = \frac{\tan\varphi}{1-e^2}.$$

which ensues from functions (2.20) and (2.25). From equality (2.107), it is easy to find in turn the expression for the square of the sin of geographic latitude:

(2.108)

$$\sin^2\varphi' = \sin^2\varphi(1+2e^2\cos^2\varphi).$$

By substituting expression (2.108) into formula (2.106), we find

(2.109)

$$r^2 = a^2\left(1+\frac{2h}{a}+\frac{h^2}{a^2}-\frac{h}{a}e^2\sin^2\varphi - e^2\sin^2\varphi - e^4\sin^2\varphi\cos^2\varphi\right).$$

Then,

(2.110)

$$\left(\frac{a}{r}\right)^2 = 1-\frac{2h}{a}+\frac{3h^2}{a^2}-3\frac{h}{a}e^2\sin^2\varphi + e^2\sin^2\varphi + e^4\sin^2\varphi.$$

Let us substitute expression (2.110) into formulas (2.98). After obvious transformations we find

$$
\left.\begin{aligned}
F_{y_2} &= \frac{g_e(q-e^2)}{2}\sin 2\varphi\left[1-\frac{4h}{a}-\right.\\
&\left.-\frac{e^4}{q-e^2}+e^2\frac{3e^4-6q}{2(q-e^2)}\sin^2 2\varphi\right],\\
F_{z_2} &= -g_e\left[1-\frac{e^2}{2}\sin^2\varphi+q\left(1+\frac{3}{2}\sin^2\varphi\right)+\right.\\
&+e^4\left(-\frac{1}{8}\sin^2\varphi-\frac{11}{32}\sin^2 2\varphi\right)+\\
&+e^2q\left(-\frac{17}{28}\sin^2\varphi+\frac{13}{16}\sin^2 2\varphi\right)+\\
&+\frac{h}{a}e^2(3\sin^2\varphi-1)+\frac{hq}{a}(-1-6\sin^2\varphi)-\\
&\left.-\frac{2h}{2}+\frac{3h^2}{a^2}\right].
\end{aligned}\right\}
\tag{2.111}
$$

These formulas with an accuracy up to values of the order of e^4 yield expressions for Fy_2 and Fz_2 in φ and h.

Let us turn in formulas (2.111) to the geographic latitude φ'. Since the trigonometric functions of the geocentric latitude φ are contained in formulas (2.111) with factors having an order of e^2 and e^4, then the values of $\sin^2\varphi$, $\sin 2\varphi$ and $\sin^2 2\varphi$, expressed by the trigonometric functions of geographic latitude φ' with an accuracy only up to values of the order of e^2, must be substituted into these formulas. From equality (2.107), we find

$$
\left.\begin{aligned}
\sin^2\varphi &= \sin^2\varphi'(1-2e^2\cos^2\varphi'),\\
\sin 2\varphi &= \sin 2\varphi'[1+e^2(2\sin^2\varphi'-1)],\\
\sin^2 2\varphi &= \sin^2 2\varphi'[1+e^2(4\sin^2\varphi'-2)].
\end{aligned}\right\}
\tag{2.112}
$$

By substituting relations (2.112) into formulas (2.111), we arrive at the equalities:

(2.113)

$$
\left.\begin{aligned}
F_{y_2} = \frac{g_e(q-e^2)}{2}\sin 2\varphi'\Big[1 - 4\frac{h}{a} - \frac{qe^2}{q-e^2} - \\
- \frac{e^2(e^2+2q)}{2(q-e^2)}\sin^2\varphi'\Big]. \\
F_{z_2} = -g_e\Big[1 - \frac{e^2}{2}\sin^2\varphi' + q\left(1 + \frac{3}{2}\sin^2\varphi'\right) + \\
+ e^4\left(-\frac{1}{8}\sin^2\varphi' - \frac{3}{32}\sin^2 2\varphi'\right) + \\
+ e^2q\left(-\frac{17}{28}\sin^2\varphi' + \frac{1}{16}\sin^2 2\varphi'\right) + \\
+ \frac{h}{a}e^2(3\sin^2\varphi' - 1) + \\
+ \frac{hq}{a}(-1 - 6\sin^2\varphi') - \frac{2h}{a} + \frac{3h^2}{a^2}\Big].
\end{aligned}\right|
$$

The right sides of these equalities are expressions in φ' and h of the projection of Fy_2 and Fz_2 of the earth's gravitational field intensity on the y_2 and z_2 axes of the geocentric moving trihedron $x_2y_2z_2$.

It is now easy to find the expressions in φ' and h of the projections of Fy_1 and Fz_1 of the gravitational field intensity on the y_1 and z_1 axes of the geographic moving trihedron. According to table (2.24) of the direction cosines, we find

(2.114)

$$
\left.\begin{aligned}
F_{y_1} &= F_{y_2}\cos(\varphi' - \varphi) - F_{z_2}\sin(\varphi' - \varphi). \\
F_{z_1} &= F_{y_2}\sin(\varphi' - \varphi) + F_{z_2}\cos(\varphi' - \varphi).
\end{aligned}\right\}
$$

In order to write the explicit expressions for Fy_1 and Fz_1 only in h and φ', we need to have with an accuracy up to values of the order of e^4 of the value of sin (φ' - φ) and cos (φ' - φ), expressed by these variables.

From (2.25), we find

$$\tan(\varphi' - \varphi) = e^2 \sin\varphi' \cos\varphi' \left(1 + e^2 \sin^2\varphi' - \frac{h}{a}\right). \tag{2.115}$$

Hence,

$$\left.\begin{aligned} \sin(\varphi' - \varphi) &= e^2 \sin\varphi' \cos\varphi' \left(1 + e^2 \sin^2\varphi' - \frac{h}{a}\right), \\ \cos(\varphi' - \varphi) &= 1 - \frac{e^4}{2} \sin^2\varphi' \cos^2\varphi'. \end{aligned}\right\} \tag{2.116}$$

By substituting these expressions together with equalities (2.113) into relations (2.114), we arrive at the following formulas:

$$\left.\begin{aligned} F_{y_1} &= g_e \sin 2\varphi' \left[\frac{q}{2}\left(1 + \frac{e^2}{2}\sin^2\varphi'\right) + \frac{h}{a}\left(\frac{e^2}{2} - 2q\right)\right], \\ F_{z_1} &= -g_e \left[1 - \frac{e^2}{2}\sin^2\varphi' + q\left(1 + \frac{3}{2}\sin^2\varphi'\right) + \right. \\ &\quad + e^4\left(-\frac{1}{8}\sin^2\varphi' + \frac{1}{32}\sin^2 2\varphi'\right) + \\ &\quad + e^2 q\left(-\frac{17}{28}\sin^2\varphi' - \frac{3}{16}\sin^2 2\varphi'\right) + \frac{h}{a}e^2(3\sin^2\varphi' - 1) + \\ &\quad \left. + \frac{hq}{a}(-1 - 6\sin^2\varphi') - \frac{2h}{a} + \frac{3h^2}{a^2}\right]. \end{aligned}\right\} \tag{2.117}$$

Formulas (2.117) yield expressions of the projection of F_{y_1} on the direction tangent to the ellipsoid and lying within the meridional plane, and projection F_{z_1} of the gravitational field intensity on the normal to the level ellipsoid.

Having set $h = 0$ in formulas (2.117), we find the formulas which determine the projections $F^0_{y_1}$ and $F^0_{z_1}$ of the gravitational field intensity on to the earth's surface (on the level ellipsoid)

$$\left.\begin{aligned} F^0_{y_1} &= \frac{g_e q}{2}\left(1 + \frac{e^2}{2}\sin^2\varphi'\right)\sin 2\varphi', \\ F^0_{z_1} &= -g_e\left[1 - \frac{e^2}{2}\sin^2\varphi' + q\left(1 + \frac{3}{2}\sin^2\varphi'\right) + \right. \\ &\quad + e^4\left(-\frac{1}{8}\sin^2\varphi' + \frac{1}{32}\sin^2 2\varphi'\right) + \\ &\quad \left. + e^2 q\left(-\frac{17}{28}\sin^2\varphi' - \frac{3}{16}\sin^2 2\varphi'\right)\right]. \end{aligned}\right\} \tag{2.118}$$

If we now add the values of the projections onto the y_1 and z_1 axes of centrifugal acceleration, which occurs because of the earth's rotation, to the values of projections $F^0_{y_1}$ and $F^0_{z_1}$, the first sum should be equal to zero, while the second sum should lead to the formula of normal gravity.

Let us denote the projections of centrifugal acceleration onto the y_1 and z_1 axes by F'_{y_1} and F'_{z_1}. We have (Figure 2.4)

$$\begin{aligned} F'_{y_1} &= -u^2 r\cos\varphi\sin\varphi', \\ F'_{z_1} &= u^2 r\cos\varphi\cos\varphi', \end{aligned} \tag{2.119}$$

where u is the earth rate.

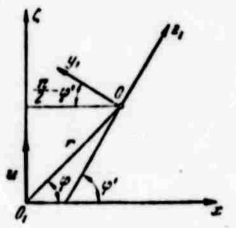

Fig. 2.4

From relations (2.112) and (2.109), with an accuracy up to terms of the order of e^2, we find

$$r\cos\varphi = a\left(1+\frac{e^2}{2}\sin^2\varphi'\right)\cos\varphi'. \tag{2.120}$$

By introducing the notation of (2.99), we find

$$\left.\begin{aligned} F'_{y_1} &= -\frac{g_e q}{2}\left(1+\frac{e^2}{2}\sin^2\varphi'\right)\sin 2\varphi', \\ F'_{z_1} &= g_e q\left(1+\frac{e^2}{2}\sin^2\varphi'\right)\cos^2\varphi'. \end{aligned}\right| \tag{2.121}$$

We now find:

(2.122)

$$
\left.\begin{aligned}
&F^0_{y_0} + F'_{y_0} = 0,\\
&F^0_{x_0} + F' r_0 - g = \\
&= -g_e\Big[1 + \sin^2\varphi'\left(\frac{5}{2}q - \frac{e^2}{2} - \frac{e^4}{8} - \frac{17}{28}e^2 q\right) + \\
&\qquad + \sin^2 2\varphi'\left(\frac{e^4}{32} - \frac{5}{16}e^2 q\right)\Big].
\end{aligned}\right|
$$

Instead of the square of first eccentricity e^2, compression α may be introduced into the second formula of (2.122). According to equalities (2.1) and (2.2), we have the expansion

(2.123)

$$\alpha = \frac{e^2}{2} + \frac{e^4}{8} + \cdots$$

By substituting the expansion (2.123) into the second equality of (2.122), we arrive at the formula

(2.124)

$$g = g_e(1 + \beta\sin^2\varphi' + \beta_1\sin^2 2\varphi'). \qquad (2.124)$$

where

(2.125)

$$
\left.\begin{aligned}
&\beta = \frac{5}{2}q - \alpha - \frac{17}{14}q\alpha,\\
&\beta_1 = \frac{\alpha^2}{8} - \frac{5}{8}q\alpha,
\end{aligned}\right|
$$

i.e., to the well-known formula of normal gravity in Helmert-Kassinis form,[11] which was required.

Calculations for the parameters of the Krasovskiy ellipsoid yields:

$$\beta = 0.0053171, \quad \beta_1 = 0.0000071.$$

Formulas (2.125) are called Clairaut's formulas. If the acceleration of gravity at the pole is denoted by g_p, then it follows from equality (2.124) that

$$\beta = \frac{g_p - g_e}{g_e}. \tag{2.126}$$

We note that if coefficient β is calculated from formula (2.124) from the results of observations of gravity at different latitudes, then Clairaut's first formula permits calculation of the compression of the terrestrial spheroid, because the value of q is known with great accuracy.

Let us also note that the formula of normal gravity was obtained only to ascertain that it results as a special case from the more general formulas which we constructed. This is the well-known check of the correctness of the calculations made above. Expression (2.124) for g may in itself be obtained simply from Somil'yan's formula.[11]

§2.3. The Earth's Motion Relative To Its Center of Mass

The earth's motion relative to distant (fixed) stars, (or, in other words, relative to the inertial reference system $O\xi_*\eta_*\zeta_*$) consists of translational motion, i.e., the motion of its center of mass and of rotation about the center of mass. If the position and velocity of the earth's center of mass are taken at some moment of time as the initial moments, then its further motion is calculated by the resulting attractive force of the earth's elementary masses by the celestial bodies. Similarly, rotation about the center of mass is determined by the moment of this resultant relative to the center of the earth.

When solving problems of autonomous inertial navigation near the earth (or rather in the system of reference bound to it), it is not necessary to know the motion of the earth's center of mass. If fact, the motion of the earth's center of mass is not contained in the fundamental equation of inertial navigation (1.88). It disappeared from this equation in view of equality (1.85).

The earth's motion about the center of mass is another matter. This motion should be known. Actually, the fundamental equation of inertial navigation (1.88) is written in the coordinate system $O_1\xi_*\eta_*\zeta_*$, whose origin coincides with the center of the earth, while orientation of the axes is identical to orientation of the axes of the inertial system of reference $O_1\xi_*\eta_*\zeta_*$. Therefore, orientation of axes ξ_*, η_* and ζ_* may be assumed to be fixed relative to the directions toward the remote stars. If we assume the earth's gravitational field intensity $g(\vec{r})$ to be given in the system of reference $O_1\xi_*\eta_*\zeta_*$, we can find the coordinates of ξ_*, η_* and ζ_* from equation (1.88). Conversion to coordinates ξ, η and ζ in the trihedron $O_1\xi\eta\zeta$ bound to the earth obviously requires knowledge of the position of trihedron $O_1\xi\eta\zeta$ relative to the trihedron $O_1\xi_*\eta_*\zeta_*$ i.e., one must know the earth's motion about its center. Moreover, as already noted in §1.4, the earth's gravitational field intensity (taking into account the non-sphericity of its gravitational field) is given in the earth body axis system $O_1\xi\eta\zeta$. Recalculation of gravitational field intensity to the coordinate trihedron $O_1\xi_*\eta_*\zeta_*$ also requires a knowledge at every moment of time of the mutual disposition of trihedrons $O_1\xi\eta\zeta$ and $O_1\xi_*\eta_*\zeta_*$.

When considering problems of the theory of autonomous inertial navigation, we may assume that the earth's center of mass coincides with the center of Clairaut's ellipsoid, while the earth's motion about the center of mass reduces to uniform rotation about the axis of symmetry of Clairaut's ellipsoid, which retains its own orientation unchanged relative to the directions toward fixed stars.

Actually, the position of the instantaneous rotational axis of the earth does not coincide with the minor axis of the terrestrial ellipsoid (the least major axis of the ellipsoid of inertia). Therefore, it follows from Euler's equations of the rotation of a solid relative to the inertial center of mass that the instantaneous axis of the earth's rotation will describe a cone about its axis in the earth's body. Euler found the period of this motion equal to approximately 305 days. S. Chandler's processing of experimental materials showed that the motion of the earth's instantaneous rotational axis in its body has two periods: the first is equal to approximately 420 days and the second is equal to one year. S. Newcomb showed that a 420-day period is Euler's period with regard to the non-rigidity of the earth. The annual period is related to the seasonal redistribution of masses on the earth's surface.[12]

The maximum deviation of the earth's instantaneous rotational axis from the direction of the minor axis of Clairaut's ellipsoid does not exceed 0.67", which yields the error of determining the latitude of the point on to the earth's surface. This error may obviously be disregarded in navigation problems.

The value of the earth rate (its modulus) is, strictly speaking, not fixed.[13] It has been noted that the length of days because of tidal friction increases by an average of 0.0016 sec per century. Moreover, seasonal variations of the length of the days by a value up to 0.0025 sec and irregular intermittent variations having values up to 0.034 sec have been observed. All these variations are small and they can be disregarded in the consideration of problems. The time determined by the earth's rotation with respect to the distant stars (stellar time) may also be assumed uniform and adequate to Newtonian dynamic time.

Orientation of vector $\vec{u}$ of the earth rate in stellar space does not remain fixed. The main cause of this is the circumstance that the earth's attraction by the sun and moon leads not only to resultant forces, directed along lines connecting the earth's center of mass to the centers of mass of the sun and moon, but also to resulting moments. This is in turn caused by the fact that compression of the earth leads to asymmetry of the earth's distribution of mass relative to the directions from its center to the sun and moon.

The vectors of the resulting moments from the sun and moon are located within the plane of the terrestrial equator and accordingly attempt to combine this plane to the plane of the ecliptic (the orbital plane of the earth) and to the plane of the lunar orbit.

The action of the indicated moments leads to precession of the earth's angular velocity vector relative to the normal to the plane of the ecliptic along a cone with an angle of $2\varepsilon=23^{\circ}27'$ at the vertex with a period approximately equal to 26,000 years, and to nutation with the main period of approximately 18.6 years, which leads to periodic variation of angle ε by the value $\Delta\varepsilon=10''$.[14]

Because of the perturbing action of the planets, the earth's orbital plane also does not remain fixed in stellar space. It rotates about an axis, lying within the orbital plane at a velocity having a value of the order of 47" per century during the current epoch. This leads to slow variation (a decrease in the current epoch) of angle ε. Moreover, because of the motion of the moon and earth about the common center of mass, the earth's orbit deviates from the plane of the ecliptic, near which the motion of the center of mass of the earth-moon system occurs by a value of the order of 1".

All the indicated effects of variation of the position of the earth's rotational axis in stellar space, which plays an important role during fundamental astronomical investigations, may obviously be disregarded in navigation problems because of smallness, and in any case if we bear in mind determination of the position of an object with an accuracy of the order of one km, and the operating time of the inertial system not exceeding, for example, one month.

Henceforth, we shall usually assume that vector $\vec{u}$ of the earth rate coincides with its axis, whose orientation we shall assume to be fixed in stellar space. Let us assume that the value of the earth rate is constant ($u=7.292116 \cdot 10^{-5}$). However, we note that, as will become clear subsequently, the problem of inertial navigation can be solved in principle and with regard to the inconstancy of the earth rate. It is sufficient to know only the projections $u_\xi(t)$, $u_\eta(t)$, and $u_\zeta(t)$ of vector $\vec{u}$ of the earth rate to its body axes ξ, η and ζ as time functions.

1. The parameters of the ellipsoids taken in other countries can be found, for example in the book: Graur L. V. Matematicheskaya Kartografiya, Isd-vo LGU im. A. A. Zhdanov, 1956

2. Mikhaylov, A. A., Kurs gravimetrii i teorii figury Zemli (Course in Gravimetry and Theory of the Shape of the Earth), Redbyuru GUGK for SNK of the USSR, 1939; Grushinskiy, N. P., Teoriya figury Zemli (Theory of the Shape of the Earth), Fizmatgiz, 1963.

3. Rashevskiy, P. K., Kurs differentsial'noy geometrii (Course in Differential Geometry), Gostekhizdat, 1956.

4. Compare, for example, Mikhaylov, A. A., op. cit., or Duboshin, G. N., Teoriya prityazheniya (Theory of Attraction), Fizmatgiz, 1961.

5. The force function is diff[illegible] sign from the gravitational field strength.

6. The proof of this theorem can be found in the works indicated in Note 4 and also in Idel'son, N. I., Teoriya potensiala i ego prilozheniya k voprosam geofiziki (Theory of the Potental and its Application to Problems of Geophysics), Gostekhizdat, 1932; Grushinskiy, N. P., op. cit.

7. Compare, for example, the literature in Note 6.

8. Andreyev, V. D., On Solving the Stokes Problem for a Reference Surface Given in the Form of a Spheroid, Prikladnaya matematika i mekhanika, Vol. XXX, Issue 2, 1966.

9. In this case (for the trihedron $Ox_2y_2z_2$) it is necessary to replace φ' by φ in Table (2.4).

10. Mikhaylov, A. A., op. cit.; Graur, A. V., Matematicheskaya kartografiya (Mathematical Cartography), A. A. Zhdanov Press of Leningrad State University, 1956.

11. Mikhaylov, A. A., op. cit.

12. Blazhko, S. N., Kurs sfericheskoy astronomii (Course in Spherical Astronomy), Gostekhizdat, 1954.

13. Kulikov, K. A., Izmenyayemost' shirot i dolgot (Variability of Latitudes and Longitudes), Fizmatgiz, 1962.

14. Blazhko, S. N., op. cit.; Subbotin, M. F., Kurs nebesnoy mekhaniki (Course in Celestial Mechanics), Vol. 2, ONTI, 1937.

Chapter 3

EQUATIONS OF THE IDEAL OPERATION OF INERTIAL NAVIGATION SYSTEMS

§3.1. Calculating the Cartesian Coordinates of an Object.[1]

3.1.1. Initial relations. Let us consider an inertial navigation system constructed in the following manner. Three newtonometers n_x, n_y and n_z (Figure 3.1) are mounted on the platform of an absolute angular-rate meter with three degrees of freedom. The directions of the axes of sensitivity of the newtonometer coincide with the directions of the x, y and z axes of the right-hand orthogonal coordinate system Oxyz, bound to the platform.[2] In the general case the platform is installed on board in a moving object in a gimbal suspension with three degrees of freedom similar to the way in which the gyrostabilized platform (Figure 1.10), considered in §1.3, was suspended. Let us assume that the task of the inertial navigation system is to calculate the Cartesian coordinates ξ_*, η_* and ζ_* of point O in the coordinate system $O_1\xi_*\eta_*\zeta_*$ (or coordinates ξ, η and ζ of this point in the coordinate system $O_1\xi\eta\zeta$), and also the parameters which determine the orientation of the object relative to the axes of this system.

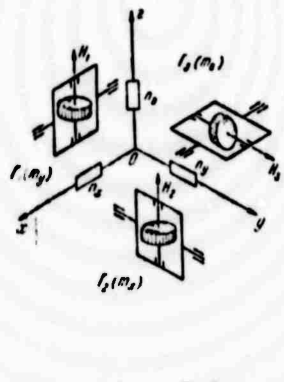

Fig. 3.1

The coordinate axes $O_1\varepsilon_*\eta_*\zeta_*$, which we introduced previously to derive the fundamental equation of inertial navigation, retain fixed directions relative to the directions to the remote stars. The origin O_1 of this coordinate system is incident with the earth's center of mass. We shall henceforth assume that the earth's center of mas coincides with its geometric center. The coordinate system $O_1\xi\eta\zeta$, also introduced previously, is rigidly bound to the earth. Its origin is incident with the earth's center and the ζ axis is directed along the vector of the earth rate. The ξ and η axes are located in the equatorial plane. Let us assume further that the ξ axis coincides with the line of intersection of the planes of the equator and of the Greenwich meridian.

Let us assume that the sensitive masses of the newtonometers are located at point O. Let us denote their readings by n_x, n_y, and n_z. Let us denote the readings of the absolute angular velocity (rate) meters by m_x, m_y and m_z, respectively. Let us assume that

$$m_x = \omega_x, \quad m_y = \omega_y, \quad m_z = \omega_z, \tag{3.1}$$

where ω_x, ω_y and ω_z are projections of the absolute angular velocity of the platform to the axes of trihedron xyz bound to it. According to the accepted disposition of the suspension axes of the gyroscope housings and the directions of their intrinsic moments of momentum , the values of m_x, m_y and m_z are calculated by relations (3.1) and (1.43) or by (3.1) and (1.45). It follows from these relations that the values of m_x, m_y and m_z are proportional to the values of deformations of elastic suspensions of gyroscopes G_2, G_1 and G_3, respectively, with great accuracy.

Let us use equation (1.88) of motion of the sensitive mass of the newtonometer to derive the equations of ideal operation. Let us first introduce the coordinate system O_1xyz, whose axes are parallel to the coordinate axes Oxyz of the same name, and let us take as the origin the earth's center O_1xyz can be obviously given in Cartesian coordinates x, y and z.

Let us turn to equation (1.88)

$$n = \frac{d^2r}{dt^2} - g(r). \tag{3.2}$$

The newtonometer readings of the considered inertial system are projections of vector $\vec{n}$ to the coordinate axes Oxyz. These projections are equal to the corresponding projections to coordinate axes O_1xyz, since axes Ox, Oy and Oz are parallel to axes O_1x, O_1y and O_1z, and, thus, trihedron Oxyz moves in a forward direction with respect to trihedron O_1xyz. Differentiation in equation (3.2) is carried out in the coordinate system $O_1\xi_*\eta_*\zeta_*$. The coordinate system Oxyz has a common origin with it and rotates with respect to it at an angular velocity $\vec{\omega}$ to axes O_1x, O_1y and O_1z are obviously equal to ω_x, ω_y and ω_z, because trihedrons Oxyz and O_1xyz, as already noted, have an identical orientation in the coordinate system $O_1\xi_*\eta_*\zeta_*$.

Having applied formula (1.14) twice to r, which yields the expression of the absolute derivative of the vector in a rotating coordinate system, we find:

$$v = \frac{dr}{dt} = \dot{r} + \omega \times r, \quad \frac{d^2r}{dt^2} = \dot{v} + \omega \times v, \tag{3.3}$$

where

$$r = x\mathbf{x} + y\mathbf{y} + z\mathbf{z}, \quad \omega = \omega_x\mathbf{x} + \omega_y\mathbf{y} + \omega_z\mathbf{z}. \tag{3.4}$$

while the dot denotes the local differentiation in the coordinate system O_1xyz, i.e., differentiation of vectors $\vec{r}$ and $\vec{\omega}$, given by relations (3.4), provided that x, y and z in these relations do not depend on time.

By substituting the second equality of (3.3) into equation (3.2), we find:

$$\mathbf{n} = \dot{\mathbf{v}} + \boldsymbol{\omega} \times \mathbf{v} - \mathbf{g}(\mathbf{r}) \tag{3.5}$$

Taking into account equalities (3.1) and introducing the vector

$$\mathbf{m} = m_x\mathbf{x} + m_y\mathbf{y} + m_z\mathbf{z}. \tag{3.6}$$

we write the equality (3.5) and the first relation of (3.3) as follows:

$$\left.\begin{aligned} \dot{\mathbf{v}} &= \mathbf{n} - \mathbf{m} \times \mathbf{v} + \mathbf{g}(\mathbf{r}), \\ \dot{\mathbf{r}} &= \mathbf{v} - \mathbf{m} \times \mathbf{r}, \end{aligned}\right\} \tag{3.7}$$

where

$$\mathbf{n} = n_x\mathbf{x} + n_y\mathbf{y} + n_z\mathbf{z}. \tag{3.8}$$

since vector $\vec{n}$ is given by its own projections on the coordinate axes Oxyz, or, which is the same thing, on the axes of system O_1xyz.

3.1.2. Integration of the fundamental equation during arbitrary rotation of the platform of an inertial system. The first group of equations of ideal operation. If we assume that vector $\vec{g}$ of the gravitational field strength can be given in the form

$$g(r) = g_x x + g_y y + g_z z, \tag{3.9}$$

where g_x, g_y and g_z are known functions of x, y and z and are time functions, then equations (3.7) can obviously be integrated. As the result of integration we find:

$$\left.\begin{aligned} v &= \int_0^t [n - m \times v + g(r)]\,dt + v(0), \\ r &= \int_0^t (v - m \times r)\,dt + r(0), \end{aligned}\right\} \tag{3.10}$$

where $\vec{r}(0)$ and $\vec{v}(0)$ are the values of the vectors $\vec{r}$ and $\vec{v}$ at t=0, i.e., the initial values of these vectors.

Equations (3.10) permit us to calculate $\vec{r}$ and $\vec{v}$ in the coordinate system O_1xyz, if we assume that vector $\vec{g}(\vec{r})$ is represented in the form of (3.9), the initial conditions of $\vec{v}(0)$ and $\vec{r}(0)$ are given and the projections of vectors $\vec{n}$ and $\vec{m}$ to axes x, y and z are known. Calculation of $\vec{r}$ in the coordinate system O_1xyz means, as follows from the first formula of (3.4), calculation of the Cartesian coordinates x, y and z of point O in the coordinate system O_1xyz.

In the considered navigational system the newtonometers and absolute angular-rate meters are located along the x, y and z axes. Vectors $\vec{n}$ and $\vec{m}$ are represented in the form of (3.6) and (3.8), and the projections of n_x, n_y, n_z, m_x, m_y and m_z, required for integration of equations (3.10), are known as time functions. This may not be said of projections g_x, g_y and g_z of vector $\vec{g}$ on axes x, y and z, because vector $\vec{g}$ is known in the general case only in the earth body axes system $O_1\xi\eta\zeta$.

In the coordinate system $O_1\xi\eta\zeta$ the earth's gravitational field is clearly determined by the representation of the power function $V(\xi,\eta,\zeta)$. Vector $\vec{g}$ of the gravitational field strength is then expressed in the coordinate system $O_1\xi\eta\zeta$ by the equality

$$\mathbf{g} = \operatorname{grad} V. \tag{3.11}$$

i.e.,

$$\mathbf{g} = \frac{\partial V}{\partial \xi}\boldsymbol{\xi} + \frac{\partial V}{\partial \eta}\boldsymbol{\eta} + \frac{\partial V}{\partial \zeta}\boldsymbol{\zeta}. \tag{3.12}$$

where $\vec{\xi}$, $\vec{\eta}$ and $\vec{\zeta}$ are unit vectors of the corresponding axes.

To find the projections g_x, g_y and g_z of vectors contained in the first equation of (3.10), the relative position of axes x, y and z and of ξ, η and ζ must be calculated from the known projections of $\partial V/\partial\xi$, $\partial V/\partial\eta$ and $\partial V/\partial\zeta$ of vector $\vec{g}$ on the earth body axes ξ, η and ζ.

It is easy to see that the relative position of axes x, y and z and of ξ, η and ζ is required to find the projections g_x, g_y and g_z only in the case of an arbitrary gravitational field. If we assume that the earth's gravitational field is spherical, then

$$V = \frac{\mu}{r}, \quad \mathbf{g} = -\frac{\mu \mathbf{r}}{r^3}. \tag{3.13}$$

where μ is the product of the earth's mass by the gravitational constant.

From the second formula of (3.13), the expressions for g_x, g_y and g_z by x, y and z follow immediately:

$$\left.\begin{array}{c} g_x = -\frac{\mu x}{r^3}, \quad g_y = -\frac{\mu y}{r^3}, \quad g_z = -\frac{\mu z}{r^3}, \\ r = (x^2 + y^2 + z^2)^{1/2}. \end{array}\right\} \tag{3.14}$$

Thus, in the case of a spherical gravitational field, formulas (3.10) together with relations (3.14) form a closed system of equations for finding x, y and z. The indicated circumstance makes it convenient for further representation of the power function of the earth's gravitational field in the form of the sum

$$V = \frac{\mu}{r} + \varepsilon(\xi, \eta, \zeta), \tag{3.15}$$

where the first term characterizes the spherical part of the earth's gravitational field, while $\varepsilon(\xi, \eta, \zeta)$ is a slight deflection of the field from a spherical shape.

Equations (3.10) with known values of g_x, g_y and g_z permit calculation, as was already noted, of the Cartesian coordinates x, y and z of the object.[6] Equations (3.10) are essential similar to equations (1.89) and they could be called equations of the ideal operation of the inertial system under consideration, if the task of the latter could be limited to finding the Cartesian coordinates x, y and z of the object in the coordinate system O_1xyz.

But trihedron xyz varies its spatial orientation arbitrarily in time, because no limiting condition of any kind has yet been applied in this relation. Therefore, a knowledge of the object's position in the coordinate system O_1xyz is inadequate for purposes of navigation. To solve navigational problems, one should either find coordinates ξ, η or ζ of the object in the earth body axes system or coordinates ξ_*, η_* or ζ_* in the fundamental Cartesian coordinate system $O_1\xi_*\eta_*\zeta_*$, whose motion with respect to the earth may be assumed known. To find coordinates ξ, η and ζ from known values of x, y and z, one must know the relative position of trihedrons xyz and $\xi\eta\zeta$ (which simultaneously solves the problem of finding g_x, g_y and g_z), and to find the coordinates ξ_*, $\eta_*$and ζ_*, one should know in turn the position of trihedrons xyz and $\xi_*\eta_*\zeta_*$ with respect to each other.

3.1.3. Determining the orientation of the platform. The second group of equations of ideal operations. Let us determine the relative position of trihedrons O_1xyz, $O_1\xi\eta\zeta$ and $O_1\xi_*\eta_*\zeta_*$. We know the relative position of these trihedrons at the initial moment of time, the angular rotational velocity $\vec{\omega}=\vec{m}$ of trihedron O_1xyz with respect to trihedron $O_1\xi_*\eta_*\zeta_*$ and the angular rotational velocity u of trihedron $O_1\xi\eta\zeta$ with respect to trihedron $O_1\xi_*\eta_*\zeta_*$. It is easy to see that the problem reduces to determining the parameters which characterize the orientation of a moving trihedron with respect to a moving object, with fixed orientation by the known projections of the absolute angular velocity on its axes. This problem leads to the well-known Poisson equations. Let us derive them.

Let us introduce the direction cosines which characterize the relative position of coordinate systems $O_1\xi_*\eta_*\zeta_*$:

(3.16)

$$\begin{array}{cccc} & x & y & z \\ \xi_* & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \eta_* & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \zeta_* & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{array}$$

Unit vectors $\vec{\xi}_*$, $\vec{\eta}_*$ and $\vec{\zeta}_*$ of axes $\xi_*\eta_*$ and ζ_* are obviously expressed by unit vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ of the x, y, and z axes in the following manner:

$$\left.\begin{aligned}\xi_* &= a_{11}x + a_{12}y + a_{13}z,\\ \eta_* &= a_{21}x + a_{22}y + a_{23}z,\\ \zeta_* &= a_{31}x + a_{32}y + a_{33}z.\end{aligned}\right\} \tag{3.17}$$

Let us differentiate the unit vectors $\vec{\xi}_*$, $\vec{\eta}_*$ and $\vec{\zeta}_*$ in the coordinate system O_1xyz. According to formula (1.14), we find:

$$\left.\begin{aligned}\frac{d\xi_*}{dt} = \dot{\xi}_* + \omega\times\xi_*,\quad \frac{d\eta_*}{dt} = \dot{\eta}_* + \omega\times\eta_*,\\ \frac{d\zeta_*}{dt} = \dot{\zeta}_* + \omega\times\zeta_*.\end{aligned}\right\} \tag{3.18}$$

But the coordinate axes $O_1\xi_*\eta_*\zeta_*$ do not change their orientation in absolute space; therefore, the absolute time derivatives of the unit vectors of these axes are equal to zero:

$$\frac{d\xi_*}{dt} = \frac{d\eta_*}{dt} = \frac{d\zeta_*}{dt} = 0.$$

By combining these equalities with those of (3.18), we come to the equations

$$\left.\begin{aligned}\dot{\xi}_* + \omega\times\xi_* = 0,\quad \dot{\eta}_* + \omega\times\eta_* = 0,\\ \dot{\zeta}_* + \omega\times\zeta_* = 0.\end{aligned}\right\} \tag{3.19}$$

By taking into account equalities (3.1), (3.4), (3.6) and (3.17), we conclude that equations (3.19) may be integrated in the coordinate system O_1xyz. As the result of integration, we find:

$$
\left.\begin{aligned}
\xi_* &= \int_0^t (\xi_* \times m)\,dt + \xi_*(0),\\
\eta_* &= \int_0^t (\eta_* \times m)\,dt + \eta_*(0),\\
\zeta_* &= \int_0^t (\zeta_* \times m)\,dt + \zeta_*(0),
\end{aligned}\right| \tag{3.20}
$$

where vectors $\vec{\xi}_*$ (0), $\vec{\eta}_*$ (0) and $\vec{\zeta}_*$ (0) characterize the relative position of the coordinate systems O_1xyz and $O_1\xi_*\eta_*\zeta_*$ at the initial moment of time.

The vector relations (3.20) are equivalent to nine scalar equations which form three groups. These equations are easily obtained by using equalities (3.6) and (3.17). They have the form:

$$
\left.\begin{aligned}
a_{11} &= \int_0^t (a_{12}m_z - a_{13}m_y)\,dt + a_{11}(0),\\
a_{12} &= \int_0^t (a_{13}m_x - a_{11}m_z)\,dt + a_{12}(0),\\
a_{13} &= \int_0^t (a_{11}m_y - a_{12}m_x)\,dt + a_{13}(0);
\end{aligned}\right| \tag{3.21}
$$

$$
\left.\begin{aligned}
a_{21} &= \int_0^t (a_{22}m_z - a_{23}m_y)\,dt + a_{21}(0),\\
a_{22} &= \int_0^t (a_{23}m_x - a_{21}m_z)\,dt + a_{22}(0),\\
a_{23} &= \int_0^t (a_{21}m_y - a_{22}m_x)\,dt + a_{23}(0);
\end{aligned}\right| \tag{3.22}
$$

$$
\left.\begin{aligned}
a_{31} &= \int_0^t (a_{32}m_z - a_{33}m_y)\,dt + a_{31}(0),\\
a_{32} &= \int_0^t (a_{33}m_x - a_{31}m_z)\,dt + a_{32}(0),\\
a_{33} &= \int_0^t (a_{31}m_y - a_{32}m_x)\,dt + a_{33}(0).
\end{aligned}\right| \tag{3.23}
$$

Each of the systems of equations (3.21), (3.22) and (3.23) are also Poisson equations known in theoretical mechanics. Equations (3.21), (3.22) and (3.23) reestablish the table of direction cosines (3.16): equations (3.21) reestablish its first line and equations (3.22) and (3.23) reestablish the second and third lines, respectively.

It is now easy to find the relations through which the Cartesian coordinates ξ_*, η_* and ζ_* are expressed by x, y and z:

$$\xi_* = \vec{\xi}_* \cdot \vec{r}, \quad \eta_* = \vec{\eta}_* \cdot \vec{r}, \quad \zeta_* = \vec{\zeta}_* \cdot \vec{r}, \tag{3.24}$$

where unit vectors $\vec{\xi}_*$, $\vec{\eta}_*$ and $\vec{\zeta}_*$ are calculated by formulas (3.17), while vector $\vec{r}$ is given by the first equality of (3.4).

The scalar equations, corresponding to relations (3.24), obviously have the form:

$$\left.\begin{aligned} \xi_* &= a_{11}x + a_{12}y + a_{13}z, \\ \eta_* &= a_{21}x + a_{22}y + a_{23}z, \\ \zeta_* &= a_{31}x + a_{32}y + a_{33}z. \end{aligned}\right\} \tag{3.25}$$

The relations reciprocal to relations (3.25) and (3.17) are obvious. We note that equations (3.19) may easily be inverted, i.e., instead of equations (3.19) for unit vectors $\vec{\xi}_*$, $\vec{\eta}_*$ and $\vec{\zeta}_*$ in the projections on axes x, y and z, the equation for unit vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ in the projections on axes ξ_*, η_* and ζ_* can be found:

$$\frac{d\vec{x}}{dt} = \vec{m} \times \vec{x}, \quad \frac{d\vec{y}}{dt} = \vec{m} \times \vec{y}, \quad \frac{d\vec{z}}{dt} = \vec{m} \times \vec{z}. \tag{3.26}$$

Equations (3.26) are obtained, if by using the principle of Galilean relativity, we assume that trihedron xyz is fixed, while trihedron $\xi_* \eta_* \zeta_*$ is assumed to rotate with respect to trihedron xyz at an angular velocity of $-\vec{\omega} = -\vec{m}$. Now using the relations inverse to relations (3.17), we can now turn to the scalar equations from vector equations (3.26). The scalar will differ from the equations in (3.21), (3.22) and (3.23) by the fact that the first and second indices of the direction cosines α_{ij} will exchange places and $-m_{\xi_*}$, $-m_{\eta_*}$ and $-m_{\zeta_*}$ will appear instead of m_x, m_y and m_z.

Let us now turn to finding the mutual disposition of trihedrons O_1xyz and $O_1\xi\eta\zeta$. Let us introduce the table of direction cosines:

$$\begin{array}{cccc} & \xi & \eta & \zeta \\ \xi_* & \alpha'_{11} & \alpha'_{12} & \alpha'_{13} \\ \eta_* & \alpha'_{21} & \alpha'_{22} & \alpha'_{23} \\ \zeta_* & \alpha'_{31} & \alpha'_{32} & \alpha'_{33}. \end{array} \tag{3.27}$$

According to table (3.27), we have:

$$\left.\begin{array}{l} \xi_* = \alpha'_{11}\xi + \alpha'_{12}\eta + \alpha'_{13}\zeta, \\ \eta_* = \alpha'_{21}\xi + \alpha'_{22}\eta + \alpha'_{23}\zeta, \\ \zeta_* = \alpha'_{31}\xi + \alpha'_{32}\eta + \alpha'_{33}\zeta. \end{array}\right| \tag{3.28}$$

By differentiating unit vectors $\vec{\xi}_*$, $\vec{\eta}_*$ and $\vec{\zeta}_*$ in the coordinate system $O_1\xi\eta\zeta$, we find similar to equations (3.19):

$$\left.\begin{array}{c} \dot{\xi}_* + u \times \xi_* = 0, \quad \dot{\eta}_* + u \times \eta_* = 0, \\ \dot{\zeta}_* + u \times \zeta_* = 0 \end{array}\right| \tag{3.29}$$

or

$$
\left.\begin{aligned}
\xi_* &= \int_0^t (\xi_* \times u)\,dt + \xi_*(0), \quad \eta_* = \int_0^t (\eta_* \times u)\,dt + \eta_*(0),\\
\zeta_* &= \int_0^t (\zeta_* \times u)\,dt + \zeta_*(0).
\end{aligned}\right| \tag{3.30}
$$

Unit vectors $\vec{\xi}_*$, $\vec{\eta}_*$ and $\vec{\zeta}_*$ in equations (3.29) and (3.30) are given by equality (3.28), while vector $\vec{u}$ should be assumed to be represented in the form

$$u = u_\xi \xi + u_\eta \eta + u_\zeta \zeta$$

Local differentiation in expressions (3.29) and integration in formulas (3.30) were carried out in the coordinate system $O_1\xi\eta\zeta$.

Equations (3.29) can be inverted in the same manner that equalities (3.19) were inverted by equations (3.26).

From vector equations (3.30), three groups of scalar equations are obtained:

$$
\left.\begin{aligned}
\alpha'_{11} &= \int_0^t (\alpha'_{12} u_\zeta - \alpha'_{13} u_\eta)\,dt + \alpha'_{11}(0),\\
\alpha'_{12} &= \int_0^t (\alpha'_{13} u_\xi - \alpha'_{11} u_\zeta)\,dt + \alpha'_{12}(0),\\
\alpha'_{13} &= \int_0^t (\alpha'_{11} u_\eta - \alpha'_{12} u_\xi)\,dt + \alpha'_{13}(0);
\end{aligned}\right| \tag{3.31}
$$

$$
\left.
\begin{aligned}
\alpha'_{21} &= \int_0^t (\alpha'_{22} u_\zeta - \alpha'_{23} u_\eta)\, dt + \alpha'_{21}(0), \\
\alpha'_{22} &= \int_0^t (\alpha'_{23} u_\xi - \alpha'_{21} u_\zeta)\, dt + \alpha'_{22}(0), \\
\alpha'_{23} &= \int_0^t (\alpha'_{21} u_\eta - \alpha'_{22} u_\xi)\, dt + \alpha'_{23}(0).
\end{aligned}
\right| \qquad (3.32)
$$

$$
\left.
\begin{aligned}
\alpha'_{31} &= \int_0^t (\alpha'_{32} u_\zeta - \alpha'_{33} u_\eta)\, dt + \alpha'_{31}(0), \\
\alpha'_{32} &= \int_0^t (\alpha'_{33} u_\xi - \alpha'_{31} u_\zeta)\, dt + \alpha'_{32}(0), \\
\alpha'_{33} &= \int_0^t (\alpha'_{31} u_\eta - \alpha'_{32} u_\xi)\, dt + \alpha'_{33}(0).
\end{aligned}
\right| \qquad (3.33)
$$

If the value of the earth rate u is assumed to be constant and the direction of u is assumed to coincide with the axis $O_1\zeta$, then

$$u_\xi = u_\eta = 0, \quad u_\zeta = u = \text{const}. \qquad (3.34)$$

It then follows from the three equations (3.31), (3.32) and (3.33) that

$$\alpha'_{13} = \alpha'_{13}(0), \quad \alpha'_{23} = \alpha'_{23}(0), \quad \alpha'_{33} = \alpha'_{33}(0). \qquad (3.35)$$

The remaining six equations fall into three systems of second-order equations of the same type:

$$\alpha'_{11} = u \int_0^t \alpha'_{12}\, dt + \alpha'_{11}(0), \quad \alpha'_{12} = -u \int_0^t \alpha'_{11}\, dt + \alpha'_{12}(0). \qquad (3.36)$$

$$\alpha'_{21} = n\int_0^t \alpha'_{22}\,dt + \alpha'_{21}(0), \quad \alpha'_{22} = -n\int_0^t \alpha'_{21}\,dt + \alpha'_{22}(0). \tag{3.37}$$

$$\alpha'_{31} = n\int_0^t \alpha'_{32}\,dt + \alpha'_{31}(0), \quad \alpha'_{32} = -n\int_0^t \alpha'_{31}\,dt + \alpha'_{32}(0). \tag{3.38}$$

The systems of equations (3.36), (3.37) and (3.38) have constant coefficients and are easily solved. It follows from them that

$$\left.\begin{aligned}
\alpha'_{11} &= \alpha'_{11}(0)\cos nt + \alpha'_{12}(0)\sin nt,\\
\alpha'_{12} &= -\alpha'_{11}(0)\sin nt + \alpha'_{12}(0)\cos nt,\\
\alpha'_{21} &= \alpha'_{21}(0)\cos nt + \alpha'_{22}(0)\sin nt,\\
\alpha'_{22} &= -\alpha'_{21}(0)\sin nt + \alpha'_{22}(0)\cos nt,\\
\alpha'_{31} &= \alpha'_{31}(0)\cos nt + \alpha'_{32}(0)\sin nt,\\
\alpha'_{32} &= -\alpha'_{31}(0)\sin nt + \alpha'_{32}(0)\cos nt.
\end{aligned}\right\} \tag{3.39}$$

If we assume that axis ζ coincides with axis ζ_*, and axis ξ_* is directed toward the point of the vernal equinox, then

$$\alpha'_{13} = \alpha'_{23} = \alpha'_{32} = \alpha'_{31} = 0, \quad \alpha'_{33} = 1.$$

In this case the table of direction cosines (3.27) determines the relationship between the Cartesian coordinates ξ_*, η_* and ζ_* in the first equatorial coordinate system and the Cartesian coordinates ξ, η and ζ in the coordinate system $O_1\xi\eta\zeta$.

Of course, relations (3.35) and (3.39) may also be obtained directly from geometric concepts. However, it will be more convenient for us in the future to use more general equations (3.31), (3.32) and (3.33), rather than relations (3.35) and (3.39). One of the reasons for this lies in the fact that equations (3.31), (3.32) and (3.33), being equivalent to relations (3.35) and (3.39) with regard to premise (3.34), generally do not compel us to use this premise.

By knowing $\alpha_{ij}(t)$ and $\alpha'_{ij}(t)$, i.e., by knowing the relative position of coordinate systems ξ_*, $\eta_*\zeta_*$ and $\xi\eta\zeta$ and xyz, we can obviously immediately find the parameters which determine the relative position of trihedrons xyz and $\xi\eta\zeta$. In fact, let the direction cosines between the axes of these trihedrons form the table:

$$\begin{array}{cccc} & x & y & z \\ \xi & \beta_{11} & \beta_{12} & \beta_{13} \\ \eta & \beta_{21} & \beta_{22} & \beta_{23} \\ \zeta & \beta_{31} & \beta_{32} & \beta_{33}. \end{array} \tag{3.40}$$

It then follows from tables (3.16) and (3.27) that

$$\beta_{ij} = \sum_{k=1}^{3} \alpha_{ki}\alpha'_{kj}, \quad i = 1, 2, 3, \quad j = 1, 2, 3. \tag{3.41}$$

Along with expressions (3.41), the direction cosines β_{ij} may also be calculated by means of equations similar to (3.21), (3.22), (3.23) or (3.31), (3.32) and (3.33). According to table (3.40), we have:

$$
\left.\begin{aligned}
\xi &= \beta_{11}x + \beta_{12}y + \beta_{13}z,\\
\eta &= \beta_{21}x + \beta_{22}y + \beta_{23}z,\\
\zeta &= \beta_{31}x + \beta_{32}y + \beta_{33}z.
\end{aligned}\right\} \tag{3.42}
$$

Trihedron xyz rotates with respect to trihedron $\xi\eta\zeta$ at an angular velocity of

$$
\omega - u = m - u. \tag{3.43}
$$

By assuming that trihedron $\xi\eta\zeta$ is fixed and by differentiating the unit vectors $\vec{\xi}$, $\vec{\eta}$ and $\vec{\zeta}$ of its axes in the coordinate system xyz, we find:

$$
\begin{gathered}
\dot{\xi} + (m - u) \times \xi = 0, \quad \dot{\eta} + (m - u) \times \eta = 0, \\
\dot{\zeta} + (m - u) \times \zeta = 0.
\end{gathered} \tag{3.44}
$$

By integrating equations (3.44) in the coordinates system xyz, we find:

$$
\left.\begin{aligned}
\xi &= \int_0^t \xi \times (m - u)\,dt + \xi(0),\\
\eta &= \int_0^t \eta \times (m - u)\,dt + \eta(0),\\
\zeta &= \int_0^t \zeta \times (m - u)\,dt + \zeta(0).
\end{aligned}\right| \tag{3.45}
$$

hence, similar to equations (3.21), (3.22) and (3.23), we have the following scalar equations:

$$\left.\begin{aligned}
\beta_{11} &= \int_0^t [\beta_{12}(m_z - u_z) - \beta_{13}(m_y - u_y)]\, dt + \beta_{11}(0),\\
\beta_{12} &= \int_0^t [\beta_{13}(m_x - u_x) - \beta_{11}(m_z - u_z)]\, dt + \beta_{12}(0),\\
\beta_{13} &= \int_0^t [\beta_{11}(m_y - u_y) - \beta_{12}(m_x - u_x)]\, dt + \beta_{13}(0);
\end{aligned}\right| \tag{3.46}$$

$$\left.\begin{aligned}
\beta_{21} &= \int_0^t [\beta_{22}(m_z - u_z) - \beta_{23}(m_y - u_y)]\, dt + \beta_{21}(0),\\
\beta_{22} &= \int_0^t [\beta_{23}(m_x - u_x) - \beta_{21}(m_z - u_z)]\, dt + \beta_{22}(0),\\
\beta_{23} &= \int_0^t [\beta_{21}(m_y - u_y) - \beta_{22}(m_x - u_x)]\, dt + \beta_{23}(0);
\end{aligned}\right| \tag{3.47}$$

$$\left.\begin{aligned}
\beta_{31} &= \int_0^t [\beta_{32}(m_z - u_z) - \beta_{33}(m_y - u_y)]\, dt + \beta_{31}(0),\\
\beta_{32} &= \int_0^t [\beta_{33}(m_x - u_x) - \beta_{31}(m_z - u_z)]\, dt + \beta_{32}(0),\\
\beta_{33} &= \int_0^t [\beta_{31}(m_y - u_y) - \beta_{32}(m_x - u_x)]\, dt + \beta_{33}(0).
\end{aligned}\right| \tag{3.48}$$

According to table (3.40), in these equations

$$\left.\begin{aligned}
u_x &= u_\xi \beta_{11} + u_\eta \beta_{21} + u_\zeta \beta_{31},\\
u_y &= u_\xi \beta_{12} + u_\eta \beta_{22} + u_\zeta \beta_{32},\\
u_z &= u_\xi \beta_{13} + u_\eta \beta_{23} + u_\zeta \beta_{33}
\end{aligned}\right\} \tag{3.49}$$

Now, when the direction cosines $\beta_{ij}(t)$ are found which characterize the relative position of trihedrons xyz and $\xi\eta\zeta$ and which permit calculation of the Cartesian coordinates ξ, η and ζ from the known coordinates x, y and z according to tables (3.40), we can go on to calculating the projections g_x, g_y and g_z of vector $\vec{g}$, contained in the first equation of (3.10).

It follows from formulas (3.14) and (3.15) that:

$$\left.\begin{aligned} g_x &= -\frac{\mu x}{r^3} + \mathrm{grad}_x\,\varepsilon, \quad g_y = -\frac{\mu y}{r^3} + \mathrm{grad}_y\,\varepsilon, \\ g_z &= -\frac{\mu z}{r^3} + \mathrm{grad}_z\,\varepsilon. \end{aligned}\right\} \tag{3.50}$$

The projections of grad ε on axes x, y and z are equal to:

$$\left.\begin{aligned} \mathrm{grad}_x\,\varepsilon &= \frac{\partial\varepsilon}{\partial\xi}\beta_{11} + \frac{\partial\varepsilon}{\partial\eta}\beta_{21} + \frac{\partial\varepsilon}{\partial\zeta}\beta_{31}, \\ \mathrm{grad}_y\,\varepsilon &= \frac{\partial\varepsilon}{\partial\xi}\beta_{12} + \frac{\partial\varepsilon}{\partial\eta}\beta_{22} + \frac{\partial\varepsilon}{\partial\zeta}\beta_{32}, \\ \mathrm{grad}_z\,\varepsilon &= \frac{\partial\varepsilon}{\partial\xi}\beta_{13} + \frac{\partial\varepsilon}{\partial\eta}\beta_{23} + \frac{\partial\varepsilon}{\partial\zeta}\beta_{33}. \end{aligned}\right\} \tag{3.51}$$

The factors $\partial\varepsilon/\partial\xi$, $\partial\varepsilon/\partial\eta$ and $\partial\varepsilon/\partial\zeta$, contained in equalities (3.51), are functions of coordinates ξ, η and ζ. The integrand of the first equation of (3.10) should contain only time functions. Therefore, coordinates ξ, η and ζ in the arguments of the derivatives should be expressed by x, y and z, i.e., instead of ξ, η and ζ the following expressions should be substituted

$$\left.\begin{aligned} \xi &= \beta_{11}x + \beta_{12}y + \beta_{13}z, \\ \eta &= \beta_{21}x + \beta_{22}y + \beta_{23}z, \\ \zeta &= \beta_{31}x + \beta_{32}y + \beta_{33}z. \end{aligned}\right\} \tag{3.52}$$

3.1.4. The complete system of equations of ideal operation. By combining equalities (3.10), (3.11), (3.15), (3.20), (3.24), (3.45) and (3.52), we find the complete system of equations of the ideal operation of the considered diagram in vector form:

$$\left.\begin{aligned} v &= \int_0^t (n - m \times v + g)\,dt + v(0),\\ r &= \int_0^t (v - m \times r)\,dt + r(0); \end{aligned}\right| \tag{3.53}$$

$$\left.\begin{aligned} \xi_0 &= \int_0^t (\xi_0 \times m)\,dt + \xi_0(0),\\ \eta_0 &= \int_0^t (\eta_0 \times m)\,dt + \eta_0(0),\\ \zeta_0 &= \int_0^t (\zeta_0 \times m)\,dt + \zeta_0(0); \end{aligned}\right| \tag{3.54}$$

$$\left.\begin{aligned} \xi &= \int_0^t [\xi \times (m - u)]\,dt + \xi(0),\\ \eta &= \int_0^t [\eta \times (m - u)]\,dt + \eta(0),\\ \zeta &= \int_0^t [\zeta \times (m - u)]\,dt + \zeta(0); \end{aligned}\right| \tag{3.55}$$

$$g = -\frac{\mu r}{r^3} + \operatorname{grad} \varepsilon(\xi, \eta, \zeta); \tag{3.56}$$

$$\xi_0 = r \cdot \xi_0, \quad \eta_0 = r \cdot \eta_0, \quad \zeta_0 = r \cdot \zeta_0; \tag{3.57}$$

$$\xi = r \cdot \xi, \quad \eta = r \cdot \eta, \quad \zeta = r \cdot \zeta. \tag{3.58}$$

All the vectors in this system of equations are calculated in coordinate system O_1xyz. Integration in equations (3.53), (3.54) and (3.55) is also carried out in this coordinate system. Equations (3.53)-(3.58) are a closed system of equations, which, according to the values of m and n, obtained as the result of measurements, according to initial conditions $r(0)$, $v(0)$, $\xi_*(0)$, $\eta_*(0)$, $\zeta_*(0)$, $\xi(0)$, $\eta(0)$, and $\zeta(0)$ and according to given values of ν, ε and u, permit us to find simultaneously the Cartesian coordinates of the object: x, y and z in moving trihedron O_1xyz; ξ, η and ζ in trihedron $O_1\xi\eta\zeta$, bound to the earth, and ξ_*, η_* and ζ_* in the fundamental Cartesian coordinate system.

In completing the derivation of equations of ideal operation, we turn from vector equations (3.53)-(3.58) to scalar equations. Taking into account the first equality of (3.4), equalities (3.6) and (3.8), relations (3.21)-(3.23), (3.25) and (3.46)-(3.48) and formulas (3.49)-(3.52), instead of the vector equations (3.53)-(3.58), we find the following scalar equations:

$$\left.\begin{aligned}
v_x &= \int_0^t [n_x - (m_y v_z - m_z v_y) + g_x]\, dt + v_x(0), \\
v_y &= \int_0^t [n_y - (m_z v_x - m_x v_z) + g_y]\, dt + v_y(0), \\
v_z &= \int_0^t [n_z - (m_x v_y - m_y v_x) + g_z]\, dt + v_z(0), \\
x &= \int_0^t [v_x - (m_y z - m_z y)]\, dt + x(0), \\
y &= \int_0^t [v_y - (m_z x - m_x z)]\, dt + y(0), \\
z &= \int_0^t [v_z - (m_x y - m_y x)]\, dt + z(0);
\end{aligned}\right| \tag{3.59}$$

$$\left.\begin{aligned}
\alpha_{11} &= \int_0^t (\alpha_{12} m_z - \alpha_{13} m_y)\, dt + \alpha_{11}(0),\\
\alpha_{12} &= \int_0^t (\alpha_{13} m_x - \alpha_{11} m_z)\, dt + \alpha_{12}(0),\\
\alpha_{13} &= \int_0^t (\alpha_{11} m_y - \alpha_{12} m_x)\, dt + \alpha_{13}(0),\\
\alpha_{21} &= \int_0^t (\alpha_{22} m_z - \alpha_{23} m_y)\, dt + \alpha_{21}(0),\\
\alpha_{22} &= \int_0^t (\alpha_{23} m_x - \alpha_{21} m_z)\, dt + \alpha_{22}(0),\\
\alpha_{23} &= \int_0^t (\alpha_{21} m_y - \alpha_{22} m_x)\, dt + \alpha_{23}(0),\\
\alpha_{31} &= \int_0^t (\alpha_{32} m_z - \alpha_{33} m_y)\, dt + \alpha_{31}(0),\\
\alpha_{32} &= \int_0^t (\alpha_{33} m_x - \alpha_{31} m_z)\, dt + \alpha_{32}(0),\\
\alpha_{33} &= \int_0^t (\alpha_{31} m_y - \alpha_{32} m_x)\, dt + \alpha_{33}(0);
\end{aligned}\right| \tag{3.60}$$

$$\left.\begin{aligned}
\beta_{11} &= \int_0^t [\beta_{12}(m_z - u_z) - \beta_{13}(m_y - u_y)]\, dt + \beta_{11}(0),\\
\beta_{12} &= \int_0^t [\beta_{13}(m_x - u_x) - \beta_{11}(m_z - u_z)]\, dt + \beta_{12}(0),\\
\beta_{13} &= \int_0^t [\beta_{11}(m_y - u_y) - \beta_{12}(m_x - u_x)]\, dt + \beta_{13}(0),\\
\beta_{21} &= \int_0^t [\beta_{22}(m_z - u_z) - \beta_{23}(m_y - u_y)]\, dt + \beta_{21}(0),\\
\beta_{22} &= \int_0^t [\beta_{23}(m_x - u_x) - \beta_{21}(m_z - u_z)]\, dt + \beta_{22}(0),\\
\beta_{23} &= \int_0^t [\beta_{21}(m_y - u_y) - \beta_{22}(m_x - u_x)]\, dt + \beta_{23}(0),\\
\beta_{31} &= \int_0^t [\beta_{32}(m_z - u_z) - \beta_{33}(m_y - u_y)]\, dt + \beta_{31}(0),\\
\beta_{32} &= \int_0^t [\beta_{33}(m_x - u_x) - \beta_{31}(m_z - u_z)]\, dt + \beta_{32}(0),\\
\beta_{33} &= \int_0^t [\beta_{31}(m_y - u_y) - \beta_{32}(m_x - u_x)]\, dt + \beta_{33}(0).
\end{aligned}\right| \tag{3.61}$$

$$\left.\begin{aligned}\xi_0 &= \alpha_{11}x + \alpha_{12}y + \alpha_{13}z,\\ \eta_0 &= \alpha_{21}x + \alpha_{22}y + \alpha_{23}z,\\ \zeta_0 &= \alpha_{31}x + \alpha_{32}y + \alpha_{33}z;\end{aligned}\right\} \qquad (3.62)$$

$$\left.\begin{aligned}\xi &= \beta_{11}x + \beta_{12}y + \beta_{13}z,\\ \eta &= \beta_{21}x + \beta_{22}y + \beta_{23}z,\\ \zeta &= \beta_{31}x + \beta_{32}y + \beta_{33}z;\end{aligned}\right\} \qquad (3.63)$$

$$\left.\begin{aligned}u_x &= u_\xi\beta_{11} + u_\eta\beta_{21} + u_\zeta\beta_{31},\\ u_y &= u_\xi\beta_{12} + u_\eta\beta_{22} + u_\zeta\beta_{32},\\ u_z &= u_\xi\beta_{13} + u_\eta\beta_{23} + u_\zeta\beta_{33};\end{aligned}\right\} \qquad (3.64)$$

$$\left.\begin{aligned}g_x &= -\frac{\mu x}{r^3} + \frac{\partial\varepsilon}{\partial\xi}\beta_{11} + \frac{\partial\varepsilon}{\partial\eta}\beta_{21} + \frac{\partial\varepsilon}{\partial\zeta}\beta_{31},\\ g_y &= -\frac{\mu y}{r^3} + \frac{\partial\varepsilon}{\partial\xi}\beta_{12} + \frac{\partial\varepsilon}{\partial\eta}\beta_{22} + \frac{\partial\varepsilon}{\partial\zeta}\beta_{32},\\ g_z &= -\frac{\mu z}{r^3} + \frac{\partial\varepsilon}{\partial\xi}\beta_{13} + \frac{\partial\varepsilon}{\partial\eta}\beta_{23} + \frac{\partial\varepsilon}{\partial\zeta}\beta_{33},\\ \varepsilon &= \varepsilon(\xi, \eta, \zeta), \quad r = (x^2 + y^2 + z^2)^{1/2}.\end{aligned}\right\} \qquad (3.65)$$

If we use premise (3.34), then the direction cosines β_{ij} in relations (3.63)-(3.65) may be substituted for their expressions by α_{ij} and α'_{ij} according to equalities (3.35), (3.39) and (3.41). Relations (3.61) are superfluous in this case.

The equations similar to (3.62) and (3.63) may be joined to the derived equations to calculate v_{ξ_*}, v_{η_*}, v_{ζ_*} and v_ξ, v_η and v_ζ from the known values of v_x, v_y and v_z.

Equations (3.59)-(3.65), equivalent to vector equations (3.53)-(3.58), also permit calculation of Cartesian coordinates ξ_*, η_* and ζ_* and ξ, η and ζ along with coordinates x, y and z.

By knowing the Cartesian coordinates ξ_*, η_* and ζ_* or ξ, η and ζ, we can generally find the curvilinear and moving coordinates of the object in coordinates systems $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$ or in any other coordinate system moving in a known manner with respect to system $O_1\xi_*\eta_*\zeta_*$ or $O_1\xi\eta\zeta$ by using the corresponding calculations. To do this, it is necessary only that the relations which link the Cartesian coordinates ξ_*, η_* and ζ_* or ξ, η and ζ to the curvilinear coordinates being introduced be given. Obviously, time may be contained in this relationship in an explicit manner.

Let us turn to the second problem which should be solved by the considered inertial system, i.e., let us turn to determining the orientation of the object in the coordinate system $O_1\xi_*\eta_*\zeta_*$. To solve this problem, it is sufficient to determine the orientation of trihedron OXYZ, rigidly bound to the object, in this coordinate system. The position of the axes of trihedron OXYZ with respect to trihedron OXYZ is completely characterized by angles α, β and γ of the revolutions of the gimbal rings of the inertial system platform with the object. These angles can be measured. The following values of direction cosines between axes x, y and z and axes X, Y, Z(x_*, y_*, z_*) are easily found from tables (1.50) by multiplying out the three matrices included in these tables:

(3.66)

	x	y	z
X	$\cos\beta\cos\gamma$	$-\cos\beta\sin\gamma$	$\sin\beta$
Y	$\sin\alpha\sin\beta\cos\gamma + {}$ $+\cos\gamma\sin\gamma$	$-\sin\alpha\sin\beta\sin\gamma + {}$ $+\cos\alpha\cos\gamma$	$-\sin\alpha\cos\beta$
Z	$-\cos\alpha\sin\beta\cos\gamma + {}$ $+\sin\alpha\sin\gamma$	$-\sin\alpha\sin\beta\sin\gamma + {}$ $+\sin\alpha\cos\gamma$	$\cos\alpha\cos\beta.$

These direction cosines together with the table of direction cosines (3.16) obtained from equations (3.21), (3.22) and (3.23) obviously give the direction cosines between axes ξ_*, η_* and ζ_* and X, Y and Z, which also determine the orientation of the object with respect to the coordinate system $O_1\xi_*\eta_*\zeta_*$.

By using table (3.27), we can easily find the orientation of the object with respect to the earth body axis system $O_1\xi\eta\zeta$.

If we measured the derivatives of angles α, β and γ with respect to time, i.e., the values of $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$, we can also find the projections of the absolute angular velocity of the object on the axes bound to it. Actually, by noting that the relative angular velocities of $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ are directed along the axes of the gimbal mount, i.e., along axes x", y" and z", and by turning to tables of direction cosines (1.50) and (3.66), we find:

(3.67)

$$
\begin{aligned}
\omega_x &= m_x\cos\beta\cos\gamma - m_y\cos\beta\sin\gamma + m_z\sin\beta + \dot{\alpha}, \\
\omega_y &= m_x(\sin\alpha\sin\beta\cos\gamma + \cos\alpha\sin\gamma) + \\
&\quad + m_y(-\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma) - \\
&\quad - (m_z + \dot{\gamma})\sin\alpha\cos\beta + \dot{\beta}\cos\alpha, \\
\omega_z &= m_x(-\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma) + \\
&\quad + m_y(\cos\alpha\sin\beta\sin\gamma + \sin\alpha\cos\gamma) + \\
&\quad + (m_z + \dot{\gamma})\cos\alpha\cos\beta + \dot{\beta}\sin\alpha.
\end{aligned}
$$

The equations (3.59) of ideal operation derived above were based on the fundamental equation of inertial navigation, taken in the form of (1.88). Equation (1.88) differs from the exact equation (1.86) by the fact that the difference

$$\Delta F_1 = F_1(0) - F_1(r) \tag{3.68}$$

of the attractive forces of celestial bodies (except the earth), determined by expression (1.87), at point O_1 (the center of the earth) and at point O (the location of the sensitive masses of the newtonometers) is discarded.

It is also essentially not possible to introduce this simplification. Let us show how the equations of ideal operation of type (3.59) can be constructed according to the exact equation of motion (1.86) of the sensitive masses of the newtonometers.

Having assumed for simplicity that the gravitational fields k of the celestial bodies being taken into account are spherical, according to equality (1.93), we find:

$$\left.\begin{aligned}
\Delta F_{1x} &= \sum_{i=1}^{k} \mu_i \left(\frac{x_i}{r_i^3} - \frac{x - x_i}{|r - r_i|^3} \right), \\
\Delta F_{1y} &= \sum_{i=1}^{k} \mu_i \left(\frac{y_i}{r_i^3} - \frac{y - y_i}{|r - r_i|^3} \right), \\
\Delta F_{1z} &= \sum_{i=1}^{k} \mu_i \left(\frac{z_i}{r_i^3} - \frac{z - z_i}{|r - r_i|^3} \right).
\end{aligned}\right\} \tag{3.69}$$

where, by analogy to equation (3.17) and (3.25),

$$\left.\begin{aligned}
x_i &= a_{11}\xi_{0i} + a_{21}\eta_{0i} + a_{31}\zeta_{0i}, \\
y_i &= a_{12}\xi_{0i} + a_{22}\eta_{0i} + a_{32}\zeta_{0i}, \\
z_i &= a_{13}\xi_{0i} + a_{23}\eta_{0i} + a_{33}\zeta_{0i}
\end{aligned}\right\} \tag{3.70}$$

The values of $\xi_{*i}^{i}(t)$, $\eta_{*i}^{i}(t)$ and $\zeta_{*i}^{i}(t)$ in formulas (3.70) are coordinates of the i-th celestial body in the coordinate system $O_1\xi_*\eta_*\zeta_*$. These coordinates should be known time functions.

Thus, to find the equations of ideal operation, which correspond to exact equation (1.86), ΔF_{1x}, ΔF_{1y} and ΔF_{1z} and the given formulas (3.69), respectively, should be added to the integrands of the first three equations of (3.59) and, moreover, relations (3.70) should be included in the system of equations of ideal operation. Consideration of the asphericity of the gravitational fields of the celestial bodies is for the time being only of strictly theoretical interest, although it may be performed. Except for complicating the relations obtained in this manner, this consideration does not cause any essential difficulties.

It is easy to discern that the constructed system of equations of ideal operation (3.59) is not the only one possible. It turns out that several systems of integral equations, essentially equivalent but differing in form, which may be equations of ideal operation, can be constructed without altering the functional diagram of the device described above. Let us indicate the main variants.

By using the solution of the second group of equations of ideal operation, i.e., equations (3.21), (3.22) and (3.23), independent of equations (3.59), the newtonometer readings could be projected on axes "ξ_*,η_*" and ζ_* and vector $\vec{n}$ in projections on these axes could be obtained and double integration of the fundamental equation could be carried out in the coordinate system $O_1\xi_*\eta_*\zeta_*$. This method is one of the most difficult to realize, because calculating operations with the newtonometer readings must be performed until integration of them.

Double integration in equations (3.59) is carried out in the same coordinate system as that which measured the components of the absolute angular velocities and newtonometer readings, i.e., in coordinate system O_1xyz. Some variation of equations (3.59) is also possible. Having turned to vector equations (3.53), from which were found the scalar equations (3.59), we note that the two equations (3.53) can be combined into a single equation:

$$r = \int_0^t \int_0^t [n - \omega \times (\dot{r} + \omega \times r) + g]\,dt\,dt + [\dot{r}(0) + \omega(0) \times r(0)]\,t + r(0). \tag{3.71}$$

This variant is interesting in that coordinates x, y and z are obtained by double integration and, consequently, double integrating devices can be used here. However, to find the velocity $\dot{\vec{r}}$, which is contained in the integrand (3.71), we must differentiate the derived coordinates x, y and z. Moreover, along with the coordinates the velocity of the object may be a necessary navigational parameter and the derivatives of coordinates $\dot{x}$, $\dot{y}$ and $\dot{z}$ may also be necessary to calculate it.

This variant of constructing the equations of ideal operation is also possible. First integration is carried out along the axes x, y and z, i.e., the first three equations of (3.59) remain unchanged. The projections v_x, v_y and v_z of the absolute velocity of the object are recalculated to other directions, for example, by using direction cosines (3.21), (3.22) and (3.23) to the direction of axes ξ_*, η_* and ζ_*, and the second integration is accomplished along the coordinate axes $O_1\xi_*\eta_*\zeta_*$. This variant usually does not yield any advantage in the number of calculating operations, and it is always more difficult to perform calculations with derivatives of the coordinates than with the coordinates themselves, because the former are more rapidly variable time functions than the latter.

The second group of equations of ideal operation (3.60) may also be represented in other forms. Instead of direction cosines, we can obviously take any other parameters which determine the orientation of trihedron O_1xyz with respect to trihedron $O_1\xi_*\eta_*\zeta_*$. For example, these parameters may be Euler angles or equivalent angles or Rodrigues-Hamilton or Cayley- Klein parameters.[5] Construction of the integral equations by which the value of these parameters can be found from known values of m_x, m_y and m_z presents no difficulty, and we will not dwell on this. The more so since equations (3.60) are more convenient than the others when working in Cartesian coordinate systems and since they have a very useful symmetry which facilitates their use as equations of the ideal operation of an inertial system.

The foregoing is also applicable to equations (3.61). It was pointed out earlier that equations (3.61) can be substituted for equations (3.31), (3.32) and (3.33) and relations (3.41), and if the assumption (3.34) on the constancy of the value and direction of the vector of the earth rate is taken, then equations (3.61) can be substituted for relations (3.35), (3.39) and (3.41). In the latter case formulas (3.64) fall out of the equations of ideal operation.

Attention should also be given to one characteristic feature of equations (3.59)-(3.65). Equations (3.60) are a closed system which can be solved separately from the remaining equations. Equations (3.61) and relations (3.64) taken together also form a closed system which can be solved independently. Equations (3.59) form a closed system with relations (3.63) and (3.65), solution of which at each time interval is possible only after solving equations (3.61) and (3.64). Solution of equations (3.60) is not required for this. Finally, coordinates ξ_*, η_* and ζ_* are calculated by formulas (3.62) only after solving equations (3.59) and (3.60).

The indicated relationships of the equations and, consequently, the required sequence of their solution are caused by the fact that the earth's gravitational field is given in coordinates ξ, η and ζ. If we assume that the earth's gravitational field is spherical, i.e., if we assume that $\varepsilon=0$, then equations (3.59) with formulas (3.65) also form a group of equations separate from the remaining ones. Thus, three groups of equations split off from the system of equations (3.59)-(3.65): the first group comprises equations (3.59) and (3.65); the second group comprises equations (3.60) and the third group comprises equations (3.61) and (3.64). These three groups of equations are solved independently. After solution of them, coordinates ξ_*, η_* and ζ_* and ξ, η and ζ are found from formulas (3.62) and (3.63).

3.1.5. Special case: fixed orientation of the platform in space and orientation of one of its axes along the direction toward the center of the earth. When deriving the first group of integral equations of ideal operation (3.56) or (3.59) of the inertial navigation system being considered, it was assumed that the platform of a three-component absolute angular-rate meter (coordinate system Oxyz) was oriented arbitrarily both with respect to inertial space and with respect to the object. Various special cases are possible here.

If the platform is invariant relative to the inertial coordinate system, for example, if the direction of axes x, y and z and ξ_*, η_* and ζ_* are combined, then equations (3.59) are transformed, as one can easily discern, into the previously derived equations (1.89). Equations (3.60) and (3.62) then drop out. The orientation of the object is obviously defined immediately by the table of direction cosines (3.66), and m_x, m_y and m_z in expressions (3.67) for projections of the absolute angular velocity the object on its axes should be assumed equal to zero n formulas (3.61). The corresponding orientation of the platm may be realized in this case by using a free gyrostabilized platform or a system of free gyroscopes.

The platform of the angular-rate meter can be rigidly bound to the body of the object, for example, by combining the coordinate systems xyz and XYZ. In this case the gimbal **mount of** the platform on to the object is not required. The equations of ideal operations will be equations (3.59)-(3.65). Relations (3.67) drop out, because the orientation of the object coincides with the orientation of the platform and is given by equations (3.60) and (3.61), while the projections of the absolute angular velocity of the object on its axes are directly the readings m_x, m_y and m_z of the gyroscopic angular-rate meter.

A case intermediate between the two preceding ones is possible, where the orientation of the platform in the inertial coordinate system $O_1\xi_*\eta_*\zeta_*$ will be a known time function, and also a function of the specific navigational coordinate system and the rate of their variation in time. This orientation of the platform can be provided only by using a controlled gyrostabilized platform considered in § 1.3, or by using a special functional circuit, which is mounted on a free stabilized platform and which gives the position of trihedron xyz, along whose axes the newtonometers are mounted, relative to the stabilized platform. The stabilized platform, as in the preceding case, can be naturally replaced by a system of free gyroscopes.

An example where orientation of a controlled gyrostabilized platform is a given time function may be orientation of it in which the axes of the platform retain their directions relative to the earth. Without loss of generality, we may assume that these directions are the directions of the coordinate axes $O_1\xi\eta\zeta$. Then the controlled gyroplatform should rotate relative to the inertial coordinate system such that the position of the platform is characterized at each moment of time by the direction cosines given in table (3.27).

Let us form the expressions of controlling moments M^4_{1x}, M^5_{1y} and M^6_{1x}, required for this case, by using relations (1.78), which in the considered case assume the form:

$$\omega_\xi = -\frac{M^5_{1y}}{H}, \quad \omega_\eta = \frac{M^4_{1x}}{H}, \quad \omega_\zeta = \frac{M^6_{1x}}{H}. \tag{3.72}$$

Relations (1.77), which take into account the finiteness of the values of δ_1, δ_2 and δ_3, may also be used of course, by first inverting these relations, i.e., by solving them with respect to M^4_{1x}, M^5_{1y} and M^6_{1x}. The latter does not cause any essential difficulties; therefore, we shall limit ourselves to more simple equalities (1.78).

Having taken into account that the axis $O_1\zeta$ is directed along vector $\vec{u}$ of the earth rate about its own axis and, consequently,

$$\omega_\xi = \omega_\eta = 0, \quad \omega_\zeta = u. \tag{3.73}$$

we find:

$$M_{1x}^{*} = 0, \quad M_{1y}^{*} = 0, \quad M_{1z}^{*} = uH. \tag{3.74}$$

Of course, besides fulfilling conditions (3.73), the initial conditions should be observed, namely, the initial position of the coordinate systems $O_1\xi\eta\zeta$ and xyz should correspond to table (3.27) of direction cosines $\alpha_{ij}(0)$.

Having turned to equations (3.60), we note that m_x, m_y and m_z are now known time functions in them:

$$m_x = m_y = 0, \quad m_z = u. \tag{3.75}$$

It is easy to see that integration of equations (3.59) in the considered case immediately yields coordinates ξ, η and ζ. Equations (3.60), (3.61), (3.63) and (3.64) drop out, because even if calculation of coordinates ξ_*, η_* and ζ_* along with ξ, η and ζ is also required, they are obtained algebraically from the coordinates ξ, η and ζ by using expressions (3.35), (3.39) and (3.62).

Orientation of the object with respect to the earth is determined by angles α, β and γ of the rotations of the gimbal rings of the platform, i.e., by the table of direction cosines (3.66). Table (3.27) should also be used to find the parameters of orientation relative to the inertial coordinate system.

If orientation of the coordinate axes $Oxyz$ is accomplished by a special functional diagram located on a free gyrostabilized platform rather than by a controlled gyrostabilized platform, this functional diagram should continuously provide disposition of the coordinate systems $O\xi_{*}\eta_{*}\zeta_{*}$ (stabilized platform) and $Oxyz$ (a trihedron along whose axes the newtonometers are installed) so that the direction cosines between the axes of the mentioned coordinate systems corresponded to table (3.27). In this case angles α, β and γ of the rotations of the gimbal rings of the stabilized platform determine the orientation of the object relative to the coordinate system $O_1\xi\eta\zeta$ can be calculated by direction to the earth body axis system $O_1\xi\eta\zeta$ can be calculated by direction cosines (3.27) and (3.66).

Let us now consider a case where the orientation of the controlled gyrostabilized platform is dependent on the coordinates calculated by the inertial system. Let us require, for example, that the z axis of the platform is constantly directed along the radius vector $\vec{r}$.

If the z axis of the platform coincides with $\vec{r}$, then

$$r = rz, \quad x = y = 0 \tag{3.76}$$

and

$$v = \dot{r}z + (\omega_y x - \omega_x y) \tag{3.77}$$

From equalities (3.76) and (3.77), we find:

$$r\omega_y = v_x, \quad r\omega_x = -v_y, \quad v_z = \dot{r}. \tag{3.78}$$

Turning to relations (1.78), we find the following expressions for the controlling moments M^1_{1x} and M^5_{1y}:

$$M^1_{1x} = \frac{Hv_x}{r}, \quad M^5_{1y} = \frac{Hv_y}{r}. \tag{3.79}$$

Moment M^6_{1x}, like ω_z, remains arbitrary, because condition (3.76) permits this arbitrariness. The value of this moment may therefore be ordered to simplif·· the equations of ideal operation. For example, we can assume

$$M^6_{1z} = 0, \tag{3.80}$$

which is obviously equivalent to the condition

$$\omega_z = 0. \tag{3.81}$$

The diagrams obtained in this case, in which the projection of the absolute angular rotational velocity of the platform to direction $\vec{r}$ is equal to zero, are sometimes called "azimuth-free" diagrams.[6]

From the condition of (3.76), equation (3.59) assume the form:

$$\left.\begin{aligned}
v_x &= \int_0^t \left(n_x - \frac{v_x \dot{r}}{r} + m_z v_y + g_x\right) dt + v_x(0),\\
v_y &= \int_0^t \left(n_y - m_z v_x - \frac{v_y \dot{r}}{r} + g_y\right) dt + v_y(0),\\
\dot{r} &= \int_0^t \left(n_z + \frac{v_x^2 + v_y^2}{r} + g_z\right) dt + \dot{r}(0),\\
x &= y = 0,\\
r &= \int_0^t \dot{r}\, dt + r(0).
\end{aligned}\right| \tag{3.82}$$

Equations (3.60) can be written in the following manner:

$$\left.\begin{aligned}
a_{11} &= \int_0^t \left(a_{12} m_z - a_{13} \frac{v_x}{r}\right) dt + a_{11}(0),\\
a_{12} &= \int_0^t \left(-a_{13} \frac{v_y}{r} - a_{11} m_z\right) dt + a_{12}(0),\\
a_{13} &= \int_0^t \frac{a_{11} v_x + a_{12} v_y}{r}\, dt + a_{13}(0);
\end{aligned}\right| \tag{3.83}$$

$$\left.\begin{aligned}
a_{21} &= \int_0^t \left(a_{22} m_z - a_{23} \frac{v_x}{r}\right) dt + a_{21}(0),\\
a_{22} &= \int_0^t \left(-a_{23} \frac{v_y}{r} - a_{21} m_z\right) dt + a_{22}(0),\\
a_{23} &= \int_0^t \frac{a_{21} v_x + a_{22} v_y}{r}\, dt + a_{23}(0);
\end{aligned}\right| \tag{3.84}$$

$$
\left.
\begin{aligned}
a_{31} &= \int_0^t \left(a_{32} m_z - a_{33} \frac{v_x}{r}\right) dt + a_{31}(0), \\
a_{32} &= \int_0^t \left(-a_{33} \frac{v_y}{r} - a_{31} m_z\right) dt + a_{32}(0), \\
a_{33} &= \int_0^t \frac{a_{31} v_x + a_{32} v_y}{r} dt + a_{33}(0).
\end{aligned}
\right| \qquad (3.85)
$$

If conditions (3.80) and (3.81) also occur as well, then

$$m_z = 0. \qquad (3.86)$$

should also be placed in equations (3.82)-(3.85).

Since x=y=0, then relations (3.25) are also simplified.

Orientation of the object relative to the coordinate system $O_1 \xi_* \eta_* \zeta_*$ is obviously determined by table (3.66) and by direction cosines (3.83)-(3.85).

Projections of the absolute angular velocity of the object on the X, Y and Z axes are found from formulas (3.67), if the values of v_y/r and v_x/r are substituted in them according to (3.78) instead of M_x and M_y, and if $M_z=0$ is also placed in them under the condition of (3.80).

If the orientation of trihedron Oxyz, to which the newtonometers are linked, is accomplished by using a free gyrostabilized platform and special functional diagram which gives the position of trihedron Oxyz relative to the gyrostabilized platform rather than by means of a controlled gyrostabilized platform, the functional diagram should provide relative position of trihedrons Oxyz and $O_1\xi_*\eta_*\zeta_*$, which corresponds to direction cosines (3.83)-(3.85). The position of the object in the coordinate system $O_1\xi_*\eta_*\zeta_*$. i.e., the relative position of the coordinate system $O_1\xi_*\eta_*\zeta_*$ and the system XYZ, linked to the object, is defined in this case by angles α, β and γ of the rotations of the gimbal rings of the stabilized platform on the object.

It should be noted that we are talking for the time being about determining the orientation of the object relative to the coordinate system $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$, i.e., about orientation of it in the main Cartesian system or in an earth body axis system. It also makes sense to talk about determination of the orientation of the object with respect to trihedron Oxyz, along whose axes the newtonometers are arranged. The relations which we obtained obviously permit solution of this problem as well.

§ 3.2. Determining the curvilinear coordinates of the object[7]

3.2.1. Initial assumptions. Let the position of point O of the object be given in the coordinate system $O_1\xi_*\eta_*\zeta_*$ by some curvilinear coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ non-orthogonal and moving in the general case (here and henceforth the superscripts are used to denote different coordinates). The transient nature of coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ is understood as the circumstance that the surfaces of equal value of coordinates, i.e., surfaces $\varkappa^i$= const, may vary their position with time with respect to trihedron $O_1\xi_*\eta_*\zeta_*$.

Radius vector $\vec{r}$ of point O of the object in the coordinate system $O_1\xi_*\eta_*\zeta_*$ is equal to:

$$\vec{r} = \xi_*\vec{\xi}_* + \eta_*\vec{\eta}_* + \zeta_*\vec{\zeta}_*. \tag{3.87}$$

where $\vec{\xi}_*$, $\vec{\eta}_*$ and $\vec{\zeta}_*$, as previously, are unit vectors of the corresponding axes and ξ_*, η_* and ζ_* are Cartesian coordinates of point O in the coordinate system $O_1\xi_*\eta_*\zeta_*$ or, which is the same thing, projections of vector $\vec{r}$ to the axes of this coordinate system.

For convenience in further exposition, let us replace the notations of axes ξ_*, η_* and ζ_* of the coordinate system $O_1\xi_*\eta_*\zeta_*$ by ξ^1, ξ^2 and ξ^3. Let us call the coordinate system $O_1\xi^1\xi^2\xi^3$ the fundamental Cartesian system. Then,

$$\vec{r} = \xi^1\vec{\xi}_1 + \xi^2\vec{\xi}_2 + \xi^3\vec{\xi}_3, \tag{3.88}$$

where the subscripts are retained to notate the unit vectors of axes ξ^1, ξ^2 and ξ^3.

It is obvious that ξ^1, ξ^2 and ξ^3 are functions of curvilinear coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ and of time:

$$\left.\begin{aligned} \xi^1 = \xi^1(\varkappa^1, \varkappa^2, \varkappa^3, t), \quad \xi^2 = \xi^2(\varkappa^1, \varkappa^2, \varkappa^3, t), \\ \xi^3 = \xi^3(\varkappa^1, \varkappa^2, \varkappa^3, t). \end{aligned}\right\} \tag{3.89}$$

Equalities (3.89) can be taken to calculate coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$.

By solving equations (3.89) with respect to $\varkappa^1$, $\varkappa^2$ a[illegible]^3, we find:

$$\left.\begin{array}{c}\varkappa^1 = \varkappa^1(\xi^1, \xi^2, \xi^3, t), \quad \varkappa^2 = \varkappa^2(\xi^1, \xi^2, \xi^3, t), \\ \varkappa^3 = \varkappa^3(\xi^1, \xi^2, \xi^3, t)\end{array}\right\} \tag{3.90}$$

In order that there be clear matching of coordinates ξ^1, ξ^2 and ξ^3 and $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$, relations (3.87) and (3.90) should be uniquely inveritable.

The necessary and sufficient conditions for unique invertability consists, as is well known, in the Jacobians of the functions ξ^1, ξ^2, and ξ^3 in the variables $\varkappa^1$, $\varkappa^2$, and $\varkappa^3$ being different from zero:

$$\frac{D(\xi^1, \xi^2, \xi^3)}{D(\varkappa^1, \varkappa^2, \varkappa^3)} = \begin{vmatrix} \frac{\partial \xi^1}{\partial \varkappa^1} & \frac{\partial \xi^1}{\partial \varkappa^2} & \frac{\partial \xi^1}{\partial \varkappa^3} \\ \frac{\partial \xi^2}{\partial \varkappa^1} & \frac{\partial \xi^2}{\partial \varkappa^2} & \frac{\partial \xi^2}{\partial \varkappa^3} \\ \frac{\partial \xi^3}{\partial \varkappa^1} & \frac{\partial \xi^3}{\partial \varkappa^2} & \frac{\partial \xi^3}{\partial \varkappa^3} \end{vmatrix} \neq 0 \tag{3.91}$$

and of functions $\varkappa^1$, $\varkappa^2$, and $\varkappa^3$ in variables ξ^1, ξ^2, and ξ^3 also being different from zero:

$$\frac{D(\varkappa^1, \varkappa^2, \varkappa^3)}{D(\xi^1, \xi^2, \xi^3)} \neq 0. \tag{3.92}$$

We shall henceforth assume that this condition is always fulfilled.

Let us consider an inertial navigation system whose task will be to determine the curvilinear coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ of the object.

Of course, the diagram of this type of system could be represented as a development of the preceding one. By calculating the coordinates ξ_*, η_* and ζ_* (or ξ, η and ζ) by using the diagram described in the preceding section, we can find the coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ by recalculation from formulas (3.90), and we can also calculate the parameters of the object with respect to any directions, which are a function of coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ in its known orientation in the coordinate system $O_1\xi_*\eta_*\zeta_*$ (or $_1\xi\eta\zeta$). This method is obvious.

We shall now pose a more general problem whose solution includes the above indicated method of obtaining coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$.

Let us represent the diagram of the system in the following manner. A free gyrostabilized platform, whose x, y and z axes coincide with the directions of axes ξ^1, ξ^2 and ξ^3 of the coordinate system $O_1\xi^1\xi^2\xi^3$, is used as its basis. Three newtonometers, the unit vectors of the directions of the axes of sensitivity of which are denoted by $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, are mounted on the gyrostabilized platform by using a special functional diagram. Let us assume that this diagram is such that it can provide the given dependence of orientation of the axes of sensitivity of the newtonometers on the coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$, calculated by the system, and on time:

$$\left.\begin{aligned} e_1 &= e_1(\varkappa^1, \varkappa^2, \varkappa^3, t), \\ e_2 &= e_2(\varkappa^1, \varkappa^2, \varkappa^3, t), \\ e_3 &= e_3(\varkappa^1, \varkappa^2, \varkappa^3, t). \end{aligned}\right\} \tag{3.93}$$

3.2.2. <u>The general case of constructing the equations of ideal operation.</u> Let us derive the equations of ideal operation of the described diagram of the inertial navigation system, i.e., relations of the type obtained in the preceding section, which would determine the coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ of the object by the readings of newtonometers $\varkappa_{e_1}$, $\varkappa_{e_2}$, and $\varkappa_{e_3}$ and the parameters which provide the orientation of the directions of the axes of sensitivity of the newtonometers, required for this.

During the derivation and analysis of the equations of ideal operation of the considered class of inertial navigation systems, it is convenient to use the symbolism and methods of tensor analysis.[8]

Let us introduce the fundamental coordinate basis, formed by the vectors

$$r_1 = \frac{\partial r}{\partial x^1}, \quad r_2 = \frac{\partial r}{\partial x^2}, \quad r_3 = \frac{\partial r}{\partial x^3}. \tag{3.94}$$

Vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ are non-coplanar. In fact, in order that vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ be coplanar, the value

$$J = r_1 \cdot (r_2 \times r_3) = r_2 \cdot (r_3 \times r_1) = r_3 \cdot (r_1 \times r_2) \tag{3.95}$$

should vanish.

But it follows from expressions (3.88) and (3.94) that

$$J = \frac{D(\xi^1, \xi^2, \xi^3)}{D(x^1, x^2, x^3)} \neq 0 \tag{3.96}$$

n view of condition (3.91) of the failure of the Jacobian of functions ξ^1, ξ^2 and ξ^3 in variables $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ to vanish . In the general case the base vectors are not orthogonal to each other, but their moduli are distinct from unity.

Since the three base vectors are non-coplanar, any vector, given in the coordinate system $O_1\xi^1\xi^2\xi^3$, for example, vectors $\vec{r}$, $d\vec{r}/dt$ and $d^2\vec{r}/dt^2$, can obviously be represented by using them. The arbitrary vector $\vec{b}$ can be represented by using the base vectors by two different methods. It can be represented either in the form of an expansion by the base vectors

$$\mathbf{b} = \sum_{i=1}^{3} b^i \mathbf{r}_i, \tag{3.97}$$

or it can be given by three scalar products

$$b_s = \mathbf{b} \cdot \mathbf{r}_s \quad (s = 1, 2, 3). \tag{3.98}$$

The values of b^s are called contravariant components of vector $\vec{b}$, and the values of b_s are called covariant components. It is easy to see that if vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ are orthogonal, while their moduli are equal, the difference between the contravariant and covariant components disappears.

Along with the fundamental coordinate basis, formed by vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$, let us calculate the basis, reciprocal to the main basis, having defined it by vectors $\vec{r}^1$, $\vec{r}^2$ and $\vec{r}^3$, related to the vectors of the main basis by the equalities

$$\mathbf{r}^1 = \frac{1}{J}\mathbf{r}_2 \times \mathbf{r}_3, \quad \mathbf{r}^2 = \frac{1}{J}\mathbf{r}_3 \times \mathbf{r}_1, \quad \mathbf{r}^3 = \frac{1}{J}\mathbf{r}_1 \times \mathbf{r}_2 \tag{3.99}$$

Let us assume that the value of J is positive and that it is always possible to provide proper selection of the order of numeration of the variables of x^i. Let us note that if the vectors of the main basis are orthogonal and units, the mutual coordinate basis coincides with the reciprocal basis.

Let us now introduce metric space tensor A, determined by the curvilinear coordinates x^1, x^2 and x^3. The covariant components a_{sk} of the metric tensor are equal to:

$$a_{sk} = r_s \cdot r_k. \tag{3.100}$$

Tensor A determines the metrics of the space given by the curvilinear coordinates x^1, x^2 and x^3. It follows from relations (3.88) and (3.94) that vector $d\vec{r}$ of the distance between two infinitely close points of space is expressed by the base vectors in the following manner:

$$dr = \sum_{s=1}^{3} \frac{\partial r}{\partial x^s} dx^s = \sum_{s=1}^{3} r_s dx^s. \tag{3.101}$$

Consequently, the square of the distance between these points is equal to:

$$dS^2 = dr \cdot dr = \sum_{s=1}^{3} \sum_{k=1}^{3} a_{sk} dx^s dx^k. \tag{3.102}$$

Thus, the covariant components of the metric tensor are coefficients of quadratic form in the expression for the square of dS, which also define the metrics in the coordinate system $x^1 x^2 x^3$ in the neighborhood of the point being considered.

Henceforth, as is used in tensor calculus (Einstein's rule), we will omit the summation signs in expressions of type (3.97) and (3.102), in which the superscripts and subscripts are repeated (umbral indicies and summation indicies), by writing these expressions in the form

$$b = b^s r_s, \quad dS^2 = a_{sk}\, dx^s\, dx^k \tag{3.103}$$

and by assuming that this writing assumes summation by umbral indices from one to three. Let us also assume that the non-repeating indices pass through values from one to three without mentioning this each time. Thus, instead of (3.98), we will simply write

$$b_s = b \cdot r_s. \tag{3.104}$$

Metric tensor A may also be given by its contravariant a^{sk} and mixed a_s^k components:

$$a^{sk} = r^s \cdot r^k, \quad a_s^k = r_s \cdot r^k. \tag{3.105}$$

Matricies $||a_{sk}||$ and $||a^{sk}||$ by definition are symmetrical and reciprocal to each other. Matrix $||a_s^k||$ is a unit matrix. It is easy to show that the determinants of these matricies are equal to:

$$|a_{sk}| = J^2, \quad |a^{sk}| = \frac{1}{J^2}, \quad |a_s^k| = 1. \tag{3.106}$$

From equalities (3.100) and (3.104) ensue the following relations, which link the vectors of the main and reciprocal basis:

$$r^i = a^{ik} r_k, \quad r_i = a_{ik} r^k. \tag{3.107}$$

It is sufficient to multiply both sides of the first relation by $\vec{r}^{\,l}$, and the second by $\vec{r}_l$ to ascertain the validity of the last statement.

Now, from equalities (3.97), (3.98) and (3.107), the following formulas ensue, which relate the covariant and contravariant components of vector $\vec{b}$:

$$b_i = a_{ik} b^k, \quad b^i = a^{ik} b_k. \tag{3.108}$$

And, from relations (3.107) and (3.108), we find:

$$b = b^k r_k = b^k a_{ki} r^i = b_i r^i. \tag{3.109}$$

Equality (3.109) together with the second equality of (3.108) means that the contravariant and covariant components of the vector in the main base are its covariant and contravariant components, respectively, in the reciprocal base.

Let us indicate the geometric sense of covariant b_s and contravariant b^s components of vector $\vec{b}$, for example, in the main base. The segments

$$\frac{b_1}{|r_1|} = \frac{b_1}{\sqrt{a_{11}}}, \quad \frac{b_2}{\sqrt{a_{22}}}, \quad \frac{b_3}{\sqrt{a_{33}}} \tag{3.110}$$

are projections of vector $\vec{b}$ to the vectors of the main base, while the segments

$$b^1|r_1| = b^1\sqrt{a_{11}}, \quad b^2\sqrt{a_{22}}, \quad b^3\sqrt{a_{33}} \tag{3.111}$$

are equal to the sides of a parallelepiped, constructed on vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ and having vector $\vec{b}$ as its diagonal.

To derive the equations of ideal operation we must find the expressions for the values measured by the newtonometers, whose axes of sensitivity are oriented along the directions $\vec{e}_s$, given by equalities (3.93).

The values measured by the newtonometers will be projections the vector $\vec{n}$ on the axes of sensitivity of the newtonometers. Since $\vec{e}_s$ are the unit vectors of the axes of sensitivity, then the measured values are equal to the covariant components of vector $\vec{n}$ along axes $\vec{e}_s$, i.e.,

$$n_{e_s} = n \cdot e_s. \tag{3.112}$$

The scalar product of the two vectors $\vec{b}$ and $\vec{c}$ can be given by their components in the main or reciprocal basis in the following manner:

$$b \cdot c = b^s c_s = b_s c^s. \tag{3.113}$$

By applying relation (3.113) to the scalar products of (3.112), we find:

$$n_{e_s} = n^k e_{sk} = a^{kl} n_k e_{sl}. \tag{3.114}$$

where n_k and n^k are the covariant and contravariant components of vector $\vec{n}$ in the main basis and e_{sl} are the covariant components of the vector $\vec{e}_s$.

Let us turn to finding n_k and n^k, the components of vector $\vec{n}$. According to formula (1.88)

$$n = \frac{d^2 r}{dt^2} - g(r). \tag{3.115}$$

Let us introduce the notation :

$$\frac{dr}{dt} = v, \quad \frac{d^2 r}{dt^2} = \frac{dv}{dt} = w. \tag{3.116}$$

It follows from (3.88), (3.89) and (3.94) that:

$$v = r_i \dot{x}^i + \frac{\partial r}{\partial t}. \tag{3.117}$$

Therefore, taking into account relations (3.98) and (3.108), we find:

$$v^i = \dot{x}^i + \frac{\partial r}{\partial t} \cdot r^i, \quad v_i = a_{ij} v^j. \tag{3.118}$$

By differentiating equality (3.117) again, we find:

$$w = r_i \ddot{x}^i + \frac{\partial r_i}{\partial x^j} \dot{x}^i \dot{x}^j + 2 \frac{\partial r_i}{\partial t} \dot{x}^i + \frac{\partial^2 r}{\partial t^2}. \tag{3.119}$$

To find the components w_k and w^k from the vectors of the main basis, we must obviously find the components of vectors $\partial\vec{r}_s/\partial x^k$, $\partial\vec{r}_s/\partial t$ and $\partial^2\vec{r}/\partial t^2$.

Vectors

$$r_{sk} = \frac{\partial r_s}{\partial x^k} = \frac{\partial^2 r}{\partial x^s \partial x^k} \tag{3.120}$$

can be represented by vectors of the main basis in the following manner:

$$r_{sk} = \Gamma_{sk}^m r_m. \tag{3.121}$$

where coefficients Γ^m_{sk} are essentially, as can be seen from comparison of equalities (3.121) and (3.97), contravariant components of vector $\vec{r}_{sk}$ in the main basis and are called Christoffel symbols of the second kind.

It follows from relation (3.120) that the Christoffel symbols of the second kind are symmetrical in their subscripts, i.e.

$$\Gamma_{sk}^m = \Gamma_{ks}^m. \tag{3.122}$$

And multiplying both sides of equality (3.121) scalarly by r_l, we find:

$$r_{sk} \cdot r_l = a_{ml}\Gamma_{sk}^m. \tag{3.123}$$

The scalar products on the left side of relation (3.123), i.e., the covariant components of vector $\vec{r}_{sk}$ in the main basis, are called Christoffel symbols of the first kind and are denoted by $\Gamma_{sk,l}$. It is easy to see that they are also symmetrical in their first two subscripts.

Relation (3.123) yields the expression of Christoffel symbols of the first kind in terms of symbols of the second kind:

$$\Gamma_{sk,l} = a_{ml}\Gamma^{m}_{sk}. \tag{3.124}$$

By multiplying equality (3.124) by a^{lt} and by recalling that

$$a_{mt}a^{lt} = \delta^{l}_{m}, \tag{3.125}$$

we find a relation reciprocal to relation (3.124):

$$\Gamma^{l}_{sk} = a^{lt}\Gamma_{sk,t}. \tag{3.126}$$

Christoffel symbols of the first and second kind can be expressed simply by the derivatives of the covariant components of the metric tensor. From formula (3.100) we find:

$$\frac{\partial}{\partial x^{t}}\mathbf{r}_{s}\cdot\mathbf{r}_{k} = \frac{\partial a_{sk}}{\partial x^{t}} = \Gamma_{st,k} + \Gamma_{kt,s}. \tag{3.127}$$

By changing the subscripts s, k and t in a cyclic order, we also have:

$$\frac{\partial a_{kt}}{\partial x^{s}} = \Gamma_{ks,t} + \Gamma_{ts,k}, \quad \frac{\partial a_{ts}}{\partial x^{k}} = \Gamma_{tk,s} + \Gamma_{sk,t}. \tag{3.128}$$

Now subtracting equality (3.127) from the sum of the two equalities of (3.128), we find:

(3.129)

$$\Gamma_{sk,t}=\frac{1}{2}\left(\frac{\partial a_{st}}{\partial x^{k}}+\frac{\partial a_{kt}}{\partial x^{s}}-\frac{\partial a_{sk}}{\partial x^{t}}\right).$$

By analogy with Christoffel symbols $\Gamma_{sk,t}$ and Γ^{t}_{sk}, let us introduce symbols of the first kind

$$\Gamma_{00,s}=\frac{\partial^{2}r}{\partial t^{2}}\cdot r_{s},\quad \Gamma_{0k,s}=\frac{\partial^{2}r}{\partial t\,\partial x^{k}}\cdot r_{s} \qquad (3.130)$$

and symbols of second kind Γ^{s}_{00} and Γ^{s}_{0k}. The zeros in the subscripts indicate that the time clearly contained in a function $r(x^{1}, x^{2}, x^{3}, t)$ should be taken instead of the corresponding coordinate when calculating the derivatives. The symbols of (3.130) are naturally equal to zero in fixed coordinates. Relations (3.124) and (3.126) remain valid for the symbols introduced. But the symbols of (3.130) are of course not expressed by the components of the metric tensor similar to Christoffel symbols.

Returning to equality (3.119), we find the following expressions for the contravariant and covariant components of acceleration in the main basis:

(3.131)

$$\left.\begin{aligned}w^{s}&=\ddot{x}^{s}+\Gamma^{s}_{mn}\dot{x}^{m}\dot{x}^{n}+2\Gamma^{s}_{0n}\dot{x}^{n}+\Gamma^{s}_{00},\\ w_{s}&=a_{sk}\ddot{x}^{k}+\Gamma_{mn,s}\dot{x}^{m}\dot{x}^{n}+2\Gamma_{0n,s}\dot{x}^{n}+\Gamma_{00,s}.\end{aligned}\right\}$$

It now follows from equality (3.115) that

$$\left.\begin{array}{l} n^s = \ddot{x}^s + \Gamma^s_{mk}\dot{x}^m\dot{x}^k + 2\Gamma^s_{\tau k}\dot{x}^k + \Gamma^s_{\tau\tau} - g^s, \\ n_s = a_{sk}n^k. \end{array}\right\} \tag{3.132}$$

Here g^s are the components of vector $\vec{g}$ of the earth's gravitational field strength in the main basis:

$$g^s = g \cdot r^s. \tag{3.133}$$

Vector $\vec{g}$ is given in the rigid earth body-axis system $O_1\eta^1\eta^2\eta^3$. The coordinate system $O_1\eta^1\eta^2\eta^3$ is identical to the coordinate system $O_1\xi\eta\zeta$ introduced previously. Therefore, according to expression (3.11),

$$g = \mathrm{grad}\, V(\eta^1, \eta^2, \eta^3). \tag{3.134}$$

It follows from equality (3.134) that

$$g_s = \mathrm{grad}^l V \eta_l \cdot r_s, \quad g^s = \mathrm{grad}^l V \eta_l \cdot r^s. \tag{3.135}$$

But

$$\eta_l = \eta_{lk} r^k = \eta_l^k r_k, \tag{3.136}$$

where η_{lk} and η_l^k are the covariant and contravariant components of unit vector η_l in the main basis. Therefore,

$$g_s = \mathrm{grad}^l V \eta_{ls}, \quad g^s = \mathrm{grad}^l V \eta_l^s. \tag{3.137}$$

Formulas (3.1[illegible]1) determine the components of vector $\vec{n}$ in the main basis. Turning to equalities (3.114), we find on the basis of formulas (3.132) and (3.137) the following expressions for the values measured by the newtonometers, whose unit vectors $\vec{e}_s$ of the directions of the axes of sensitivity are given by equality (3.93):

(3.138)

$$n_{e_s} = (\ddot{\varkappa}^k + \Gamma^k_{mn}\dot{\varkappa}^m\dot{\varkappa}^n + 2\Gamma^k_{0n}\dot{\varkappa}^n + \Gamma^k_{00} - \operatorname{grad}^k V\eta^0_0)e_{sk},$$

where e_{sk} are the covariant components of vector $\vec{e}_s$.

From equalities (3.138), we find:

(3.139)

$$\dot{\varkappa}^k e_{sk} = \int_0^t \left[n_{e_s} + \dot{\varkappa}^k \dot{e}_{sk} - (\Gamma^k_{mn}\dot{\varkappa}^m\dot{\varkappa}^n + 2\Gamma^k_{0n}\dot{\varkappa}^n + \Gamma^k_{00} - \operatorname{grad}^k V\eta^0_0)e_{sk}\right]dt + \dot{\varkappa}^k(0)e_{sk}(0).$$

By solving the left sides of equations (3.139) with respect to $\dot{\varkappa}^k$, we have:

(3.140)

$$\dot{\varkappa}^k = \frac{(\dot{\varkappa}^m e_{sm})E^{sk}}{E},$$

where E is the determinant and E^{sk} is the cofactor of the s-th line and of the k-th column of matrix $||e_{sk}||$. Now,

(3.141)

$$\varkappa^k = \int_0^t \dot{\varkappa}^k\,dt + \varkappa^k(0).$$

The elements of matrix $||e_{sk}||$ which define the orientation of the axes of sensitivity $\vec{e}_s$ of the newtonometers, should, of course, be known. If direction cosines of unit vectors $\vec{e}_s$ are known as functions of $\varkappa^s$ and time with respect to the axes of the stabilized platform, i.e., with respect to the unit vectors $\vec{\xi}_k$ of the main Cartesian system $O_1\xi^1\xi^2\xi^3$, then, by denoting these direction cosines by γ_s^l, we find:

$$e_s = \gamma_s^l \xi_l.$$

On the other hand, it follows from relations (3.88) and (3.89) that

$$r_k = \frac{\partial \xi^l}{\partial \varkappa^k} \xi_l. \tag{3.143}$$

Therefore,

$$e_{sk} = e_s \cdot r_k = \gamma_s^l \frac{\partial \xi^l}{\partial \varkappa^k}. \tag{3.144}$$

The equations which determine η_l^k and the relationship of η^l to ξ^s and time, must be added to equations (3.139), (3.140) and (3.144). We can turn to equations (3.30) to obtain the required relations, which, by taking into account the conformity of the notation introduced and the notation used previously, we write:

$$\xi_l = \int_0^t (\xi_l \times u)\,dt + \xi_l(0). \tag{3.145}$$

Here,

$$u = u^s_\eta \eta_s, \quad \xi_l = \alpha'^k_l \eta_k. \tag{3.146}$$

where u^s_η are the projections of vector $\vec{u}$ of the earth rate on axes $O_1\eta^s$, and α'^k_l are the direction cosines between the axes ξ^1, ξ^2, ξ^3 and η^1, η^2 and η^3. These are the same direction cosines as α'_{lk}, which form table (3.27), except that the second subscript in them has been converted for convenience of writing into a superscript. Equations (3.145) are equivalant to equations (3.31), (3.32) and (3.33), from which the direction cosines α'^k_l are also obtained.

From the second group of equalities (3.146) and formulas (3.88), (3.89) and (3.107) we have:

$$\left.\begin{array}{l} \eta^k_l = \eta_l \cdot r^k = \alpha'^s_l a^{km} \frac{\partial \xi^s}{\partial x^m}, \\ \eta^k = \alpha'^k_s \xi^s. \end{array}\right\} \tag{3.147}$$

Relations (3.147), in which ξ^s are given by equalities (3.89), together with equations (3.31), (3.32), and (3.33) fully determine η^k_ℓ and η^k in the integrands of (3.139).

The contravariant components η^k_ℓ of the unit vectors $\vec{\eta}_\ell$ in the main basis and the coordinates η^k may also be calculated in a somewhat different manner. Inverting equations (3.145), we may write them in differential form as follows:

$$\frac{d\eta_l}{dt} + \eta_l \times u = 0. \tag{3.148}$$

From (3.148), (3.136), (3.121) and (3.130) we will then obtain:

$$\eta_l^k = -\int_0^t [\eta_l^m(\Gamma^k_{sm}\dot{x}^s + \Gamma^k_{0m}) + (\eta_l \times u)\cdot r^k]\,dt + \eta_l^k(0). \qquad (3.149)$$

To expand the mixed products of the vectors $\vec{\eta}_l$, $\vec{u}$ and $\vec{r}^k$ it is convenient to introduce the Levi-Civita symbols ϵ_{kns} and ϵ^{kns}, defined as follows:

$$\epsilon_{kns} = r_k \cdot (r_n \times r_s), \quad \epsilon^{kns} = r^k \cdot (r^n \times r^s). \qquad (3.150)$$

The Levi-Civita symbols are non-zero only when the indices n, s and k are non-identical. If the indices are different and follow in the order 1, 2, 3 or in the order obtained from the standard cyclic permutation,

$$\epsilon_{kns} = J, \quad \epsilon^{kns} = \frac{1}{J}, \qquad (3.151)$$

where J is the Jacobian determinant (3.96).

If the order of the indices is different from the standard order we have:

$$\epsilon_{kns} = -J, \quad \epsilon^{kns} = -\frac{1}{J}. \qquad (3.152)$$

From relations (3.15) we obtain:

$$\epsilon_{kns} r^s = r_k \times r_n, \quad \epsilon^{kns} r_s = r^k \times r^n. \qquad (3.153)$$

Since the vector u may be represented in the form

$$u = u_s r^s = u^s r_s, \qquad (3.154)$$

it follows from equalities (3.150) and (3.136) that the mixed products in the integrands of (3.149) may be written as follows:

$$(\eta_i \times u) \cdot r^k = \epsilon^{rlk} \eta_{ir} u_l. \tag{3.155}$$

But

$$u_l = u \cdot r_l = u^l_\eta \eta^n_l a_{nl}, \quad \eta_{ir} = \eta^q_i a_{qr}. \tag{3,156}$$

and therefore equalities (3.155) take the form:

$$(\eta_i \times u) \cdot r^k = \epsilon^{rlk} \eta^q_i \eta^n_l a_{qr} a_{nl} u^l_\eta. \tag{3.157}$$

In expressions (3.156) and (3.157) u^i_η designate the projections of the vector $\vec{u}$ of the earth rate on the $O_1\eta^i$ axes.

Introducing equalities (3.157) and (3.149), we obtain:

$$\eta^k_i = -\int_0^t \left[\eta^m_i (\Gamma^k_{sm} \dot{x}^s + \Gamma^k_{0m}) + \epsilon^{rlk} \eta^q_i a_{qr} a_{nl} \eta^n_l u^l_\eta\right] dt + \eta^k_i(0) \tag{3.158}$$

Finally, from relations (3.136) and the obvious equality,

$$r = \eta^k \eta_k \tag{3.159}$$

we find:

$$\eta^k = \frac{1}{2} \eta^i_k \frac{\partial r^k}{\partial x^i}. \tag{3.160}$$

Formulas (3.158) and (3.160) may be used instead of formulas (3.147), (3.31), (3.32) and (3.33) introduced above.

We note that use of the Levi-Civita symbols enables us to write equations (3.31), (3.32) and (3.33) in a more compact form:

$$a_i^{\prime s} = \int_0^t \frac{1}{2}\epsilon_{ijs} a_i^{\prime j} u_k^j \, dt + a_i^{\prime s}(0). \tag{3.161}$$

Thus, the portion of the ideal equations relating to the determination of the curvilinear non-stationary coordinates $\varkappa^s$ and their rates of change $\dot{\varkappa}^s$ may be written in the form of the following system of equations:

$$\left.\begin{aligned} \dot{\varkappa}^s e_{is} = \int_0^t [n_{e_i} + \dot{\varkappa}^s \dot{e}_{is} - (\Gamma^s_{mn}\dot{\varkappa}^m\dot{\varkappa}^n + 2\Gamma^s_{0n}\dot{\varkappa}^n + \\ + \Gamma^s_{00} - \mathrm{grad}^s V \eta_i^s) e_{is}] \, dt + \dot{\varkappa}^s(0) e_{is}(0). \\ \dot{\varkappa}^s = \frac{(\dot{\varkappa}^m e_{im}) E^{is}}{E}, \quad \varkappa^s = \int_0^t \dot{\varkappa}^s \, dt + \varkappa^s(0); \end{aligned}\right| \tag{3.162}$$

$$\left.\begin{aligned} \eta_i^s = -\int_0^t [\eta_i^m(\Gamma^s_{rm}\dot{\varkappa}^r + \Gamma^s_{0m}) + \\ + \epsilon^{rls}\eta_l^q a_{qr} a_{ol}\eta_i^o u_\eta^l] \, dt + \eta_i^s(0). \\ \eta^s = \frac{1}{2}\eta_0^i \frac{\partial r^s}{\partial \alpha^i}. \end{aligned}\right| \tag{3.163}$$

Equations (3.163) may be replaced by the equivalent equations:

$$\left.\begin{aligned} a_i^{\prime s} = \int_0^t \frac{1}{2}\epsilon_{ijs} a_i^{\prime j} u_k^j \, dt + a_i^{\prime s}(0). \\ \eta_i^s = a_i^{\prime j} a^{sn} \frac{\partial r^j}{\partial \alpha^n}, \quad \eta^s = a_i^{\prime s}\xi^i. \end{aligned}\right| \tag{3.164}$$

In the inertial navigation system under consideration a free gyrostabilized platform was taken as the basis of the functional diagram The rotation angles α, β, and λ of the gimbal rings determine, clearly, the orientation of the object in the basic Cartesian coordinate system. The direction cosines retain the angles X, Y and Z of the object and the ξ^1, ξ^2, ξ^3 axes are given by table (3.66). The only change required is to replace the x, y, z axes by the ξ^1, ξ^2, ξ^3 axes. Since relations (3.88), (3.89) and (3.94) give the orientation of the vectors of the main basis, these relations, together with table (3.66), define the orientation of the object relative to the basic coordinate system.

3.2.3. Orientation of the newtonometers along the normals to the coordinate surfaces. The system discussed above was one in which the directions of the axes of sensitivity $\vec{e}_s$ occupy an arbitrary position. The only conditions imposed were that these directions should be non-coplaner and that their direction cosines with the $\xi^1\xi^2\xi^3$ axes should be known at each moment of time. A free gyrostabilized platform, relative to the axes of which the directions of the $\vec{e}_s$ axes are given, was taken as the basis of the system. It is not difficult, however, to extend the results obtained for this system to the case of a three-component gauge of absolute angular velocity (or a maneuverable gyrostabilized platform) as the basis, the directions of $\vec{e}_s$ being given relative to the axes of the gauge platform (or the maneuverable gyroplatform).

For the system in question equations (3.138) were integrated by isolating the total derivatives from the sums $\ddot{x}^k e_{sk}$. Separation of variables was performed after the first integration.

It was noted in Chapter 1 that there are two possible ways of solving the basic inertial navigation equation in curvilinear coordinates. Both possibilities are based on the assumption that the first operation performed on the newtonometer readings is that of integration. The first possibility was discussed above. The second is based on considering the directions of the axes of sensitivity of the newtonometers as no longer arbitrary, but as given such that each newtonometer reading should contain the second derivative of only one of the coordinates $\ddot{x}^s$, i.e., such that relations (3.138) should be solvable for the first derivatives.

This condition may be satisfied by choosing e_{sk} such that

$$\left.\begin{array}{l} e_{sk} = 0, \text{ если } s \neq k, \\ e_{sk} = e_{ss} \neq 0, \text{ если } s = k. \end{array}\right\} \tag{3.165}$$

This choice implies that $\vec{e}_s$ are normal to $\vec{r}_k$ for $k \neq s$, and therefore coincide with the vectors $\vec{r}^s$ of the reciprocal basis.

The correctness of this statement follows from the definition (3.104) of the covariant component and from formulas (3.99), giving the vectors $\vec{r}^s$ of the reciprocal basis. This result is to be expected, since the vectors of the reciprocal basis are, by definition, normal to the surfaces of equal values of the coordinates, i.e., are gradient vectors.

If condition (3.165) is satisfied, i.e., if the axes of sensitivity of the newtonometers are situated along the vectors of the reciprocal basis, we find from relations (3.138):

$$n_{e_s} = (\ddot{\varkappa}^s + \Gamma^s_{mn}\dot{\varkappa}^m\dot{\varkappa}^n + 2\Gamma^s_{0n}\dot{\varkappa}^n + \Gamma^s_{00} - \operatorname{grad}^s V\eta_0^s)\, e_{ss}. \qquad (3.166)$$

Since $\vec{e}_s$ are unit vectors and are oriented along the vectors $\vec{r}^s$, it follows from equalities (3.110) that

$$e_s = \frac{r^s}{\sqrt{a^{ss}}}, \quad e_{ss} = \frac{1}{\sqrt{a^{ss}}}. \qquad (3.167)$$

Now from (3.165) and (3.167) we obtain:

$$\ddot{\varkappa}^s = \sqrt{a^{ss}}\, n_{e_s} - \Gamma^s_{mn}\dot{\varkappa}^m\dot{\varkappa}^n - 2\Gamma^s_{0n}\dot{\varkappa}^n - \Gamma^s_{00} + \operatorname{grad}^s V\eta_0^s. \qquad (3.168)$$

Integrating equations (3.168), we obtain the relations

$$\left.\begin{aligned} \dot{\varkappa}^s &= \int_0^t \left[\sqrt{a^{ss}}\, n_{e_s} - \Gamma^s_{mn}\dot{\varkappa}^m\dot{\varkappa}^n - 2\Gamma^s_{0n}\dot{\varkappa}^n - \right.\\ &\qquad \left. - \Gamma^s_{00} + \operatorname{grad}^s V\eta_0^s\right] dt + \dot{\varkappa}^s(0),\\ \varkappa^s &= \int_0^t \dot{\varkappa}^s\, dt + \varkappa^s(0) \end{aligned}\right| \qquad (3.169)$$

Relations (3.169) enable us to determine the current values of the coordinates $\varkappa^s$ and their rates of change $\dot{\varkappa}^s$ using the known values of n_{e_s} and the initial conditions. These relations could also be taken as the ideal equations of the inertial system under consideration. However, in equalities (3.169) the magnitudes n_{e_s} of the newtonometer readings must be multiplied by $\sqrt{a^{ss}}$ before integration. The diagonal elements of the metric tensor may be, in the general case, variable. Therefore, this multiplication is undesirable according to the considerations presented in §1.4 (p. 55).

In order to avoid computational operations on the newtonometer readings before their integration, let us transform equalities (3.168). Let us first divide the right and left sides of these equalities by $\sqrt{a^{ss}}$ and then let us subtract from both sides the quantity

$$\frac{1}{2}\frac{\dot{x}^s \dot{a}^{ss}}{(a^{ss})^{3/2}}.$$

We then obtain (not summing over s!):

$$\frac{d}{dt}\left(\frac{\dot{x}^s}{\sqrt{a^{ss}}}\right) = n_{\varepsilon_s} - \left(\Gamma^s_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^s_{0n}\dot{x}^n + \right.$$
$$\left. + \Gamma^s_{00} - \mathrm{grad}^l V \eta^s_l\right)\frac{1}{\sqrt{a^{ss}}} - \frac{1}{2}\frac{\dot{x}^s\dot{a}^{ss}}{(a^{ss})^{3/2}} \tag{3.170}$$

or

$$\frac{d}{dt}\left(\frac{\dot{x}^s}{\sqrt{a^{ss}}}\right) = n_{\varepsilon_s} - \frac{1}{\sqrt{a^{ss}}}\left(\dot{x}^s \frac{d}{dt}\ln\sqrt{a^{ss}} + \right.$$
$$\left. + \Gamma^s_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^s_{0n}\dot{x}^n + \Gamma^s_{00} - \mathrm{grad}^l V \eta^s_l\right) \tag{3.171}$$

Integrating these equations, we find:

$$\frac{\dot{x}^s}{\sqrt{a^{ss}}} = \int_0^t \left[n_{\varepsilon_s} - \frac{1}{\sqrt{a^{ss}}}\left(\dot{x}^s \frac{d}{dt}\ln\sqrt{a^{ss}} + \right.\right.$$
$$+ \Gamma^s_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^s_{0n}\dot{x}^n + \Gamma^s_{00} -$$
$$\left.\left. - \mathrm{grad}^l V \eta^s_l\right)\right]dt + \frac{\dot{x}^s(0)}{\sqrt{a^{ss}(0)}},$$
$$\dot{x}^s = \left(\frac{\dot{x}^s}{\sqrt{a^{ss}}}\right)\sqrt{a^{ss}}, \quad x^s = \int_0^t \dot{x}^s dt + x^s(0). \tag{3.172}$$

The systems of equations (3.163) and (3.164) remain valid, clearly, for the determination of η^s_l and η^k.

Formulas (3.162) and (3.163) or (3.172) and (3.164) are the portion of the ideal equations of the system in question which relate to the determination of the coordinates x^s of the object and their rates of change $\dot{x}^s$. To determine the orientation parameters of the object table (3.66) must be added to these formulas, and also the table of the direction cosines between $\vec{e}_s$ and ξ_k:

$$\begin{array}{cccc} & \xi_1 & \xi_2 & \xi_3 \\ e_s & \frac{e^s \cdot \xi_1}{\sqrt{a^{ss}}} & \frac{e^s \cdot \xi_2}{\sqrt{a^{ss}}} & \frac{e^s \cdot \xi_3}{\sqrt{a^{ss}}}. \end{array} \tag{3.173}$$

Table (3.173) together with (3.66) enables us to obtain the orientation parameters of the object in the coordinate system along the axes of which the newtonometers are oriented, i.e., in the reciprocal coordinate system. Using the definition (3.94) of the vectors of the main basis, we can find the orientation of the object in the coordinate system defined by the vectors of the main basis.

3.2.4. <u>Orthogonal coordinates</u>. Let us consider the case in which the coordinates $\varkappa^1$, $\varkappa^2$, $\varkappa^3$ are orthogonal.

In this case the vectors of the main basis are perpendicular. The directions of the vectors of the reciprocal basis coincide with those of the vectors of the main basis. Only the diagonal elements a^{ss} and a_{ss} of the matrices of the contravariant and covariant components of the metric tensor are non-zero. Introducing the Lame coefficients h_s, we obtain the following expressions for a_{ss} and a^{ss}:

$$a_{ss} = \frac{1}{a^{ss}} = h_s^2. \tag{3.174}$$

For orthogonal coordinates only the following Christoffel symbols of the first and second kinds (not summing over s!) are non-zero:

$$\left.\begin{aligned} &\Gamma_{ss,\beta} = -h_s\frac{\partial h_s}{\partial \varkappa^\beta}, \quad \Gamma_{s\beta}^{s} = \Gamma_{\beta s}^{s} = \frac{\partial \ln h_s}{\partial \varkappa^\beta},\\ &\Gamma_{s\beta,s} = \Gamma_{\beta s,s} = h_s\frac{\partial h_s}{\partial \varkappa^\beta}, \quad \Gamma_{ss}^{\beta} = -\frac{h_s}{h_\beta^2}\frac{\partial h_s}{\partial \varkappa^\beta},\\ &\Gamma_{ss,s} = h_s\frac{\partial h_s}{\partial \varkappa^s}, \quad \Gamma_{ss}^{s} = \frac{\partial \ln h_s}{\partial \varkappa^s}. \end{aligned}\right\} \tag{3.175}$$

Now, taking into account relations (3.174) and (3.175), we obtain:

$$\frac{d}{dt}\ln\sqrt{a^{ss}} = -\frac{d}{dt}\ln h_s = -\Gamma_{s\beta}^{s}\dot{\varkappa}^\beta + \Gamma_{s0}^{s}.$$

In accordance with (3.175) for orthogonal coordinates only $\Gamma_{ks}^{s} = \Gamma_{sk}^{s}$ of the Γ_{mn}^{s} symbols are non-zero. Taking this into account, along with the expressions obtained above for $\frac{d}{dt}\ln\sqrt{a^{ss}}$, we may represent equalities (3.171) in the following form:

$$\frac{d}{dt}(h_s\dot{\varkappa}^s) = n_{e_s} - h_s\left[\Gamma_{0\beta}^{s}\dot{\varkappa}^\beta + \Gamma_{\alpha\beta}^{s}\dot{\varkappa}^\alpha\dot{\varkappa}^\beta + \Gamma_{ss}^{s}(\dot{\varkappa}^s)^2 + 2\Gamma_{s0}^{s}\dot{\varkappa}^s + \Gamma_{00}^{s} - \operatorname{grad}^s V\eta_s\right]. \tag{3.176}$$

(3.176)

where the summation is carried out over all k different from s.

According to expressions (3.130) and (3.174) we have:

$$\left.\begin{aligned}\Gamma_{\omega\omega,s} &= \frac{\partial^2 r}{\partial t^2}\cdot r_s, & \Gamma_{\omega\omega}^{s} &= \frac{1}{h_s^2}\,\frac{\partial^2 r}{\partial t^2}\cdot r_s,\\ \Gamma_{\omega k,s} &= \frac{\partial^2 r}{\partial t\,\partial x^k}\cdot r_s, & \Gamma_{\omega k}^{s} &= \frac{1}{h_s^2}\,\frac{\partial^2 r}{\partial t\,\partial x^k}\cdot r_s\end{aligned}\right| \tag{3.177}$$

while for orthogonal coordinates for s ≠ k

$$\Gamma_{\omega k,s} = -\Gamma_{\omega s,k} \tag{3.178}$$

This follows from the fact that, for orthogonal vectors $\vec{r}_s$, in this case the equalities

$$\left.\begin{aligned} r_s\cdot r_k &= 0 \quad \text{при} \quad s\neq k,\\ r_s\cdot r_k &= 1 \quad \text{при} \quad s=k.\end{aligned}\right\} \tag{3.179}$$

obtain.

Taking the partial time derivatives of these equalities, we find:

$$\frac{\partial^2 r}{\partial t\,\partial x^s}\cdot r_k + \frac{\partial^2 r}{\partial t\,\partial x^k}\cdot r_s = 0, \tag{3.180}$$

which is equivalent to equalities (3.178).

We now obtain from relations (3.176) (not summing over s!):

$$\left.\begin{aligned} h_s\dot{x}^s = \int_0^t \{w_s - h_s[\Gamma_{\omega k}^{s}\dot{x}^k + \Gamma_{ks}^{s}\dot{x}^k\dot{x}^s +{} \\ + \Gamma_{kk}^{s}(\dot{x}^k)^2 + 2\Gamma_{\omega s}^{s}\dot{x}^s + \Gamma_{\omega\omega}^{s} - \operatorname{grad}^s V\eta_l^s]\}\,dt +{} \\ + h_s(0)\,\dot{x}^s(0), \\ \dot{x}^s = (h_s\dot{x}^s)\frac{1}{h_s}, \quad x^s = \int_0^t \dot{x}^s\,dt + x^s(0), \end{aligned}\right| \tag{3.181}$$

where, as in expressions (3.176), k ≠ s.

To obtain formulas for η_l^s and η^k, we will once again use the systems of equations (3.163) or (3.164).

Taking into account equalities (3.174) and (3.175), formulas (3.163) take the form:

$$\left.\begin{aligned} \eta_k^s &= -\int_0^t \left[\eta_j^s\left(\Gamma_{sk}^j \dot{x}^s + \Gamma_{kk}^j + \Gamma_{mk}^j \dot{x}^m\right) + \right. \\ &\quad + \eta_j^m\left(\Gamma_{km}^j + \Gamma_{sm}^j \dot{x}^s + \Gamma_{mm}^j \dot{x}^m\right) + \\ &\quad\quad \left. + \epsilon^{pnk} \eta_j^p \eta_l^n h_s^2 h_p^2 u_q^l \right] dt + \eta_k^s(0), \\ \eta_j^s &= \frac{1}{2} \eta_i^j \frac{\partial v^s}{\partial x^j}. \end{aligned}\right| \tag{3.182}$$

where the summation is carried out over all m different from k.

Since in the case of an orthogonal reference grid

$$J = h_1 h_2 h_3, \tag{3.182a}$$

it follows that

$$\epsilon^{mnk} = \pm \frac{1}{h_1 h_2 h_3}, \quad \epsilon_{mnk} = \pm h_1 h_2 h_3, \tag{3.183}$$

where the plus or minus sign is selected as a function of the order of the indices, as discussed with regard to relations (3.151) and (3.152). Therefore formulas (3.164) take the form:

$$\left.\begin{aligned} \alpha_i'^s &= \int_0^t \frac{1}{h_j h_l h_s} \epsilon_{jls} \alpha_i'^l u_q^j \, dt + \alpha_i'^s(0), \\ \eta_j^s &= \alpha_i'^i \frac{1}{h_s^2} \frac{\partial \xi^i}{\partial x^s} \xi, \quad \eta^s = \alpha_i'^s \xi^i. \end{aligned}\right| \tag{3.184}$$

and the table of direction cosines (3.173) is written in the following form:

	ξ_1	ξ_2	ξ_3
e_1	$\frac{1}{h_1}\frac{\partial \xi^1}{\partial x^1}$	$\frac{1}{h_1}\frac{\partial \xi^2}{\partial x^1}$	$\frac{1}{h_1}\frac{\partial \xi^3}{\partial x^1}$.

(3.185)

For the case of the orthogonal curvilinear coordinates x^s, the functional diagram of the system may be constructed, clearly, on the basis of a maneuverable gyrostabilized platform, since in this case $\vec{e}_s$ form a rigid orthogonal trihedron which could be rigidly attached to the platform of the inertial system.

Let us form the moments M^4_{1x}, M^5_{1y}, M^6_{1x} required for control of the gyroplatform.

To do this we first need to express the projections of the absolute angular velocity of the basis trihedron $\vec{r}_1\vec{r}_2\vec{r}_3$ on the directions of the vectors forming this trihedron in terms of the coordinates $\varkappa^s$ and their derivatives $\dot{\varkappa}^s$.

Let us introduce the notation

$$\varepsilon_s = \frac{dr_s}{dt}. \qquad (3.186)$$

Recalling the definition (3.94) of the vectors $\vec{r}_s$, we may represent the vectors $\vec{\varepsilon}_s$ as follows:

$$\varepsilon_s = \frac{\partial^2 r}{\partial \varkappa^s \partial \varkappa^\alpha}\dot{\varkappa}^\alpha + \frac{\partial^2 r}{\partial t\, \partial \varkappa^s}. \qquad (3.187)$$

According to the definition of the Cristoffel symbols and the symbols Γ^m_{0s}:

$$\frac{\partial^2 r}{\partial \varkappa^s \partial \varkappa^\alpha} = \Gamma^m_{s\alpha} r_m, \quad \frac{\partial^2 r}{\partial t\, \partial \varkappa^s} = \Gamma^m_{0s} r_m. \qquad (3.188)$$

Therefore

$$\varepsilon_s = (\Gamma^m_{s\alpha}\dot{\varkappa}^\alpha + \Gamma^m_{0s}) r_m. \qquad (3.189)$$

Let us use ε_{sk} and ε^k_s to denote the covariant and contravariant components of $\vec{\varepsilon}_s$ relative to vectors $\vec{r}_k$ of the main basis. From relations (3.189) we have:

$$\varepsilon^k_s = \Gamma^k_{s\alpha}\dot{\varkappa}^\alpha + \Gamma^k_{0s}, \quad \varepsilon_{sk} = \Gamma_{s\alpha, k}\dot{\varkappa}^\alpha + \Gamma_{0s, k}. \qquad (3.190)$$

On the other hand,

$$\frac{dr_s}{dt} = \frac{d}{dt}\left[\sqrt{a_{ss}}\left(\frac{r_s}{\sqrt{a_{ss}}}\right)\right], \qquad (3.191)$$

where $\vec{r}_s/\sqrt{a_{ss}}$ is a unit vector of direction $\vec{r}_s$. Consequently,

$$\varepsilon_s = \frac{\frac{d}{dt}\sqrt{a_{ss}}}{\sqrt{a_{ss}}} r_s + \sqrt{a_{ss}}\,\frac{d}{dt}\left(\frac{r_s}{\sqrt{a_{ss}}}\right). \tag{3.192}$$

If the vectors $\vec{r}_s$ of the main basis form a rigid trihedron, then, using $\vec{\omega}$ to denote the absolute angular velocity of this trihedron, we obtain:

$$\frac{d}{dt}\left(\frac{r_s}{\sqrt{a_{ss}}}\right) = \omega \times \frac{r_s}{\sqrt{a_{ss}}}. \tag{3.193}$$

Substituting expressions (3.193) into equalities (3.192), we find the relation between the vectors $\vec{\varepsilon}_s$ and the absolute angular velocity of the rotation of the basis trihedron:

$$\varepsilon_s = \frac{\frac{d}{dt}\sqrt{a_{ss}}}{\sqrt{a_{ss}}} r_s + \omega \times r_s. \tag{3.194}$$

We now have:

$$\varepsilon_{sk} = \frac{a_{sk}\frac{d}{dt}\sqrt{a_{ss}}}{\sqrt{a_{ss}}} + (\omega \times r_s)\cdot r_k. \tag{3.195}$$

For orthogonal vectors $\vec{r}_s$ the non-diagonal components of the metric tensor are equal to 0, and so for $s \neq k$ equalities (3.195) simplify and take the form:

$$\varepsilon_{sk} = (\omega \times r_s)\cdot r_k, \tag{3.196}$$

or

$$\varepsilon_{sk} = \omega^{\alpha}(r_{\alpha} \times r_s)\cdot r_k. \tag{3.197}$$

Let us expand the mixed products on the rights sides of relations (3.197), using the Levi- Civita symbols given by equalities (3.150), (3.151) and (3.152). We have:

$$\varepsilon_{sk} = \omega^{\alpha}\epsilon_{\alpha s k}. \tag{3.198}$$

Multiplying the right and left sides of equalities (3.198) by ϵ^{nsk} we obtain (not summing over s or k!):

$$e_{sk}\epsilon^{nsk} = \omega^n. \tag{3.199}$$

Turning now to formulas (3.190), we find expressions for the components ω^n and ω_ℓ of the vector $\vec{\omega}$ (not summing over s or k!) as follows:

$$\left.\begin{aligned} \omega^n &= \epsilon^{nsk}(\Gamma_{sm,k}\dot{x}^m + \Gamma_{0s,k}), \\ \omega_l &= a_{ln}\omega^n. \end{aligned}\right\} \tag{3.200}$$

For the case under consideration, in which the $\vec{r}_s$ are orthogonal,

$$\omega_l = a_{ll}\epsilon^{lsk}(\Gamma_{sm,k}\dot{x}^m + \Gamma_{0s,k}). \tag{3.201}$$

Using the known covariant components ω_ℓ of the vector $\vec{\omega}$ and (3.110), the projections $\vec{\omega}_{(\ell)}$ of the vector $\vec{\omega}$ on $\vec{r}_\ell$ are easily found (not summing over ℓ!):

$$\omega_{(l)} = \frac{\omega_l}{\sqrt{a_{ll}}} = \sqrt{a_{ll}}\,\epsilon^{lsk}(\Gamma_{sm,k}\dot{x}^m + \Gamma_{0s,k}). \tag{3.202}$$

or

$$\omega_{(l)} = \sqrt{a_{ll}}\,\epsilon^{lsk}a_{kk}(\Gamma^k_{sm}\dot{x}^m + \Gamma^k_{0s}). \tag{3.203}$$

In expressions (3.202) and (3.203) the indices s and k are different.

In accordance with equalities (3.175) only those Christoffel symbols Γ^k_{sm} are non-zero in which either s = m or k = m. Since according to formulas (3.174) and (3.182a)

$$h_l = \sqrt{a_{ll}}, \quad J = h_1h_2h_3, \tag{3.204}$$

we obtain from relations (3.202):

$$\left.\begin{aligned}\omega_{(1)} &= \frac{1}{h_2 h_3}\left(\Gamma_{2m,3}\dot{x}^m + \Gamma_{02,3}\right) = \\ &= -\frac{1}{h_2 h_3}\left(\Gamma_{3m,2}\dot{x}^m + \Gamma_{03,2}\right). \\ \omega_{(2)} &= \frac{1}{h_3 h_1}\left(\Gamma_{3m,1}\dot{x}^m + \Gamma_{03,1}\right) = \\ &= -\frac{1}{h_3 h_1}\left(\Gamma_{1m,3}\dot{x}^m + \Gamma_{01,3}\right). \\ \omega_{(3)} &= \frac{1}{h_1 h_2}\left(\Gamma_{1m,2}\dot{x}^m + \Gamma_{01,2}\right) = \\ &= -\frac{1}{h_1 h_2}\left(\Gamma_{2m,1}\dot{x}^m + \Gamma_{02,1}\right).\end{aligned}\right\} \tag{3.205}$$

In relations (3.205) the first expressions on the right sides correspond to the following orders of the indices ℓ, s, k: 1, 2, 3; 2, 3, 1; 3, 1, 2. The second expressions correspond to the orders 1, 3, 2; 2, 1, 3; 3, 2, 1.

The two expressions given by formulas (3.205) for each projection $\omega_{(\ell)}$ are identical. This is easily demonstrated by noting that, in accordance with equalities (3.175) and (3.187),

$$\Gamma_{sm,k} = -\Gamma_{km,s}, \quad \Gamma_{0s,k} = -\Gamma_{0k,s}. \tag{3.206}$$

Specifically, if the symbols $\Gamma_{0\,s,k}$ are equal to 0, i.e., if the reference grid κ^s does not change its position in the main Cartesian coordinate system, then

$$\left.\begin{aligned}\omega_{(1)} &= \frac{1}{h_2 h_3}\Gamma_{2m,3}\dot{x}^m = -\frac{1}{h_2 h_3}\Gamma_{3m,2}\dot{x}^m, \\ \omega_{(2)} &= \frac{1}{h_3 h_1}\Gamma_{3m,1}\dot{x}^m = -\frac{1}{h_3 h_1}\Gamma_{1m,3}\dot{x}^m, \\ \omega_{(3)} &= \frac{1}{h_1 h_2}\Gamma_{1m,2}\dot{x}^m = -\frac{1}{h_1 h_2}\Gamma_{2m,1}\dot{x}^m.\end{aligned}\right\} \tag{3.207}$$

In this case, taking into account the expressions (3.175) of the Christoffel system in terms of the Lame coefficients, we arrive at the following expressions for $\omega_{(\ell)}$:

$$\left.\begin{aligned}\omega_{(1)} &= -\frac{1}{h_3}\frac{\partial h_2}{\partial x^3}\dot{x}^2 + \frac{1}{h_2}\frac{\partial h_3}{\partial x^2}\dot{x}^3, \\ \omega_{(2)} &= \frac{1}{h_3}\frac{\partial h_1}{\partial x^3}\dot{x}^1 - \frac{1}{h_1}\frac{\partial h_3}{\partial x^1}\dot{x}^3, \\ \omega_{(3)} &= -\frac{1}{h_2}\frac{\partial h_1}{\partial x^2}\dot{x}^1 + \frac{1}{h_1}\frac{\partial h_2}{\partial x^1}\dot{x}^2.\end{aligned}\right\} \tag{3.208}$$

Formulas (3.205) for the projections $\omega_{(\ell)}$ of the vector $\vec{\omega}$ on the $\vec{r}_\ell$ directions enable us to form the controlling moments M^4_{1x}, M^5_{1y} and M^6_{1x} of the gyrostabilized platform. Assuming, for example, that the Ox, Oy, Oz axes of the platform coincide with $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_1$, respectively, and using relations (1.78), we find:

$$M^4_{1x} = H\omega_{(1)}, \quad M^5_{1y} = -H\omega_{(2)}, \quad M^6_{1x} = H\omega_{(3)}. \tag{3.209}$$

Let us summarize the results obtained for orthogonal curvilinear coordinates and collect together the formulas defining the operational algorithm of the inertial system.

For the case of a free gyrostabilized platform as the basis of the system, the ideal equations have the form:

$$\left.\begin{aligned} h_s\dot{x}^s = \int_0^t \bigl[n_{e_s} - h_s\bigl(\Gamma^s_{0j}\dot{x}^j + \Gamma^s_{kj}\dot{x}^k\dot{x}^j + {} \\ + \Gamma^s_{kk}(\dot{x}^k)^2 + 2\Gamma^s_{0k}\dot{x}^k + \Gamma^s_{00} - {} \\ - \operatorname{grad}^l V\eta^s_l\bigr)\bigr]\,dt + h_s(0)\,\dot{x}^s(0), \\ \dot{x}^s = (h_s\dot{x}^s)\frac{1}{h_s}, \quad x^s = \int_0^t \dot{x}^s\,dt + x^s(0); \end{aligned}\right\} \tag{3.210}$$

$$\begin{aligned} \eta^s_l = -\int_0^t \bigl[\eta^s_l\bigl(\Gamma^s_{js}\dot{x}^j + \Gamma^s_{0s} + \Gamma^s_{ks}\dot{x}^k\bigr) + {} \\ + \eta^k_l\bigl(\Gamma^s_{kk}\dot{x}^k + \Gamma^s_{sk}\dot{x}^s + \Gamma^s_{0k}\bigr) + \epsilon^{rtl}\eta^r_l\eta^t_l h^2_l h^2_p u^l_\eta\bigr]\,dt + \eta^s_l(0); \end{aligned} \tag{3.211}$$

$$\eta^k = \frac{1}{g}\eta^s_k\frac{\partial r^s}{\partial x^k}, \tag{3.212}$$

$$e_s \cdot \xi_k = \frac{1}{h_s}\frac{\partial \xi^k}{\partial x^s}. \tag{3.213}$$

Formulas (3.211) and (3.212) may be replaced by the equivalent formulas (3.184). In formulas (3.210) and (3.211) the summation over s is not performed; the summation over k, however, is performed for all values of this index different from s.

For the case of a maneuverable gyrostabilized platform as the basis of the system, relations (3.213) drop out, relations (3.205) and (3.209) taking their place.

3.2.5. <u>Comparison with the results obtained in 3.2.1.</u> It is interesting to compare the first group of formulas (3.210) with the first three formulas (3.59). Formulas (3.210) define the operational algorithm of an inertial system operating in orthogonal curvilinear coordinates, while formulas (3.59) define the operational algorithm for Cartesian coordinates. Since in the latter case the position of the xyz trihedron is arbitrary, it is possible to superpose the unit vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ with the vectors $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_1$ of the main basis. Then, clearly, it is possible to move directly from the first three equations (3.59) to the first group of equations (3.210). Let us demonstrate this.

Using the Levi-Civita symbols and the indexation which we have been using in this section, the first three formulas (3.59) may be represented in the following form:

$$v_{(s)} = \int_0^t \left[n_{(s)} - \omega_{(l)} v_{(k)} \epsilon_{lks} \frac{1}{h_l h_k h_s} + g_{(s)} \right] dt + v_{(s)}(0). \quad (3.214)$$

(3.214)

where $v_{(s)}$, $\omega_{(s)}$ and $g_{(s)}$ are the projections of the vector of the absolute velocity of the point O, the vector of thé absolute rate of the rotation of the basis trihedron and the vector of the strength of the gravitational field on the direction of the vector $\vec{r}_s$, respectively.

The projections $v_{(s)}$ and $g_{(s)}$ may be expressed in terms of the covariant components of the vectors $\vec{v}$ and $\vec{g}$ in the main basis.

According to relations (3.110) and (3.174)

$$v_{(s)} = \frac{v_s}{h_s}, \quad g_{(s)} = \frac{g_s}{h_s}. \tag{3.215}$$

From formulas (3.118)

$$v_s = \left(\dot{x}^s + \frac{\partial r}{\partial t} \cdot r^s \right) h_s^2. \tag{3.216}$$

and from their expressions (3.174) and (3.203):

$$\omega_{(l)} \epsilon_{lks} = h_s^2 h_l \left(\Gamma^s_{kn} \dot{x}^n + \Gamma^s_{tk} \right). \tag{3.217}$$

Substituting relations (3.215) -- (3.217) into equalities (3.214), we find:

$$h_s\left(\dot{x}^s + \frac{\partial r}{\partial t}\cdot r^s\right) = \int_0^t \left[n_{(s)} - h_s\left(\Gamma_{sn}^{s}\dot{x}^n + \Gamma_{0s}^{s}\right)\left(\dot{x}^s + \frac{\partial r}{\partial t}\cdot r^s\right) + g^s h_s\right]dt + \left[h_s\left(\dot{x}^s + \frac{\partial r}{\partial t}\cdot r^s\right)\right]_{t=0} \tag{3.218}$$

According to equalities (3.217) containing $\epsilon_{\ell ks}$, k are here different from s.

Let us now leave only $h_s \dot{x}^s$ on the left, differentiating the terms $h_s \frac{\delta \vec{r}}{\delta t} \cdot \vec{r}^s$ and moving them to the right side under the integral sign.

Differentiating, we obtain:

$$\frac{d}{dt}\left(h_s \frac{\partial r}{\partial x^s}\cdot r^s\right) = \frac{d}{dt}\left(\frac{1}{h_s}\frac{\partial r}{\partial t}\cdot r_s\right) = \frac{1}{h_s}\left(\frac{\partial^2 r}{\partial t\,\partial x^n}\cdot r_s \dot{x}^n + \frac{\partial^2 r}{\partial t^2}\cdot r_s + \frac{\partial r}{\partial t}\cdot\frac{\partial^2 r}{\partial x^s\,\partial x^n}\dot{x}^n + \frac{\partial r}{\partial t}\cdot\frac{\partial^2 r}{\partial t\,\partial x^s}\right) - \frac{1}{h_s^2}\left(\frac{\partial h_s}{\partial t}\frac{\partial r}{\partial t}\cdot r_s - \frac{\partial h_s}{\partial x^n}\frac{\partial r}{\partial t}\cdot r_s \dot{x}^n\right) \tag{3.219}$$

Taking into account formulas (3.121), (3.130) and (3.175) and noting, in addition, that from relations

$$h_s^2 = r_s \cdot r_s$$

there follow the equalities

$$\frac{\partial h_s}{\partial t} = h_s \Gamma_{0s}^{s}, \tag{3.220}$$

expressions (3.219) may be reduced to the form, where $k \neq s$:

$$\frac{d}{dt}\left(h_s \frac{\partial r}{\partial t}\cdot r^s\right) = \frac{1}{h_s}\left(\frac{\partial r}{\partial t}\cdot r_k \Gamma_{sn}^{k}\dot{x}^n + h_s^2\Gamma_{0s}^{s} + h_s^2\Gamma_{0n}^{s}\dot{x}^n + \frac{\partial r}{\partial t}\cdot r_k \Gamma_{0s}^{k}\right) \tag{3.221}$$

Substituting these expressions into equalities (3.218), we obtain:

$$h_s\dot{x}^s = \int_0^t \Big[n_{(s)} - h_s\Big(\Gamma^s_{kn}\dot{x}^k\dot{x}^n + \Gamma^s_{0k}\dot{x}^k + \frac{1}{h_k^2}\frac{\partial r}{\partial t}\cdot r_k\left(\Gamma^k_{sn}\dot{x}^n + \Gamma^k_{0s}\right)\Big) - \frac{1}{h_s}\left(\frac{\partial r}{\partial t}\cdot r_k\Gamma^k_{sn}\dot{x}^n + h_s^2\Gamma^s_{00} + h_s^2\Gamma^s_{0n}\dot{x}^n + \frac{\partial r}{\partial t}\cdot r_k\Gamma^k_{0s}\right) + g^s h_s\Big]\,dt + h_s(0)\dot{x}^s(0). \tag{3.222}$$

or, after grouping in the integrands,

$$h_s\dot{x}^s = \int_0^t \Big[n_{(s)} - h_s\left(\Gamma^s_{kn}\dot{x}^k\dot{x}^n + \Gamma^s_{0k}\dot{x}^k + \Gamma^s_{00} + \Gamma^s_{0n}\dot{x}^n + g^s\right) - \frac{\partial r}{\partial t}\cdot r_k\Big(\frac{h_s}{h_k^2}\Gamma^k_{sn}\dot{x}^n + \frac{h_s}{h_k^2}\Gamma^k_{0s} + \frac{1}{h_s}\Gamma^k_{sn}\dot{x}^n + \frac{1}{h_s}\Gamma^k_{0s}\Big)\Big]\,dt + h_s(0)\dot{x}^s(0). \tag{3.223}$$

Since it is orthogonal curvilinear coordinates which are being considered, in accordance with formulas (3.175) the only Christoffel symbols in the integrands of (3.223) which are non-zero are those of the form

$$\Gamma^l_{lp} = \Gamma^l_{pl},\quad \Gamma^l_{pp},\quad \Gamma^p_{pp},$$

where $l \neq p$. Taking this into account and noting that in equalities (3.223) $s \neq k$, we rewrite them in the following form:

$$h_s\dot{x}^s = \int_0^t \Big\{ n_{(s)} - h_s\left[\Gamma^s_{0s}\dot{x}^s + \Gamma^s_{ks}\dot{x}^k\dot{x}^s + \Gamma^s_{kk}(\dot{x}^k)^2 + 2\Gamma^s_{0k}\dot{x}^k + \Gamma^s_{00} - g^s\right] - \frac{\partial r}{\partial t}\cdot r_k\Big(\frac{h_s}{h_k^2}\Gamma^k_{ss}\dot{x}^s + \frac{h_s}{h_k^2}\Gamma^k_{kk}\dot{x}^k + \frac{h_s}{h_k^2}\Gamma^k_{0s} + \frac{1}{h_s}\Gamma^k_{ss}\dot{x}^s + \frac{1}{h_s}\Gamma^k_{sk}\dot{x}^k + \frac{1}{h_s}\Gamma^k_{0s}\Big)\Big\}\,dt + h_s(0)\dot{x}^s(0). \tag{3.224}$$

But the expressions in parentheses in the integrand are identically equal to 0, as follows from relations (3.206) and (3.175). Therefore the formulas reduce to the form

$$h_s\dot{x}^s = \int_0^t \left\{ n_{(s)} - h_s\left[\Gamma^s_{0s}\dot{x}^s + \Gamma^s_{ks}\dot{x}^k\dot{x}^s + \Gamma^s_{kk}(\dot{x}^k)^2 + 2\Gamma^s_{0k}\dot{x}^k + \Gamma^s_{00} - g^s\right]\right\}dt + h_s(0)\dot{x}^s(0) \tag{3.225}$$

Since, clearly,

$$n_{(s)} = n_{c_s},\quad g^s = \operatorname{grad}^s V\eta^s_c. \tag{3.226}$$

equations (3.225) coincide with the first group of equations (3.210), as required.

We note that this conversion from equations (3.59) to equations (3.210) enables us to write the first group of equations (3.210) for orthogonal coordinates also in the following form:

$$
\begin{aligned}
h_s\left(\dot{\varkappa}^s + \frac{\partial r}{\partial t}\cdot r^s\right) = \int_0^t \Big[n_{e_s} - h_s\big(\Gamma^s_{0s}\dot{\varkappa}^s &+ \\
+ \Gamma^s_{as}\dot{\varkappa}^a\dot{\varkappa}^s + \Gamma^s_{aa}(\dot{\varkappa}^a)^2 + 2\Gamma^s_{0a}\dot{\varkappa}^a &+ \\
+ \Gamma^s_{00} - \mathrm{grad}^s V\eta^s_t\big) + \frac{d}{dt}\left(\frac{\partial r}{\partial t}\cdot r^s\right)\Big] dt &+ \\
+ \left[h_s\left(\dot{\varkappa}^s + \frac{\partial r}{\partial t}\cdot r^s\right)\right]_{t=0}&.
\end{aligned}
\qquad (3.227)
$$

This form will frequently prove useful.

In discussing non-stationary curvilinear coordinates, we have hitherto considered the general case of non-stationaryness. The surfaces of equal values of $\varkappa^s$ = const could freely change their position in the main Cartesian coordinate system $O_1\xi^1\xi^2\xi^3$. This is reflected in (3.89) and (3.90) by the fact that time explicitly appears on the right sides of these formulas. The nature of this explicit time dependency is not stipulated, however.

There is a particular special case of this dependency which will be of special interest below. This is the case in which the curvilinear coordinates $\varkappa^1$, $\varkappa^2$, and $\varkappa^3$ define the position of the object in the $O_1\eta^1\eta^2\eta^3$ coordinate system rigidly bound to the earth, such that

$$
\left.
\begin{aligned}
\eta^1 = \eta^1(\varkappa^1, \varkappa^2, \varkappa^3), \quad \eta^2 = \eta^2(\varkappa^1, \varkappa^2, \varkappa^3), \\
\eta^3 = \eta^3(\varkappa^1, \varkappa^2, \varkappa^3)
\end{aligned}
\right\}
\qquad (3.228)
$$

The coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ are in this case stationary relative to the $O_1\eta^1\eta^2\eta^3$ coordinate system. But since this coordinate system is rigidly bound to the earth, the coordinates $\varkappa^1$, $\varkappa^2$, and $\varkappa^3$ are stationary relative to the earth, i.e., the coordinate lines and surfaces occupy a fixed position relative to the earth.

Relative to the main Cartesian coordinate system $O_1\xi^1\xi^2\xi^3$, the coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ are non-stationary.

From equalities (3.35) and (3.39), and table (3.27) of direction cosines we have:

$$\left.\begin{aligned}
\xi^1 &= \eta^1\left[\alpha'_{11}(0)\cos ut + \alpha'_{12}(0)\sin ut\right] + \\
&\quad + \eta^2\left[-\alpha'_{11}(0)\sin ut + \alpha'_{12}(0)\cos ut\right] + \eta^3\alpha'_{13}(0), \\
\xi^2 &= \eta^1\left[\alpha'_{21}(0)\cos ut + \alpha'_{22}(0)\sin ut\right] + \\
&\quad + \eta^2\left[-\alpha'_{21}(0)\sin ut + \alpha'_{22}(0)\cos ut\right] + \eta^3\alpha'_{23}(0), \\
\xi^3 &= \eta^1\left[\alpha'_{31}(0)\cos ut + \alpha'_{32}(0)\sin ut\right] + \\
&\quad + \eta^2\left[-\alpha'_{31}(0)\sin ut + \alpha'_{32}(0)\cos ut\right] + \eta^3\alpha'_{33}(0).
\end{aligned}\right\} \tag{3.229}$$

If we assume that the $O_1\xi^3$ axis at all times coincides with the $O_1\eta^3$ axis, and that the $O_1\eta^1$ and $O_1\eta^2$ axes coincide with the $O_1\xi^1$ and $O_1\xi^2$ axes only at the initial moment of time,

$$\alpha'_{33}(0) = \alpha'_{11}(0) = \alpha'_{22}(0) = 1, \tag{3.230}$$

of the direction cosines α'_{ij} (0) are non-zero and formulas (3.229) take the form:

$$\left.\begin{aligned}
\xi^1 &= \eta^1\cos ut - \eta^2\sin ut, \\
\xi^2 &= \eta^1\sin ut + \eta^2\cos ut, \\
\xi^3 &= \eta^3.
\end{aligned}\right\} \tag{3.231}$$

In calculating the elements of the metric tensor, the Lamé coefficients and the Christoffel symbols for this case we may let $t = 0$. The validity of this assumption derives from the fact that the reference grid $\varkappa^1$, $\varkappa^2$, $\varkappa^3$ moves as a unit and the properties of the space defined by the curvilinear coordinates $\varkappa^1$, $\varkappa^2$, and $\varkappa^3$ are not functions of time. Of course, the validity of this assumption may also be demonstrated by direct computation.

For the symbols $\Gamma_{0k,s}$ and $\Gamma_{00,s}$ we obtain, using relations (3.130) and (3.131), the following expressions, which are also independent of time:

$$\left.\begin{aligned} \Gamma_{ta,\,s} &= u\left[\frac{\partial\eta^1}{\partial x^k}\frac{\partial\eta^2}{\partial x^s} - \frac{\partial\eta^2}{\partial x^k}\frac{\partial\eta^1}{\partial x^s}\right], \\ \Gamma_{tt,\,s} &= -\frac{u^2}{2}\frac{\partial}{\partial x^s}[(\eta^1)^2 + (\eta^2)^2]. \end{aligned}\right\} \quad (3.232)$$

In concluding our consideration of the ideal equations the following points should be noted:

In considering systems operating in Cartesian coordinates in §3.1, we took as the basis of the functional diagram a maneuverable gyroplatform or a three-dimensional gauge of angular velocity [angular rate meter]. It was assumed here that the directions of the axes of sensitivity of the newtonometers and the gyroscopic elements formed a single rigid orthogonal trihedron.

In §3.2 it was assumed that the basis of the functional diagram was a free gyrostabilized platform, i.e., it was assumed that the directions of the axes of sensitivity of the gyroscopic elements were fixed, but that the directions of the axes of sensitivity of the newtonometers were given with the aid of some functional diagram relative to the gyrostabilized platform. The construction of a system on the basis of a maneuverable gyroplatform was also considered for the case of orthogonal curvilinear coordinates and the resulting equations were compared with the equations derived in the preceding section.

Affine coordinates, clearly, are a special case of non-orthgonal curvilinear coordinates. Systems operating in affine coordinates and constructed on the basis of a gyrostabilized platform are, therefore, a special case of the systems under consideration in this section.

It is possible, of course, to imagine a maneuverable gyroplatform with non-orthogonal positioning of the axes of the gyroscopic sensing elements, these axes not necessarily forming a rigid trihedron. The relative positions of the axes, as well as their orientation in space, mat be a function of time and the coordinates determined by the system. In this case, a system determining affine coordinates may be constructed on the basis of a maneuverable platform, and the directions of the axes of the gyroscopic elements and the newtonometers may be mutually superposed. Systems for determining arbitrary curvilinear coordinates may also be constructed on the basis of this type of "non-rigid" maneuverable platform.

Finally, the trihedra formed by the axes of sensitivity of the gyroscopic elements and newtonometers, being variable as a function of time and the coordinates determined by the inertial system, may not coincide with one another.

Analysis of all of the alternatives noted here and construction of the ideal equations for these alternative systems yield results which are not in any way fundamentally different from the results obtained previously, since simple summation of the results obtained above enables us to write all of the relations required to obtain the ideal equations for these alternatives.

§3.3. Examples of Ideal Equations for Inertial Systems Operating in Various Generally Used Coordinates.

3.3.1. Affine and Cartesian coordinates. In the previous section we derived the ideal equations for inertial navigation systems operating in the curvilinear non-stationary coordinates $\varkappa^1$, $\varkappa^2$, and $\varkappa^3$.

For the sake of illustration we will now obtain from these general formulas the operational equations for inertial navigation systems operating in certain more widely used reference grids.

Specifically, let us consider spherical coordinate systems: -- geocentric and geodetic , as well as geographic coordinate systems and an example of non-orthogonal coordinates. We will limit ourselves here to the positioning of the axes of sensitivity of the newtonometers along the vectors of the basis trihedron, i.e., for non-orthogonal coordinates, to equations (3.172), (3.163), (3.164) and table (3.173), and for orthogonal coordinates, to equations (3.210) -- (3.213), (3.205), (3.209) and (3.134). The ideal equations corresponding to an arbitrary positioning of the newtonometers and deriving from relations (3.162), (3.163), (3.164) and (3.93), will not be considered here. If necessary these equations could easily be written, since expressions for all of the terms entering into these equations will be obtained.

Before proceeding to the special cases of curvilinear coordinates, we note that the equations for affine coordinates may easily be obtained from equalities (3.172), (3.163), (3.164) and table (3.173), and the equations for Cartesian coordinates from relations (3.210) -- (3.213), (3.205), (3.209) and (3.184).

If the coordinates $\varkappa^1$, $\varkappa^2$, and $\varkappa^3$ are affine but not stationary, i.e., if the directions of the axes of sensitivity of the newtonometers change their orientation relative to the main Cartesian system $O_1\xi^1\xi^2\xi^3$ only as an explicit function of time, then in formulas (3.172) and (3.163) terms containing Christoffel symbols disappear, since for affine coordinates these symbols are equal to 0. The components of the metric tensor become functions of time only, as do the elements of table (3.173).

For the case in which the trihedron formed by $\vec{e}_s$ (the axes of sensitivity of the newtonometers) is rigid, the elements of the metric tensor become constant and in formulas (3.172) the terms $\dot{\kappa}^s \frac{d}{dt}\ln\sqrt{a^{ss}}$ drop out. In this case the trihedron formed by $\vec{e}_s$ rotates in the main Cartesian coordinate system as a unit; the absolute rate of its rotation is determined by formulas (3.200), in which, of course, the $\Gamma_{sm,k}$ symbols must be set equal to 0.

For affine stationary coordinates the symbols $\Gamma_{00,s}$ and $\Gamma_{0k,s}$ are also equal to 0.

If the coordinates $\varkappa^1$, $\varkappa^2$ and $\varkappa^3$ are Cartesian but not stationary, the equations deriving from the general formulas (3.210) -- (3.213), (3.205), (3.209) and (3.184) reduce to the two equations (3.59) -- (3.65), derived in §3.1.

Indeed, in this case

$$\left.\begin{aligned} r = \xi^a \xi_a = \varkappa^s r_s(t), \\ a^{ss} = r_{ss} = a^s_s = 1, \quad a^{sk} = a_{sk} = a^k_s = 0 \quad (s \neq k), \end{aligned}\right\} \tag{3.233}$$

12*

Consequently,

$$\frac{\partial r}{\partial t} = \frac{\partial r_s}{\partial t}\dot{x}^s, \quad \frac{\partial^2 r}{\partial t^2} = \frac{\partial^2 r_s}{\partial t^2}x^s. \tag{3.234}$$

But

$$\frac{\partial r_s}{\partial t} = \frac{dr_s}{dt} = \omega \times r_s \tag{3.235}$$

and therefore

$$\frac{\partial^2 r_s}{\partial t^2} = \dot{\omega} \times r_s + (\omega \times \omega) \times r_s + \omega \times (\omega \times r_s) = \\ = \dot{\omega} \times r_s + \omega \times (\omega \times r_s). \tag{3.236}$$

Recalling definition (3.130) of the symbols $\Gamma_{00,k}$ and $\Gamma_{0n,k}$, we find:

$$\Gamma_{00,k} + 2\Gamma_{0n,k}\dot{x}^n = r_k \cdot [\dot{\omega} \times r + \omega \times (\omega \times r) + 2\omega \times \dot{r}]. \tag{3.237}$$

Substituting expressions (3.237) and (3.233) into the first group of equations (3.210) and noting that for the case under consideration

$$\operatorname{grad}^j V \eta_j^s = g \cdot r_s, \tag{3.238}$$

we arrive at the equations

$$\dot{x}^s = \int_0^t \{n_{e_s} - r_s \cdot [\dot{\omega} \times r + \omega \times (\omega \times r) + \\ + 2\omega \times \dot{r} + g]\}\, dt + \dot{x}^s(0).$$

or the equations

$$\dot{x}^s = \int_0^t \left\{n_s - r_s \cdot \left[(\omega \times r) + \omega \times \frac{dr}{dt} + g\right]\right\} dt + \dot{x}^s(0). \tag{3.239}$$

Since

$$\frac{dr}{dt} = r_s\dot{x}^s + \omega \times r = v, \tag{3.240}$$

it follows from equations (3.239) that

$$\frac{dr}{dt} = \int_0^t \left[n - \omega \times \frac{dr}{dt} + g\right] dt + \frac{dr(0)}{dt}. \tag{3.241}$$

Equation (3.241) coincides with the first equation (3.53), from which the first three equations (3.59) were derived.

The second equation (3.53) follows from the second and third groups of equations (3.210) and equality (3.240), i.e., the fourth, fifth and sixth equations (3.59).

From expressions (3.211), (3.235), and (3.130) it follows that

$$\eta_k^i = \int_0^t [\eta_l \times (\omega - u)] \cdot r_i \, dt + \eta_k^i(0). \tag{3.242}$$

or

$$\eta_k = \int_0^t \eta_l \times (\omega - u)\, dt + \eta_k(0). \tag{3.243}$$

which coincides with the vector equations (3.55), from which the scaler equations (3.61) and (3.64) are obtained.

From equations (3.212) we obtain:

$$\eta^k = r \cdot \eta_k, \tag{3.244}$$

which corresponds with (3.63). Relations (3.65), clearly, are equivalent to relations (3.238).

Finally, from equalities (3.213)

$$\frac{de_s}{dt} = \frac{\partial^2 r}{\partial t\, \partial x^s} = \omega \times r_s = \omega \times e_s, \tag{3.245}$$

which is equivalent to equalities (3.54), from which formulas (3.60) are obtained.

Formulas (3.62) do not follow directly from equations (3.210) -- (3.213), since the latter do not presuppose the transition to the Cartesian coordinates ξ^1, ξ^2, ξ^3.

It is easy to see that formulas (3.61), (3.63) and (3.64) also follow from equations (3.184), which are replaced by equations (3.211) and (3.212) in the system of equations (3.210) -- (3.213).

Equalities (3.205) and (3.209) are satisfied identically on the basis of equalities (3.233) and (3.235). This concludes the transformation from equations (3.210) -- (3.213), (3.205), (3.209) and (3.184) to equations (3.59) -- (3.65).

For the case in which the coordinates are Cartesian, and the orientation of the trihedron $\vec{r}_1\vec{r}_2\vec{r}_3$ coincides with trihedron $O_1\eta^1\eta^2\eta^3$, the derived coordinates are, of course, the Cartesian coordinates η^1, η^2, η^3.

If the Cartesian coordinates $\varkappa^s$ are stationary, the trihedron $\vec{r}_1\vec{r}_2\vec{r}_3$ may be considered, without loss of generality, as coinciding with the main Cartesian coordinate system $O_1\xi^1\xi^2\xi^3$. Equations (3.210) together with equalities (3.59), then reduce to equations (.189).

3.3.2. Geocentric coordinates independent of the earth's rotation. Let the position of the point of the center of the sensitivity masses of the newtonometers O be defined in the main Cartesian coordinate system $O_1\xi^1\xi^2\xi^3$ by the distance $\varkappa^1 = r$ of the point O from the center of the earth and the two angles $\varkappa^2 = \lambda_1$ and $\varkappa^3 = \varphi$ (Figure 3.2). The angle φ is taken to be the angle between the plane $O_1\xi^1\xi^2$ and the line O_1O. The symbol λ_1 designates the angle in the plane $O_1\xi^1\xi^2$ between the $O_1\xi^1$ axis and the line of intersection of the $O_1\xi^1\xi^2$ plane with the ξ^3O_1O plane. If the circle formed by the intersection of the $O_1\xi^1\xi^2$ plane and the sphere of radius r centered at the point O_1 is termed the equator, then the angles φ and λ_1 will be the latitude and longitude of the point O on this sphere.

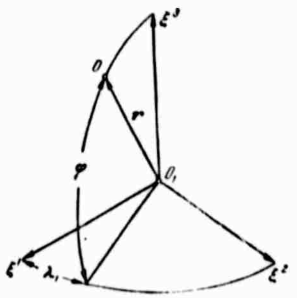

Figure 3.2

It follows from the definition of angles φ and λ_1 that

$$\left.\begin{array}{c}\xi^1 = r\cos\varphi\cos\lambda_1, \quad \xi^2 = r\cos\varphi\sin\lambda_1, \\ \xi^3 = r\sin\varphi,\end{array}\right\} \tag{3.246}$$

from which

$$\left.\begin{array}{lll}\dfrac{\partial\xi^1}{\partial r} = \cos\varphi\cos\lambda_1, & \dfrac{\partial\xi^2}{\partial r} = \cos\varphi\sin\lambda_1, & \dfrac{\partial\xi^3}{\partial r} = \sin\varphi, \\ \dfrac{\partial\xi^1}{\partial\lambda_1} = -r\cos\varphi\sin\lambda_1, & \dfrac{\partial\xi^2}{\partial\lambda_1} = r\cos\varphi\cos\lambda_1, & \dfrac{\partial\xi^3}{\partial\lambda_1} = 0, \\ \dfrac{\partial\xi^1}{\partial\varphi} = -r\sin\varphi\cos\lambda_1, & \dfrac{\partial\xi^2}{\partial\varphi} = -r\sin\varphi\sin\lambda_1, & \dfrac{\partial\xi^3}{\partial\varphi} = r\cos\varphi.\end{array}\right| \tag{3.247}$$

In this case the Jacobian determinant is

$$J = \frac{D(\xi^1, \xi^2, \xi^3)}{D(r, \lambda_1, \varphi)} = r^2\cos\varphi \tag{3.248}$$

Since, clearly, $r \neq 0$, the reference grid degenerates only on the straight line $\varphi = \pm\pi/2$, on which $J = 0$.

The vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ of the main basis are equal to:

$$r_1 = \frac{\partial r}{\partial r}, \quad r_2 = \frac{\partial r}{\partial\lambda_1}, \quad r_3 = \frac{\partial r}{\partial\varphi}. \tag{3.249}$$

From expressions (3.249), (3.247) and (3.88) are found the diagonal elements of the metric tensor A:

$$a_{11} = 1, \quad a_{22} = r^2\cos^2\varphi, \quad a_{33} = r^2. \tag{3.250}$$

The nondiagonal elements of the metric tensor are equal to 0. The reference grid r, λ_1, φ is orthogonal.

The directions of $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ coincide with the directions of $\vec{r}$ tangential to the parallel and tangential to the meridian, respectively. The vectors $\vec{r}_2$ and $\vec{r}_3$ are directed, clearly, towards increasing λ_1 and φ.

The Lame coefficients follow from equalities (3.250)

$$h_1 = 1, \quad h_2 = r\cos\varphi, \quad h_3 = r. \tag{3.251}$$

Referring to relations (3.175), we find that for the case in question only the followingChristoffel symbols are non-zero:

$$
\left.\begin{array}{l}
\Gamma_{21,2}=\Gamma_{12,2}=r\cos^2\varphi, \quad \Gamma_{21}^{2}=\Gamma_{12}^{2}=\frac{1}{r}, \\
\Gamma_{23,2}=\Gamma_{32,2}=-r^2\sin\varphi\cos\varphi, \quad \Gamma_{23}^{2}=\Gamma_{32}^{2}=-\operatorname{tg}\varphi, \\
\Gamma_{31,3}=\Gamma_{13,3}=r, \quad \Gamma_{31}^{3}=\Gamma_{13}^{3}=\frac{1}{r}, \; \Gamma_{22,1}=-r\cos^2\varphi, \\
\Gamma_{22}^{1}=-r\cos^2\varphi, \quad \Gamma_{22,3}=r^2\sin\varphi\cos\varphi, \quad \Gamma_{22}^{3}=\sin\varphi\cos\varphi, \\
\Gamma_{33,1}=-r, \quad \Gamma_{33}^{1}=-r.
\end{array}\right| \tag{3.252}
$$

Because of the stability of the reference grid in question

$$\Gamma_{0k}^{s}=\Gamma_{00}^{s}=\Gamma_{0j}^{s}=0. \tag{3.253}$$

Now, substituting expressions (3.251), (3.252) and (3.253) into (3.210), we obtain equations for r, λ_1 and φ :

$$
\left.\begin{array}{l}
\dot{r}=\int_0^t\left[n_1+r(\dot{\varphi}^2+\dot{\lambda}_1^2\cos^2\varphi)+\operatorname{grad}^l V\eta_l^1\right]dt+\dot{r}(0), \\
r\dot{\lambda}_1\cos\varphi=\int_0^t\left[n_2-\dot{\lambda}_1(\dot{r}\cos\varphi-\dot{\varphi}\sin\varphi)+\right. \\
\qquad\left.+r\cos\varphi\operatorname{grad}^l V\eta_l^2\right]dt+r(0)\cos\varphi(0)\dot{\lambda}_1(0), \\
r\dot{\varphi}=\int_0^t\left[n_3-\dot{r}\dot{\varphi}-r\dot{\lambda}_1^2\sin\varphi\cos\varphi+\right. \\
\qquad\left.+r\operatorname{grad}^l V\eta_l^3\right]dt+r(0)\dot{\varphi}(0), \\
r=\int_0^t\dot{r}\,dt+r(0), \\
\lambda_1=\int_0^t\frac{1}{r\cos\varphi}(r\dot{\lambda}_1\cos\varphi)dt+\lambda_1(0), \\
\varphi=\int_0^t\frac{1}{r}(r\dot{\varphi})dt+\varphi(0).
\end{array}\right| \tag{2.354}
$$

Taking into account that the $O_1\eta^3$ and $O_1\xi^3$ axes coincide, and therefore that

$$u_\eta^1=u_\eta^2=0, \quad u_\eta^3=u, \tag{3.255}$$

together with relations (3.252) and (3.253), we obtain the following equations for η_ℓ^s from equalities (3.211):

$$\left.\begin{aligned}\eta_i^1 = -\int_0^t \Big[& -\eta_i^2 r\cos^2\varphi\,\dot{\lambda}_1 - \eta_i^3 r\dot{\varphi} + \\ & + u\left(\eta_i^2\eta_3^1 - \eta_i^1\eta_3^2\right) r^2\cos\varphi\Big]\,dt + \eta_i^1(0).\end{aligned}\right\}$$

$$\left.\begin{aligned}\eta_i^2 = -\int_0^t \Big[& \frac{\eta_i^1}{r}\dot{\lambda}_1 - \eta_i^3\dot{\lambda}_1\,\mathrm{tg}\,\varphi + \eta_i^2\left(\frac{\dot{r}}{r} - \dot{\varphi}\,\mathrm{tg}\,\varphi\right) + \\ & + u\left(\eta_i^3\eta_3^1 - \eta_i^1\eta_3^3\right)\frac{1}{\cos\varphi}\Big]\,dt + \eta_i^2(0). \\ \eta_i^3 = -\int_0^t \Big[& \eta_i^2\dot{\lambda}_1\sin\varphi\cos\varphi + \frac{\eta_i^1}{r}\dot{\varphi} + \frac{\eta_i^3}{r}\dot{r} + \\ & + u\cos\varphi\left(\eta_i^1\eta_3^2 - \eta_i^2\eta_3^1\right)\Big]\,dt + \eta_i^3(0).\end{aligned}\right\} \tag{3.256}$$

It is easily demonstrated by direct substitution that the following values of η_ℓ^s satisfy these equations:

$$\left.\begin{aligned} & \eta_1^1 = \cos\varphi\cos(\lambda_1 - ut), \quad \eta_1^2 = -\frac{\sin(\lambda_1 - ut)}{r\cos\varphi}, \\ & \eta_1^3 = -\frac{\sin\varphi\cos(\lambda_1 - ut)}{r}, \\ & \eta_2^1 = \cos\varphi\sin(\lambda_1 - ut), \quad \eta_2^2 = \frac{\cos(\lambda_1 - ut)}{r\cos\varphi}, \\ & \eta_2^3 = -\frac{\sin\varphi\sin(\lambda_1 - ut)}{r}, \\ & \eta_3^1 = \sin\varphi, \quad \eta_3^2 = 0, \quad \eta_3^3 = \frac{\cos\varphi}{r}.\end{aligned}\right\} \tag{3.257}$$

We note that the values (3.257) of η_ℓ^s may also be obtained directly from relations (3.88), (3.247), (3.251), (3.249) and (3.231), without reference to equations (3.211), by using the equalities

$$\eta_l^s = \frac{\eta_l \cdot r_s}{L_s^2}. \tag{3.258}$$

From relations (3.246) and (3.212) we also obtain:

$$\eta^s = \eta_k^s r.$$

or, taking into account relations (3.257):

$$\left.\begin{aligned}\eta^1 = r\cos\varphi\cos(\lambda_1 - ut), \quad \eta^2 = r\cos\varphi\sin(\lambda_1 - ut), \\ \eta^3 = r\sin\varphi.\end{aligned}\right\} \tag{3.259}$$

Finally, from (3.213) we obtain the direction cosines:

	ξ_1	ξ_2	ξ_3
e_1	$\cos\varphi\cos\lambda_1$	$\cos\varphi\sin\lambda_1$	$\sin\varphi$
e_2	$-\sin\lambda_1$	$\cos\lambda_1$	0
e_3	$-\sin\varphi\cos\lambda_1$	$-\sin\varphi\sin\lambda_1$	$\cos\varphi.$

(3.260)

Since for the case under consideration the vectors $\vec{e}_s$ are unit vectors of direction $\vec{r}_s$, table (3.260) also defines the directions of $\vec{r}_s$ relative to the ξ^1, ξ^2, ξ^3 axes.

If equalities (3.184) are used in place of relations (3.211) and (3.212), we obtain from the first group of formulas (3.184) for the direction cosines $\alpha_\ell^{\cdot k}$ the table

	η^1	η^2	η^3
ξ^1	$\cos ut$	$-\sin ut$	0
ξ^2	$\sin ut$	$\cos ut$	0
ξ^3	0	0	1.

In this table the superscript of $\alpha_\ell^{\cdot k}$ corresponds to the columns, and the subscript to the rows. It is easily shown that expressions (3.257) for η_ℓ^s derive from equalities (3.251) and (3.257), from the second group of equations (3.184), and from the above table.

The projections $\omega_{(1)}$, $\omega_{(2)}$, and $\omega_{(3)}$ of the absolute angular velocity of the trihedron $\vec{e}_1\vec{e}_2\vec{e}_3$ or, equivalently, the trihedron $\vec{r}_1\vec{r}_2\vec{r}_3$, are found from expressions (3.205). They are:

$$\omega_{(1)} = \dot{\lambda}_1\sin\varphi, \quad \omega_{(2)} = -\dot{\varphi}, \quad \omega_{(3)} = \dot{\lambda}_1\cos\varphi \tag{3.261}$$

Now, from formulas (3.209)

$$M^1_{1} = H\dot{\lambda}_1\cos\varphi, \quad M^3_{1} = H\dot{\varphi}, \quad M^2_{1} = H\dot{\lambda}_1\sin\varphi. \tag{3.262}$$

Thus, for the reference grid under consideration and for the case of a free gyrostabilized platform as the basis of the functional diagram the operational algorithm of the inertial system will consist of equations (3.254), (3.257) and (3.259), and table (3.260). For the case of a maneuverable gyrostabilized platform as the basis of the functional diagram, the ideal equations are (3.254), (3.257), (3.259), (3.261) and (3.262).

We note that, if the earth's gravitational field is considered to be spherical, then according to expressions (3.13), (3.257) and (3.259), in equalities (3.254)

$$\operatorname{grad}^{l} V \eta_l^{s} = -\frac{\mu}{r^3} \eta^{l} \eta_l^{s} \tag{3.263}$$

and, consequently,

$$\operatorname{grad}^{l} V \eta_l^{1} = -\frac{\mu}{r^2}, \quad \operatorname{grad}^{l} V \eta_l^{2} = \operatorname{grad}^{l} V \eta_l^{3} = 0. \tag{3.264}$$

It follows from relations (3.264) that for a spherical gravitational field formulas (3.257) and (3.259) drop out of the ideal equations.

If we assume that the vector $\vec{g}$ of the strength of the earth's gravitational field lies in the plane of the meridian, then, using g_0^{η} to designate its projection on the $\eta^1\eta^2$ plane (the plane of the earth's equator), we may represent the projections of the vector $\vec{g}$ on the η^1 and η^2 axes as follows:

$$\left.\begin{aligned} \operatorname{grad}^{1} V &= g_0^{\eta} \cos(\lambda_1 - ut), \\ \operatorname{grad}^{2} V &= g_0^{\eta} \sin(\lambda_1 - ut). \end{aligned}\right\} \tag{3.265}$$

Then, introducing g_3^{η} to designate the projection of g on the η^3 axis, i.e., taking

$$\operatorname{grad}^{3} V = g_3^{\eta},$$

and taking into account equalities (3.257), we obtain for the sums $\operatorname{grad}^{l} V \eta_l^{s}$ entering into the integrands of (3.254), the following formulas:

$$\left.\begin{aligned} \operatorname{grad}' V \eta_1^1 &= g_0^\eta \cos\varphi + g_3^\eta \sin\varphi = g^1, \\ \operatorname{grad}' V \eta_1^2 &= 0, \\ \operatorname{grad}' V \eta_1^3 &= \frac{1}{r}\left(-g_0^\eta \sin\varphi + g_3^\eta \cos\varphi\right) = \frac{g^3}{r}. \end{aligned}\right\} \tag{3.266}$$

In these equalities g^1 and g^3 are the projections of the strength vector of the earth's gravitational field on the direction of r and the direction of $\vec{r}_3$, normal to $\vec{r}$ and lying in the plane of the earth's meridian.

The functions g_0^η and g_3^η, and, consequently, g^1 and g^3, are functions only of r and φ. In terms of the notation introduced in §2.2,

$$g^1 = F_{\eta_1}, \quad g^3 = F_{\eta_3}.$$

3.3.3. Geocentric coordinates. Now let r, λ and φ be ordinary geocentric coordinates. Their determination is analogous to that of the coordinates r, λ_1, and φ. Only now r, λ and φ are referred not to the trihedron $O_1\xi^1\xi^2\xi^3$, but to $O_1\eta^1\eta^2\eta^3$, rigidly bound to the earth.

Therefore in relation to $O_1\xi^1\xi^2\xi^3$ the coordinates r, λ, φ will not be stationary. Considering, as before, that the $O_1\xi^3$ and $O_1\eta^3$ axes are oriented along the vector $\vec{u}$ of the absolute earth rate and that $O_1\xi^1\xi^2\xi^3$ coincides with $O_1\eta^1\eta^2\eta^3$ at some initial moment of time, we find from equalities (3.232), (3.246), (3.247) and (3.251) the non-zero symbols Γ_{0k}^{s}, $\Gamma_{0k,s}$ and $\Gamma_{00,s}$, Γ_{00}^{s}:

$$\left.\begin{aligned} &\Gamma_{00,1} = -u^2 r\cos^2\varphi, \quad \Gamma_{00}^1 = -u^2 r\cos^2\varphi, \\ &\Gamma_{00,3} = u^2 r^2 \sin\varphi\cos\varphi, \quad \Gamma_{00}^3 = u^2 \sin\varphi\cos\varphi, \\ &\Gamma_{01,2} = -\Gamma_{02,1} = ur\cos^2\varphi, \quad \Gamma_{01}^2 = \frac{u}{r}, \\ &\Gamma_{02}^1 = -ur\cos^2\varphi, \\ &\Gamma_{02,3} = -\Gamma_{03,2} = u^2 r\sin\varphi\cos\varphi, \\ &\Gamma_{02}^3 = u\sin\varphi\cos\varphi, \\ &\Gamma_{03}^2 = -u\operatorname{tg}\varphi. \end{aligned}\right\} \tag{3.267}$$

The Christoffel symbols (3.252) calculated for the preceding reference grid clearly remain valid.

From the first three equations (3.210) and formulas (3.251), (3.252) and (3.267) we now obtain in place of the first three formulas (3.254) the following equations:

$$\begin{aligned}
\dot{r} &= \int_0^t \{n_1 + r[\dot{\varphi}^2 + (u+\dot{\lambda})^2\cos^2\varphi] + \\
&\qquad + \operatorname{grad}^t V \eta_i^1\}\, dt + \dot{r}(0), \\
r\dot{\lambda}\cos\varphi &= \int_0^t \{n_2 - (\dot{\lambda} + 2u)(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi) + \\
&\qquad + r\cos\varphi \operatorname{grad}^t V \eta_i^2\}\, dt + r(0)\cos\varphi(0)\,\dot{\lambda}(0), \\
r\dot{\varphi} &= \int_0^t \{n_3 - [\dot{r}\dot{\varphi} + r(\dot{\lambda}+u)^2\sin\varphi\cos\varphi] + \\
&\qquad + r\operatorname{grad}^t V \eta_i^3\}\, dt + r(0)\dot{\varphi}(0).
\end{aligned} \tag{3.268}$$

They differ from the first three equations (3.254) only in that $\dot{\lambda} + u$ replaces $\dot{\lambda}_1$ everywhere in the integrands. The last three equations (3.254) remain valid for the case under consideration.

In equations (3.256) it is also necessary to replace $\dot{\lambda}_1$ by $\dot{\lambda} + u$, and to substitute λ for $\lambda_1 - ut$ in relations (3.257) and (3.259). Table (3.260) now defines the direction cosines of the vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ relative to the η^1, η^2, η^3 axes. In order to obtain the direction cosines of the vectors $\vec{e}_s$ relative to the ξ^1, ξ^2, ξ^3 axes, it is necessary, clearly, to replace λ_1 by $\lambda + ut$ in table (3.260).

From expressions (3.205), (3.251), (3.252) and (3.267) are found $\omega_{(1)}$, $\omega_{(2)}$ and $\omega_{(3)}$. They are:

$$\begin{aligned}
\omega_{(1)} &= (\dot{\lambda} + u)\sin\varphi, \quad \omega_{(2)} = -\dot{\varphi}, \\
\omega_{(3)} &= (\dot{\lambda} + u)\cos\varphi.
\end{aligned} \tag{3.269}$$

Correspondingly

$$\begin{aligned}
M^1_{1x} &= H\omega_{(3)} = H(\dot{\lambda} + u)\cos\varphi, \quad M^1_{1y} = -H\omega_{(2)} = H\dot{\varphi}, \\
M^1_{1z} &= H\omega_{(1)} = H(\dot{\lambda} + u)\sin\varphi.
\end{aligned} \tag{3.270}$$

Expressions (3.269) and (3.270) also differ from expressions (3.261) and (3.262) only in that $\dot{\lambda} + u$ appears in place of $\dot{\lambda}_1$.

For a spherical gravitational field formulas (3.264) remain valid, and if we assume that the vector $\vec{g}$ lies in the plane of the meridian, formulas (3.266) also remain valid, since λ_1 does not enter into them.

3.3.4. <u>Geodetic coordinates.</u> The geocentric reference grid considered above has a singularity at $\varphi = \pm\pi/2$, i.e., at the poles. Realization of the ideal equations in this reference grid gives rise to considerable difficulty, if the object with which the inertial system is associated is moving in the immediate vicinity of the earth's poles.

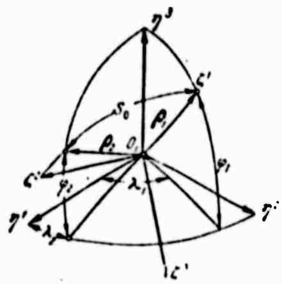

Figure 3.3

These difficulties make it expedient to convert to the so-called geodetic coordinate system. This coordinate system is also spherical. It is analogous to the geocentric coordinate system. The only difference is that the pole of the geodetic coordinate system does not coincide with the pole of the earth. Its position is selected such that it lies outside of the area of possible motion of the object with which the inertial system is associated.

It follows from the above description of geodetic coordinates that an geodetic system stationary relative to the trihedron $O_1\xi^1\xi^2\xi^3$ does not differ in any way from a stationary geocentric coordinate system. Using the arbitrariness of the orientation of the ξ^1, ξ^2, ξ^3 axes, we may always superpose the ξ^3 axis on the polar axis of a stationary geodetic coordinate system.

The first system which we will consider, therefore, will be a geodetic coordinate system rigidly bound to the earth.

Let us define it as follows. Let, in $O_1\eta^1\eta^2\eta^3$ rigidly bound to the earth and the $O_1\eta^3$ axis of which is directed along the earth's axis, two lines emerge from the point O_1, the unit vectors of which we will designate as $\vec{\rho}_1$ and $\vec{\rho}_2$ (Figure 3.3). Let us attach the geodetic (right orthogonal) trihedron $O_1\zeta^1\zeta^2\zeta^3$ to these directions, specifying the unit vectors $\vec{\zeta}^1$, $\vec{\zeta}^2$, and $\vec{\zeta}^3$ of its axes by means of the equalities:

$$\zeta_1 = \rho_1, \quad \zeta_2 = \frac{\rho_2 - \rho_1 \cos S_0}{\sin S_0}, \quad \zeta_3 = \frac{\rho_1 \times \rho_2}{\sin S_0}. \tag{3.271}$$

where S_0 is the constant angle between the vectors $\vec{\rho}_1$ and $\vec{\rho}_2$ such that

$$\cos S_0 = \rho_1 \cdot \rho_2.$$

We will consider that the vectors $\vec{\rho}_1$ and $\vec{\rho}_2$ are defined relative to the trihedron $O_1\eta^1\eta^2\eta^3$ by the geocentric coordinates φ_1, λ_1 and φ_2, λ_2. Then

$$\left.\begin{aligned} \rho_1 &= \eta_1 \cos\varphi_1 \cos\lambda_1 + \eta_2 \cos\varphi_1 \sin\lambda_1 + \eta_3 \sin\varphi_1, \\ \rho_2 &= \eta_1 \cos\varphi_2 \cos\lambda_2 + \eta_2 \cos\varphi_2 \sin\lambda_2 + \eta_3 \sin\varphi_2. \end{aligned}\right\} \tag{3.272}$$

Let δ_{ij} be the direction cosines between the η^1, η^2, η^3 axes and the ζ^1, ζ^2, ζ^3 axes, forming the table:

$$\begin{array}{cccc} & \zeta^1 & \zeta^2 & \zeta^3 \\ \eta^1 & \delta_{11} & \delta_{12} & \delta_{13} \\ \eta^2 & \delta_{21} & \delta_{22} & \delta_{23} \\ \eta^3 & \delta_{31} & \delta_{32} & \delta_{33}. \end{array} \tag{3.273}$$

Then, in accordance with relations (3.272) and (3.271), the elements of this table may be expressed in terms of φ_1, λ_1, φ_2, and λ_2 as follows:

$$\left.\begin{aligned} \delta_{11} &= \cos\varphi_1 \cos\lambda_1, \\ \delta_{21} &= \cos\varphi_1 \sin\lambda_1, \\ \delta_{31} &= \sin\varphi_1, \\ \delta_{12} &= \frac{1}{\sin S_0} (\cos\varphi_2 \cos\lambda_2 - \cos\varphi_1 \cos\lambda_1 \cos S_0), \\ \delta_{22} &= \frac{1}{\sin S_0} (\cos\varphi_2 \sin\lambda_2 - \cos\varphi_1 \sin\lambda_1 \cos S_0), \end{aligned}\right| \tag{3.274}$$

$$
\left.\begin{aligned}
\delta_{32} &= \frac{1}{\sin S_0}(\sin\varphi_2 - \sin\varphi_1\cos S_0),\\
\delta_{13} &= \frac{1}{\sin S_0}(\cos\varphi_1\sin\varphi_2\sin\lambda_1 - \cos\varphi_2\sin\varphi_1\sin\lambda_2),\\
\delta_{23} &= \frac{1}{\sin S_0}(\sin\varphi_1\cos\varphi_2\cos\lambda_2 - \cos\varphi_1\sin\varphi_2\cos\lambda_1),\\
\delta_{33} &= \frac{1}{\sin S_0}\cos\varphi_1\cos\varphi_2\sin(\lambda_2 - \lambda_1),
\end{aligned}\right|
\tag{3.274}
$$

where according to the relation $\cos S_0 = \vec{\rho}_1 \cdot \vec{\rho}_2$ and formulas (3.272)

$$
\cos S_0 = \cos\varphi_1\cos\varphi_2\cos(\lambda_2 - \lambda_1) + \sin\varphi_1\sin\varphi_2.
\tag{3.275}
$$

If we introduce the angle ψ_0 between the geodetic plane, i.e., the $\zeta^1\zeta^2$ plane, and the tangent to the geocentric meridian at the point defined by the coordinates φ_1 and λ_1, such that

$$
\left.\begin{aligned}
\sin\psi_0 &= \frac{\cos\varphi_2\sin(\lambda_2 - \lambda_1)}{\sin S_0},\\
\cos\psi_0 &= \frac{\sin\varphi_2 - \sin\varphi_1\cos S_0}{\cos\varphi_1\sin S_0},
\end{aligned}\right|
\tag{3.276}
$$

then the expressions for δ_{i2} and δ_{i3} may be represented differently than in formulas (3.274):

$$
\left.\begin{aligned}
\delta_{12} &= -\sin\varphi_1\cos\lambda_1\cos\psi_0 - \sin\lambda_1\sin\psi_0,\\
\delta_{22} &= -\sin\varphi_1\sin\lambda_1\cos\psi_0 + \cos\lambda_1\sin\psi_0,\\
\delta_{32} &= \cos\varphi_1\cos\psi_0,\\
\delta_{13} &= -\sin\varphi_1\cos\lambda_1\sin\psi_0 + \cos\psi_0\sin\lambda_1,\\
\delta_{23} &= -\sin\varphi_1\sin\lambda_1\sin\psi_0 - \cos\lambda_1\cos\psi_0,\\
\delta_{33} &= \cos\varphi_1\sin\psi_0
\end{aligned}\right|
\tag{3.277}
$$

We will define the position of the arbitrary point O in relation to the geodetic trihedron $O_1\zeta^1\zeta^2\zeta^3$ by means of its distance $\varkappa^1 = r$ from the origin O_1 of the trihedron and the two angles $\varkappa^2 = S$ and $\varkappa^3 = z$ (Figure 3.4). The angle S is measured in the $O_1\zeta^1\zeta^2$ plane from the $O_1\zeta^1$ axis in the direction of the $O_1\zeta^2$ axis, and the angle Z is measured from the $O_1\zeta^1\zeta^2$ plane in the direction of the $O_1\zeta^3$ axis. Thus, the angles S and Z are analogous to λ and φ, by means of which we define the arbitrary direction in the trihedron $\eta^1\eta^2\eta^3$. Specifically, if the trihedra $\eta^1\eta^2\eta^3$ and $\zeta^1\zeta^2\zeta^3$ coincide, then angles S and Z reduce to angles λ and φ.

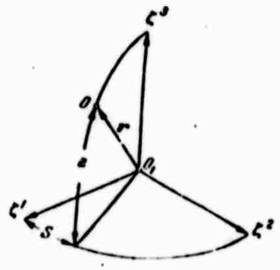

Figure 3.4

In the geodetic coordinate system introduced above, the polar axis $O_1\zeta^3$ may be selected such that the object moves far away from it, for example, in the vicinity of the $O_1\zeta^1\zeta^2$ plane. The angles S and Z may be termed the <u>distance along the geodetic</u> and the <u>distance from the geodetic.</u>

In analogy with relations (3.89) we have:

$$\left.\begin{aligned} \zeta^1 &= r\cos z\cos S, \\ \zeta^2 &= r\cos z\sin S, \\ \zeta^3 &= r\sin z. \end{aligned}\right\} \qquad (3.278)$$

Thus, the vector $\vec{r}$ in the $O_1\zeta^1\zeta^2\zeta^3$ coordinate system has the form:

$$\begin{aligned} r = {} & \zeta_1 r\cos z\cos S + \\ & + \zeta_2 r\cos z\sin S + \\ & + \zeta_3 r\sin z \end{aligned} \qquad (3.279)$$

Using relations (3.278) and (3.231), and table (3.273), we find:

$$\left.\begin{aligned} \xi^1 = {} & r(\delta_{11}\cos z\cos S + \delta_{12}\cos z\sin S + \delta_{13}\sin z)\cos ut - \\ & - r(\delta_{21}\cos z\cos S + \delta_{22}\cos z\sin S + \delta_{23}\sin z)\sin ut, \\ \xi^2 = {} & r(\delta_{11}\cos z\cos S + \delta_{12}\cos z\sin S + \delta_{13}\sin z)\sin ut + \\ & + r(\delta_{21}\cos z\cos S + \delta_{22}\cos z\sin S + \delta_{23}\sin z)\cos ut, \\ \xi^3 = {} & r(\delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z). \end{aligned}\right\} \qquad (3.280)$$

The coordinates r, S and z are orthogonal. It is obvious that the Lame coefficients have the same values as in equalities (3.251). It is necessary only to replace φ by z. Therefore

$$h_1 = 1, \quad h_2 = r\cos z, \quad h_3 = r. \qquad (3.281)$$

Analogously, from formulas (3.252) expressions for the non-zero Christoffel symbols may be obtained:

$$
\begin{array}{ll}
\Gamma_{21,2}=\Gamma_{12,2}=r\cos^2 z, & \Gamma_{21}^2=\Gamma_{12}^2=\frac{1}{r},\\
\Gamma_{23,2}=\Gamma_{32,2}=-r^2\sin z\cos z, & \Gamma_{23}^2=\Gamma_{32}^2=-\operatorname{tg} z,\\
\Gamma_{31,3}=\Gamma_{13,3}=r, & \Gamma_{31}^3=\Gamma_{13}^3=\frac{1}{r},\\
\Gamma_{22,1}=-r\cos^2 z, & \Gamma_{22}^1=-r\cos^2 z,\\
\Gamma_{22,3}=r^2\sin z\cos z, & \Gamma_{22}^3=\sin z\cos z,\\
\Gamma_{33,1}=-r. & \Gamma_{33}^1=-r.
\end{array}
\qquad (3.282)
$$

To write the ideal equations of an inertial system operating in geodetic coordinates in accordance with relations (3.210) -- (3.213), only the symbols Γ_{00}^s, Γ_{0k}^s and Γ_{0s}^s remain to be found.

From (3.280) we have:

$$
\begin{array}{l}
\frac{\partial\xi^1}{\partial t}=-u\xi^2,\quad \frac{\partial\xi^2}{\partial t}=u\xi^1,\quad \frac{\partial\xi^3}{\partial t}=0,\\
\frac{\partial^2\xi^1}{\partial t^2}=-u^2\xi^1,\quad \frac{\partial^2\xi^2}{\partial t^2}=-u^2\xi^2,\quad \frac{\partial^2\xi^3}{\partial t^2}=0,\\
\frac{\partial\xi^1}{\partial r}=\frac{\xi^1}{r},\quad \frac{\partial\xi^2}{\partial r}=\frac{\xi^2}{r},\quad \frac{\partial\xi^3}{\partial r}=\frac{\xi^3}{r},\\
\frac{\partial\xi^1}{\partial S}=r(-\delta_{11}\cos z\sin S+\delta_{12}\cos z\cos S)\cos ut-\\
\qquad -r(-\delta_{21}\cos z\sin S+\delta_{22}\cos z\cos S)\sin ut,\\
\frac{\partial\xi^2}{\partial S}=r(-\delta_{11}\cos z\sin S+\delta_{12}\cos z\cos S)\sin ut+\\
\qquad +r(-\delta_{21}\cos z\sin S+\delta_{22}\cos z\cos S)\cos ut,\\
\frac{\partial\xi^3}{\partial S}=r(-\delta_{31}\cos z\sin S+\delta_{32}\cos z\cos S),\\
\frac{\partial^2\xi^1}{\partial t\,\partial r}=-\frac{u}{r}\xi^2,\quad \frac{\partial^2\xi^2}{\partial t\,\partial r}=\frac{u}{r}\xi^1,\quad \frac{\partial^2\xi^3}{\partial t\,\partial r}=0,\\
\frac{\partial^2\xi^1}{\partial t\,\partial S}=-u\frac{\partial\xi^2}{\partial S},
\end{array}
$$

$$
\begin{array}{l}
\frac{\partial^2\xi^2}{\partial t\,\partial S}=u\frac{\partial\xi^1}{\partial S},\\
\frac{\partial^2\xi^3}{\partial t\,\partial S}=0,\\
\frac{\partial^2\xi^1}{\partial t\,\partial z}=-ur(-\delta_{11}\sin z\cos S-\delta_{12}\sin z\sin S+\\
\qquad +\delta_{13}\cos z)\sin ut-ur(-\delta_{21}\sin z\cos S-\\
\qquad\qquad -\delta_{22}\sin z\sin S+\delta_{23}\cos z)\cos ut,\\
\frac{\partial^2\xi^2}{\partial t\,\partial z}=ur(-\delta_{11}\sin z\cos S-\delta_{12}\sin z\sin S+\\
\qquad +\delta_{13}\cos z)\cos ut-ur(-\delta_{21}\sin z\cos S-\\
\qquad\qquad -\delta_{22}\sin z\sin S+\delta_{23}\cos z)\sin ut.
\end{array}
\qquad (3.283)
$$

Now, using the definition (3.130) of the symbols $\Gamma_{00,s}$ and $\Gamma_{0k,s}$ we find:

$$
\left.
\begin{aligned}
\Gamma_{00,1} &= -u^2 r\{1 - (\delta_{31}\cos z\cos S + \\
&\qquad + \delta_{32}\cos z\sin S + \delta_{33}\sin z)^2\}, \\
\Gamma_{00,2} &= -u^2 r^2\{(\delta_{31}^2 - \delta_{32}^2)\cos^2 z\cos S\sin S + \\
&\qquad + \delta_{31}\delta_{32}\cos^2 z(\sin^2 S - \cos^2 S) + \\
&\qquad + \delta_{33}(\delta_{31}\sin S - \delta_{32}\cos S)\sin z\cos z\}, \\
\Gamma_{00,3} &= -u^2 r^2\{(\delta_{31}\cos S + \delta_{32}\sin S)[(\delta_{31}\cos S + \\
&\qquad + \delta_{32}\sin S)\sin z\cos z + \delta_{33}(\sin^2 z - \cos^2 z)] - \\
&\qquad - \delta_{33}^2\sin z\cos z\}, \\
\Gamma_{01,2} &= -\Gamma_{02,1} = ur\{\delta_{33}\cos^2 z - \\
&\qquad - (\delta_{31}\cos S + \delta_{32}\sin S)\sin z\cos z\}, \\
\Gamma_{01,3} &= -\Gamma_{03,1} = ur(\delta_{31}\sin S - \delta_{32}\cos S), \\
\Gamma_{02,3} &= -\Gamma_{03,2} = ur^2\{\delta_{33}\sin z\cos z + \\
&\qquad + (\delta_{32}\sin S + \delta_{31}\cos S)\cos^2 z\}.
\end{aligned}
\right| \tag{3.284}
$$

The remaining symbols of this type are equal to zero.

Expressions (3.284) were found directly from the definition of the symbols $\Gamma_{00,s}$ and $\Gamma_{0k,s}$. They can also be calculated from formulas (3.232). To do this it is necessary only to note that, in accordance with (3.278) and table (3.273)

$$
\begin{aligned}
\eta^1 &= r(\delta_{11}\cos z\cos S + \delta_{12}\cos z\sin S + \delta_{13}\sin z), \\
\eta^2 &= r(\delta_{21}\cos z\cos S + \delta_{22}\cos z\sin S + \delta_{23}\sin z), \\
\eta^3 &= r(\delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z).
\end{aligned}
$$

Calculations performed in accordance with formulas (3.232) also lead, of course, to expressions (3.284).

From (3.284) on the basis of formulas (3.126), (3.174) and (3.281), we further find:

$$
\left.
\begin{aligned}
&\Gamma_{00}^1 = \Gamma_{00,1}, \\
&\Gamma_{00}^2 = \frac{1}{r^2\cos^2 z}\Gamma_{00,2}, \quad \Gamma_{00}^3 = \frac{1}{r^2}\Gamma_{00,3}, \\
&\Gamma_{01}^2 = \frac{1}{r^2\cos^2 z}\Gamma_{01,2}, \quad \Gamma_{01}^3 = \frac{1}{r^2}\Gamma_{01,3}, \\
&\Gamma_{02}^1 = -\Gamma_{01,2}, \qquad\quad\;\; \Gamma_{03}^1 = -\Gamma_{01,3}, \\
&\Gamma_{02}^3 = \frac{1}{r^2}\Gamma_{02,3}, \qquad\quad \Gamma_{03}^2 = -\frac{1}{r^2\cos^2 z}\Gamma_{02,3}
\end{aligned}
\right| \tag{3.285}
$$

If the geodetic trihedron $O_1\zeta^1\zeta^2\zeta^3$ coincides with $O_1\eta^1\eta^2\eta^3$, in table (3.273) the nondiagonal elements are equal to zero, while the diagonal elements are equal to one. It is evident that in this case expressions (3.284) and (3.285) for the symbols $\Gamma_{0s,k}$ and Γ_{0s}^{k} become equal to the corresponding expressions (3.267), since for the case of the correspondence of $O_1\zeta^1\zeta^2\zeta^3$ and $O_1\eta^1\eta^2\eta^3$ the angle z reduces to the angle ϕ.

Substituting expressions (3.281), (3.282), (3.284) and (3.285) into equations (3.210) and grouping terms as required, we arrive at the following relations describing the algorithm for determination of the coordinates r, S and z by the inertial system:

$$
\left.
\begin{aligned}
\dot{r} = \int_0^t \{ & n_1 + r[\dot{z} + u(\delta_{31}\sin S - \delta_{32}\cos S)]^2 + \\
& + r[\dot{S}\cos z + u(-\delta_{31}\sin z\cos S - \delta_{32}\sin z\sin S + \\
& + \delta_{33}\cos z)]^2 + \operatorname{grad}' V'\eta_i^1 \}\, dt + \dot{r}(0). \\
r\dot{S}\cos z = \int_0^t \{ & n_2 + r\dot{z}\dot{S}\sin z - \dot{r}\dot{S}\cos z - \\
& - 2u\dot{r}[\delta_{33}\cos z - \sin z(\delta_{31}\cos S + \delta_{32}\sin S)] + \\
& + 2ur\dot{z}[\delta_{33}\sin z + \cos z(\delta_{32}\sin S + \delta_{31}\cos S)] + \\
& + u^2 r[(\delta_{31}^2 - \delta_{32}^2)\cos z\cos S\sin S + \\
& + \delta_{31}\delta_{32}\cos z(\sin^2 S - \cos^2 S) + \\
& + \delta_{33}\sin z(\delta_{31}\sin S - \delta_{32}\cos S)] + \\
& + r\cos z\operatorname{grad}' V'\eta_i^2 \}\, dt + r(0)\dot{S}(0)\cos z(0). \\
r\dot{z} = \int_0^t \{ & n_3 - \dot{r}\dot{z} - r\dot{S}^2\sin z\cos z - \\
& - 2\dot{r}u(\delta_{31}\sin S - \delta_{32}\cos S) - 2ru\dot{S}[\delta_{33}\sin z\cos z + \\
& + \cos^2 z(\delta_{31}\sin S + \delta_{31}\cos S)] + u^2 r[(\delta_{31}\cos S + \\
& + \delta_{32}\sin S)(\sin z\cos z(\delta_{31}\cos S + \delta_{32}\sin S) + \\
& + \delta_{33}(\sin^2 z - \cos^2 z)) - \delta_{33}^2\sin z\cos z] + \\
& + r\operatorname{grad}' V'\eta_i^3 \}\, dt + r(0)\dot{z}(0). \\
r = \int_0^t & \dot{r}\, dt + r(0),\quad S = \int_0^t \frac{1}{r\cos z}(r\dot{S}\cos z)\, dt + S(0), \\
z = \int_0^t & \frac{1}{r}(r\dot{z})\, dt + z(0).
\end{aligned}
\right| \qquad (3.286)
$$

We note that the second and third equations (3.286) may be partially integrated. Simultaneously regrouping the expressions contained in these equations, we obtain:

$$
\left.
\begin{aligned}
& r[\dot{S}\cos z + u(-\delta_{31}\sin z\cos S - \\
& -\delta_{32}\sin z\sin S + \delta_{33}\cos z)] = \\
& = \int_0^t \{n_2 + r[\dot{z} + u(\delta_{31}\sin S - \delta_{32}\cos S)][\dot{S}\sin z + \\
& + u(\delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z)] - \\
& - \dot{r}[\dot{S}\cos z + u(-\delta_{31}\sin z\cos S - \delta_{32}\sin z\sin S + \\
& + \delta_{33}\cos z)] + r\cos z\,\mathrm{grad}' V\eta_i^2\}\,dt + \\
& + r(0)[\dot{S}(0)\cos z(0) + u[-\delta_{31}\sin z(0)\cos S(0) - \\
& \qquad - \delta_{32}\sin z(0)\sin S(0) + \delta_{33}\cos z(0)]], \\
& r[\dot{z} + u(\delta_{31}\sin S - \delta_{32}\cos S)] = \\
& = \int_0^t \{n_3 - r[\dot{S}\sin z + u(\delta_{31}\cos z\cos S + \\
& + \delta_{32}\cos z\sin S + \delta_{33}\sin z)][\dot{S}\cos z + \\
& + u(-\delta_{31}\sin z\cos S - \delta_{32}\sin z\sin S + \delta_{33}\cos z)] - \\
& - \dot{r}[\dot{z} + u(\delta_{31}\sin S - \delta_{32}\cos S)] + \\
& + r\,\mathrm{grad}' V\eta_i^3\}\,dt + r(0)[\dot{z}(0) + \\
& \qquad + u[\delta_{31}\sin S(0) - \delta_{32}\cos S(0)]].
\end{aligned}
\right| \qquad (3.287)
$$

The fifth and sixth equations of system (3.286) may be similarly altered. We note also that, since equations (3.268) are a special case of equations (3.286), partial integration is also possible in equations (3.268), as is their representation in the form of (3.287).

Form (3.287) of the ideal equations is convenient in that the groups of variables contained in them may be simply expressed in terms of the projections of the absolute angular velocity of the trihedron $\vec{r}_1\vec{r}_2\vec{r}_3$ on its axes.

Indeed, from expressions (3.205), (3.281), (3.282) and (3.284)

$$
\left.
\begin{aligned}
\omega_{(1)} &= \dot{S}\sin z + u(\delta_{31}\cos z\cos S + \\
& \qquad + \delta_{32}\cos z\sin S + \delta_{33}\sin z), \\
\omega_{(2)} &= \dot{z} + u(-\delta_{31}\sin S + \delta_{32}\cos S), \\
\omega_{(3)} &= \dot{S}\cos z + \\
& + u(\delta_{33}\cos z - \delta_{31}\sin z\cos S - \delta_{32}\sin z\sin S)
\end{aligned}
\right| \qquad (3.288)
$$

Therefore, substituting relations (3.288) into the first equation (3.286) and into both equations (3.287), we may represent them in the following form:

$$\left.\begin{aligned}
\dot{r} &= \int_0^t \{n_1 + r[\omega_{(2)}^2 + \omega_{(3)}^2] + \operatorname{grad}^i V \eta_i^1\}\, dt + \dot{r}(0), \\
r\omega_{(3)} &= \int_0^t [n_2 - r\omega_{(2)}\omega_{(1)} - \dot{r}\omega_{(3)} + \\
&\quad + r \cos z \operatorname{grad}^i V \eta_i^2]\, dt + r(0)\,\omega_{(3)}(0), \\
-r\omega_{(2)} &= \int_0^t [n_3 - r\omega_{(1)}\omega_{(3)} + \dot{r}\omega_{(2)} + \\
&\quad + r \operatorname{grad}^i V \eta_i^3]\, dt - r(0)\,\omega_{(2)}(0).
\end{aligned}\right| \qquad (3.289)$$

In doing this, clearly, in place of the last three equations (3.286) we must write:

$$\left.\begin{aligned}
r &= \int_0^t \dot{r}\, dt + r(0), \\
S &= \int_0^t \left\{ \frac{[r\omega_{(3)}]}{r \cos z} - \frac{u}{\cos z} (\delta_{33} \cos z - \delta_{31} \sin z \cos S - \right. \\
&\quad \left. - \delta_{32} \sin z \sin S) \right\} dt + S(0), \\
z &= \int_0^t \left\{ \frac{[-r\omega_{(2)}]}{r} + \right. \\
&\quad \left. + u(-\delta_{31} \sin S + \delta_{32} \cos S) \right\} dt + z(0).
\end{aligned}\right| \qquad (3.290)$$

Equations (3.289) and (3.290) are easily compared with equations (3.82), obtained in §3.1 for the case of Cartesian coordinates, with a moving trihedron on a sphere surrounding the earth taken as the trihedron Oxyz. It is necessary only to note that for the spherical coordinates r, S and z the following relations obtain:

$$v_{(1)} = \dot{r}, \quad v_{(2)} = r\omega_{(3)}, \quad v_{(3)} = -r\omega_{(2)}, \qquad (3.291)$$

with the x, v, z axes coinciding with the directions of the vectors $\vec{r}_2$, $\vec{r}_3$ and $\vec{r}_1$ respectively.

In equations (3.289), as in (3.82), the first integration gives the projections of the absolute angular velocity of the moving trihedron on its axes. From this point on, however, the solutions differ. In equations (3.82) the second integration is performed along the axes of the moving trihedron Oxyz and the Cartesian coordinates x = y = 0 and z = r are found the moving coordinate system O_1xyz. Then the projections of the absolute angular velocity on the axes of the moving trihedron are used to find the direction cosines of the x, y, z axes in relation to the ξ_*, η_*, ζ_* (ξ^1, ξ^2, ξ^3) axes and the conversion from the Cartesian coordinates x, y, z to the ξ_*, η_*, ζ_* coordinates or the ξ, η, ζ (η^1, η^2, η^3) coordinates is effected.

When equations (3.279) and (3.290) are used, the projections of the absolute angular velocity of the moving trihedron $\vec{r}_1\vec{r}_2\vec{r}_3$ are expressed using equalities (3.288) in terms of the time-derivatives of the curvilinear coordinates $\dot{S}$ and $\dot{z}$. The coordinates themselves are then found by integration.

We note further that comparison of equations (3.82) with equations (3.289) and (3.290) shows also that in the determination of coordinates S and z the first integration may be performed not only along the directions of $\vec{r}_2$ and $\vec{r}_3$, but also along the directions $\vec{r}\,'_2$ and $\vec{r}\,'_3$, rotated relative to $\vec{r}_2$ and $\vec{r}_3$ through an angle $\psi(t)$ in a plane normal to $\vec{r}$, i.e., tangential to the sphere of radius r concentric with the earth.

If

$$\left.\begin{aligned}\frac{r'_2}{|r'_2|} &= \frac{r_2}{|r_2|}\cos\psi + \frac{r_3}{|r_3|}\sin\psi,\\ \frac{r'_3}{|r'_3|} &= -\frac{r_2}{|r_2|}\sin\psi + \frac{r_3}{|r_3|}\cos\psi,\end{aligned}\right\} \tag{3.292}$$

then the corresponding ideal equations may be written in the following form:

$$\left.\begin{aligned}\dot{r} &= \int_0^t \left[n_1 + r\left(\omega'^2_{(2)} + \omega'^2_{(3)}\right) + \operatorname{grad}' V\eta^1_i\right] dt + \dot{r}(0),\\ r\omega'_{(3)} &= \int_0^t \left[n'_2 - r\omega'_{(2)}\omega'_{(1)} - \dot{r}\omega'_{(1)} + \right.\\ &\quad + (r\cos z \operatorname{grad}' V\eta^2_i)\cos\psi + \\ &\quad \left. + (r \operatorname{grad}' V\eta^3_i)\sin\psi\right] dt + r(0)\omega'_{(3)}(0),\\ -r\omega'_{(2)} &= \int_0^t \left[n'_1 - r\omega'_{(1)}\omega'_{(3)} + \dot{r}\omega'_{(2)} - \right.\\ &\quad - (r\cos z \operatorname{grad}' V\eta^2_i)\sin\psi + \\ &\quad \left. + (r \operatorname{grad}' V\eta^3_i)\cos\psi\right] dt + r(0)\omega'_{(2)}(0)\end{aligned}\right\} \tag{3.293}$$

Clearly,

$$\left.\begin{aligned}\omega'_{(1)} &= \omega_{(1)} + \dot{\psi},\\ \omega_{(2)} &= \omega'_{(2)}\cos\psi - \omega'_{(3)}\sin\psi,\\ \omega_{(3)} &= \omega'_{(2)}\sin\psi + \omega'_{(3)}\cos\psi\end{aligned}\right\} \tag{3.294}$$

Equations (3.290) may now be used to find S and z.

It is evident that $\psi(t)$ is arbitrary. Specifically, it is possible to calculate, in analogy with condition (3.81), to select $\psi(t)$ such that it satisfies the condition

$$\omega'_{11} = 0. \tag{3.295}$$

Let us now turn to equations (3.211). In accordance with relations (3.281), (3.282), (3.284) and (3.285), they reduce to the form

$$\left.\begin{aligned}
\eta_i^1 = &-\int_0^t \{-\eta_i^2 r\cos z\,[\dot S\cos z + u(-\delta_{31}\sin z\cos S + \\
&+\delta_{33}\cos z - \delta_{32}\sin z\sin S)] - \\
&-\eta_i^3 r\,[\dot z + u(\delta_{31}\sin S - \delta_{32}\cos S)] + \\
&+ u r^2\cos z\,(\eta_i^2\eta_3^3 - \eta_i^3\eta_3^2)\}\,dt + \eta_i^1(0).
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\eta_i^2 = &-\int_0^t \Big\{\eta_i^2\Big(\frac{\dot r}{r} - \dot z\,\mathrm{tg}\,z\Big) + \frac{\eta_i^1}{r\cos z}[\dot S\cos z + \\
&+u(-\delta_{31}\sin z\cos S - \delta_{32}\sin z\sin S + \delta_{33}\cos z)] + \\
&+\frac{\eta_i^3}{\cos z}[-\dot S\sin z - u(\delta_{33}\sin z + \delta_{32}\cos z\sin S + \\
&+\delta_{31}\cos z\cos S)] + \frac{u}{\cos z}(\eta_i^3\eta_3^1 - \eta_i^1\eta_3^3)\Big\}\,dt + \eta_i^2(0). \\
\eta_i^3 = &-\int_0^t \Big\{\eta_i^3\frac{\dot r}{r} + \eta_i^2\cos z\,[\dot S\cos z + u(\delta_{31}\cos z\cos S + \\
&+\delta_{32}\cos z\sin S + \delta_{31}\sin z)] + \frac{\eta_i^1}{r}[\dot z + u(\delta_{31}\sin S - \\
&-\delta_{32}\cos S)] + u\cos z\,(\eta_i^1\eta_3^2 - \eta_i^2\eta_3^1)\Big\}\,dt + \eta_i^3(0).
\end{aligned}\right|$$

$$(3.296)$$

It is not difficult to demonstrate that for constant u equations (3.296) satisfy the following values of η_ℓ^s:

$$\left.\begin{aligned}
\eta_1^1 &= (\delta_{11}\cos z\cos S + \delta_{12}\cos z\sin S + \delta_{13}\sin z). \\
\eta_1^2 &= \frac{1}{r\cos z}(-\delta_{11}\sin S + \delta_{12}\cos S). \\
\eta_1^3 &= \frac{1}{r}(-\delta_{11}\sin z\cos S - \delta_{12}\sin z\sin S + \delta_{13}\cos z). \\
\eta_2^1 &= (\delta_{21}\cos z\cos S + \delta_{22}\cos z\sin S + \delta_{23}\sin z). \\
\eta_2^2 &= \frac{1}{r\cos z}(-\delta_{21}\sin S + \delta_{22}\cos S). \\
\eta_2^3 &= \frac{1}{r}(-\delta_{21}\sin z\cos S - \delta_{22}\sin z\sin S + \delta_{23}\cos z). \\
\eta_3^1 &= (\delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z). \\
\eta_3^2 &= \frac{1}{r\cos z}(-\delta_{31}\sin S + \delta_{32}\cos S). \\
\eta_3^3 &= \frac{1}{r}(-\delta_{31}\sin z\cos S - \delta_{32}\sin z\sin S + \delta_{33}\cos z).
\end{aligned}\right| \tag{3.297}$$

In equalities (3.297) the direction cosines δ_{ij} are determined by table (3.273) and relations (3.274) or (3.277).

As in the preceding cases, the reduced values of η_ℓ^s may be obtained not from equations (3.296), but from equalities (3.258). They may also be obtained by computing the direction cosines between the vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ and $\vec{\eta}_1$, $\vec{\eta}_2$, $\vec{\eta}_3$ and then dividing them by the corresponding Lame coefficients. This is evident from the second group of equations (3.184), if we rewrite them in the form

$$\eta_l^s = \frac{1}{h_s}\left(\eta_s'^{\,l}\,\frac{1}{h_s}\,\frac{\partial \xi^l}{\partial x^s}\right).$$

The terms in parentheses on the right sides of equalities (3.297) are the direction cosines between $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ and $\vec{\eta}_1$, $\vec{\eta}_2$, $\vec{\eta}_3$, and the factors in front of the parentheses are the reciprocals of the Lame coefficients.

From equations (3.212)

$$\eta^s = \hat{\eta}^l_s r \qquad (3.298)$$

Taking into account relations (3.297), we obtain:

$$\left.\begin{aligned}
\eta^1 &= r(\delta_{11}\cos z\cos S + \delta_{12}\cos z\sin S + \delta_{13}\sin z),\\
\eta^2 &= r(\delta_{21}\cos z\cos S + \delta_{22}\cos z\sin S + \delta_{23}\sin z),\\
\eta^3 &= r(\delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z)
\end{aligned}\right\} \qquad (3.299)$$

The direction cosines between the $\vec{\xi}_k$ axes and the newtonometer axes e_α are found from formulas (3.213).

Let us denote these direction cosines by γ_{ij}. Let them form the table:

	ξ_1	ξ_2	ξ_3
e_1	γ_{11}	γ_{12}	γ_{13}
e_2	γ_{21}	γ_{22}	γ_{23}
e_3	γ_{31}	γ_{32}	γ_{33}.

(3.300)

Using expressions (3.281) for the Lame coefficients and (3.283) for the partial derivatives of the ξ^k coordinates relative to $\varkappa^s$ from relations (3.213) we obtain the following expressions for the elements of table (3.300):

$$
\begin{aligned}
\gamma_{11} &= (\delta_{11}\cos z\cos S + \delta_{12}\cos z\sin S + \delta_{13}\sin z)\cos ut - \\
&\quad - (\delta_{21}\cos z\cos S + \delta_{22}\cos z\sin S + \delta_{23}\sin z)\sin ut, \\
\gamma_{12} &= (\delta_{11}\cos z\cos S + \delta_{12}\cos z\sin S + \delta_{13}\sin z)\sin ut + \\
&\quad + (\delta_{21}\cos z\cos S + \delta_{22}\cos z\sin S + \delta_{23}\sin z)\cos ut, \\
\gamma_{13} &= \delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z, \\
\gamma_{21} &= (-\delta_{11}\sin S + \delta_{12}\cos S)\cos ut - (\delta_{21}\sin S + \delta_{22}\cos S)\sin ut, \\
\gamma_{22} &= (-\delta_{11}\sin S + \delta_{12}\cos S)\sin ut + \\
&\quad + (-\delta_{21}\sin S + \delta_{22}\cos S)\cos ut, \\
\gamma_{23} &= -\delta_{31}\sin S + \delta_{32}\cos S, \\
\gamma_{31} &= (-\delta_{11}\sin z\cos S - \delta_{12}\sin z\sin S + \delta_{13}\cos z)\cos ut - \\
&\quad - (-\delta_{21}\sin z\cos S - \delta_{22}\sin z\sin S + \delta_{23}\cos z)\sin ut, \\
\gamma_{32} &= (-\delta_{11}\sin z\cos S - \delta_{12}\sin z\sin S + \delta_{13}\cos z)\sin ut + \\
&\quad + (-\delta_{21}\sin z\cos S - \delta_{22}\sin z\sin S + \delta_{23}\cos z)\cos ut, \\
\gamma_{33} &= -\delta_{31}\sin z\cos S - \delta_{32}\sin z\sin S - \delta_{33}\cos z.
\end{aligned}
$$

It is useful to note that the direction cosines δ_{ij} may be obtained from the expressions in parentheses on the right sides of relations (3.297). To do this it is necessary to substitute λ_1 + ut and λ_2 + ut in place of λ_1 and λ_2 in formulas (3.274) defining δ_{ij}, and then to substitute these new values of δ'_{ij} into relations (3.297).

The projections of the absolute angular velocity of the trihedron $\vec{r}_1\vec{r}_2\vec{r}_3$ on the directions of the vectors forming this trihedron were obtained above. These projections are defined by equalities (3.288). It follows from these equalities and from formulas (3.209) that

$$
\left.
\begin{aligned}
M_{1x}^a &= H[\dot{S}\cos z + u(-\delta_{31}\sin z\cos S - \\
&\quad - \delta_{32}\sin z\sin S + \delta_{33}\cos z)], \\
M_{1y}^a &= H[\dot{z} - u(-\delta_{31}\sin S + \delta_{32}\cos S)], \\
M_{1z}^a &= H[\dot{S}\sin z + u(\delta_{31}\cos z\cos S + \\
&\quad + \delta_{32}\cos z\sin S + \delta_{33}\sin z)].
\end{aligned}
\right| \tag{3.301}
$$

Thus, the operational algorithm of an inertial system determining geodetic coordinates, when the basis of its functional diagram is a free gyrostabilized platform, includes formulas (3.286), (3.297), (3.299) and (3.274), as well as expressions for the direction cosines γ_{ij}. In place of formulas (3.286), formulas (3.289) and (3.290) or

(3.293), (3.294) and (3.290) may also be used, and in place of equalities (3.297) equations (3.296) may be used.

If a maneuverable gyroplatform is taken as the basis of the functional diagram, the expressions for the direction cosines γ_{ij} of the newtonometer axes relative to the axes of the stabilized platform should be replaced in the operational algorithm by equalities (3.301), in accordance with which the moments controlling the platform are formed.

Formulas (3.286), (3.289) and (3.293) contain the components of the gradient of the force function of the earth's gravitational field along the axes of the $O_1\eta^1\eta^2\eta^3$ coordinate system bound to the earth These functions should be given as functions of the coordinates η^ℓ.

If we assume that the gravitational field is spherical, then, as is easily demonstrated, equalities (3.264) obtain, and therefore formulas (3.296) and (3.297) drop out of the ideal equations.

If we consider that the vector gradV of the strength of the earth's gravitational field lies in the plane of the earth's meridian, then in accordance with formulas (3.265)

$$\left.\begin{array}{c} \operatorname{grad}^1 V = g_0^\eta \cos\lambda, \quad \operatorname{grad}^2 V = g_0^\eta \sin\lambda, \\ \operatorname{grad}^3 V = g_3^\eta. \end{array}\right| \tag{3.302}$$

where g_0^η is the projection of the vector gradV on the plane of the equator, and g_3^η is the projection of this vector on the earth's axis of rotation. The quantities g_0^η and g_3^η are functions of φ and r. The geocentric coordinates required for the formation of the right sides of (3.302) -- the latitude φ and the longitude λ -- are related to the geodetic coordinates S and z by the equalities:

$$\left.\begin{array}{r} \cos\varphi\cos\lambda = \delta_{11}\cos z\cos S + \delta_{12}\cos z\sin S + \delta_{13}\sin z, \\ \cos\varphi\cos\lambda = \delta_{21}\cos z\cos S + \delta_{22}\cos z\sin S + \delta_{23}\sin z, \\ \sin\varphi = \delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z. \end{array}\right\} \tag{3.303}$$

The components $\mathrm{grad}^{\ell} V$ of the gradient of the earth's gravitational field enter into the integrands of formulas (3.286), (3.289) and (3.293) in the form of the sums

$$\left.\begin{array}{c} \mathrm{grad}^l V \eta_l^1, \quad r \cos z \, \mathrm{grad}^l V \eta_l^2, \\ r \, \mathrm{grad}^l V \eta_l^3, \end{array}\right\} \tag{3.304}$$

where the components $\mathrm{grad}^{\ell} V$ are defined by equalities (3.302), and η_{ℓ}^{s} by equalities (3.297)

Equalities (3.302) contain the geographical longitude λ. At the same time we may assume in accordance with sums (3.304) that they may be written in a form such that the longitude λ does not appear in them. In order to find this form, we introduce g^1 and g^3, which were defined in accordance with equalities (3.266) as the projections of the strength vector of the earth's gravitational field on the direction $\vec{r}$ and the direction tangent to the geocentric meridian. From equalities (3.266)

$$\left.\begin{array}{l} g_0^{\eta} = g^1 \cos\varphi - g^3 \sin\varphi, \\ g_3^{\eta} = g^1 \sin\varphi - g^3 \cos\varphi. \end{array}\right\} \tag{3.305}$$

Let us substitute these values into equalities (3.302), and then, along with relations (3.297), into the sums (3.304). Then, after obvious simplifications using equalities (3.303) and the orthogonality of table (3.273), we arrive at the formulas:

$$\left.\begin{array}{r} \mathrm{grad}^l V \eta_l^1 = g^1, \\ r \cos z \, \mathrm{grad}^l V \eta_l^2 = \dfrac{g^3}{\cos\varphi}(-\delta_{31} \sin S + \delta_{32} \cos S), \\ r \, \mathrm{grad}^l V \eta_l^3 = \dfrac{g^3}{\cos\varphi}(-\delta_{31} \sin z \cos S - \\ - \delta_{32} \sin z \sin S + \delta_{33} \cos z), \end{array}\right\} \tag{3.306}$$

as required.

As is evident, in formulas (3.306) the quantities

$$\left.\begin{array}{l} \dfrac{1}{\cos\varphi}(-\delta_{31} \sin S + \delta_{32} \cos S), \\ \dfrac{1}{\cos\varphi}(-\delta_{31} \sin z \cos S - \delta_{32} \sin z \sin S + \delta_{33} \cos z) \end{array}\right\} \tag{3.307}$$

are simply the cosines of the angles between $\vec{r}_2 = \vec{r}_S$ and $\vec{r}_3 = \vec{r}_z$ and the bearing to the north at the current location of the object.

Comparing all of the above alternatives for representing the ideal equations of an inertial system determining the geodetic coordinates r, z and S, we may conclude that the simplest and most convenient for the case of a maneuverable gyroplatform as the basis of the system will be the set of equations (3.289), (3.290), (3.306) and (3.209), to which the third equation (3.303) and the first equation (3.288) must be added. Let us write out these equations here. Thus, assuming that the x, y, z axes of the maneuverable platform coincide with the directions of the vectors $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_1$, respectively, we alter in an appropriate manner the indexation of the quantities entering into the equations and, in addition, we introduce v_x and v_y into the formulas in accordance with the equalities $v_x = \omega_y r$, and $v_y = -\omega_y r$. As a result we obtain:

$$\left.\begin{aligned}
\dot{r} &= \int_0^t [n_z + v_x\omega_y - v_y\omega_x + g^1(r,\varphi)]\,dt + \dot{r}(0),\\
v_x &= \int_0^t \Big[n_x + v_y\omega_z - \dot{r}\omega_y - \\
&\quad - \frac{g^1(r,\varphi)}{\cos\varphi}(-\delta_{31}\sin S + \delta_{32}\cos S)\Big]\,dt + v_x(0),\\
v_y &= \int_0^t \Big[n_y - v_x\omega_z + \dot{r}\omega_x + \\
&\quad + \frac{g^1(r,\varphi)}{\cos\varphi}(-\delta_{31}\sin z\cos S - \delta_{32}\sin z\sin S + \\
&\quad + \delta_{33}\cos z)\Big]\,dt + v_y(0),\\
r &= \int_0^t \dot{r}\,dt + r(0), \quad \omega_x = -\frac{v_y}{r}, \quad \omega_y = \frac{v_x}{r}.
\end{aligned}\right|$$

$$\left.\begin{aligned}
S &= \int_0^t \Big[\frac{\omega_y}{\cos z} - \frac{u}{\cos z}(\delta_{33}\cos z - \delta_{31}\sin z\cos S - \\
&\quad - \delta_{32}\sin z\sin S)\Big]\,dt + S(0),\\
z &= \int_0^t [-\omega_x + u(-\delta_{31}\sin S + \delta_{32}\cos S)]\,dt + z(0),\\
\sin\varphi &= \delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z,\\
\omega_z &= \omega_y \operatorname{tg} z + \frac{u}{\cos z}(\delta_{31}\cos S + \delta_{32}\sin S),\\
M^s_{1y} &= -H\omega_x, \quad M^s_{1x} = H\omega_y, \quad M^s_{1z} = H\omega_z.
\end{aligned}\right|$$

(3.308)

If the basis of the system is not a maneuverable gyrostabilized platform but a free platform, the last three equations (3.308) are replaced by the table of direction cosines (3.300).

Equations (3.308) for the geodetic coordinates r, S, and z include as special cases the equations for the geocentric coordinates r, λ, and φ and the coordinates r, λ_1, and ϕ examined above. In order to convert from the geodetic coordinates r, S and z to the geocentric coordinates r, λ, and φ, it is sufficient to require that the trihedra $\eta^1\eta^2\eta^3$ and $\zeta^1\zeta^2\zeta^3$ coincide. This is the case if, of all of the direction cosines δ_{ij} of table (3.273), only

$$\delta_{11} = \delta_{12} = \delta_{13} = 1.$$

are non-zero. To obtain the ideal equations in the coordinates r, λ_1 and φ, it is necessary in addition to set u = 0.

To the above ideal equations of inertial systems determining the coordinates r, λ_1, φ; r, λ, φ; and r, S, z, it is necessary to add further the table (3.66) of the direction cosines between the X, Y, Z axes attached to the object and the x, y, z axes of the gyrostabilized platform. This table enables us to find the parameters characterizing the orientation of the object in space from the measured values of the rotation angles α, β, and γ of the gimbal ring of the gyrostabilized platform (or the platform of the gauge of absolute angular velocity). This aspect of the problem does not differ in any way from the analogous problem for the case of Cartesian coordinates considered in §3.1.

3.3.5. Geographical coordinates. Let us now consider an example of a non-spherical orthogonal reference grid, in the form of geographical coordinates: latitude φ', longitude λ, and height h over the surface of the ocean.

We will consider the surface of the ocean to be an ellipsoid of revolution. This ellipsoid is usually termed a Clairaut ellipsoid. The minor semiaxis of the ellipsoid is its axis of symmetry and coincides with the earth's axis of rotation.

The geographical latitude φ' will be defined, as usual, as the angle between the plane of the earth's equator and the external normal to the Clairaut ellipsoid.

To apply the general formulas of the preceding section, we will, as before, stipulate the equalities

$$x^1 = h, \quad x^2 = \lambda, \quad x^3 = \varphi'. \tag{3.309}$$

Since the η^1, η^2, η^3 axes are rigidly bound to the earth and the η^3 axis is directed along the earth's axis of rotation, according to (2.7) and (2.8) we have:

$$\left.\begin{aligned}
\eta^1 &= \left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\cos\varphi'\cos\lambda,\\
\eta^2 &= \left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\cos\varphi'\sin\lambda,\\
\eta^3 &= \left(\frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\sin\varphi'.
\end{aligned}\right| \tag{3.310}$$

where a is the minor semiaxis, e is the eccentricity of the Clairaut ellipsoid, and η^1, η^2, η^3 are the projections of the radius vector $\vec{r}$ of the point O at which the sensing masses of the newtonometers are located on the η^1, η^2, η^3 axes. The projections of the vector $\vec{r}$ on the ξ^1, ξ^2, ξ^3 axes of the main Cartesian system are related, according to relations (3.231), to the η^1, η^2, η^3 coordinates by the equalities

$$\left.\begin{aligned}
\xi^1 &= \eta^1\cos ut - \eta^2\sin ut,\\
\xi^2 &= \eta^1\sin ut + \eta^2\cos ut,\\
\xi^3 &= \eta^3.
\end{aligned}\right\} \tag{3.311}$$

From expressions (3.310) and (3.311) we obtain:

$$\left.\begin{aligned}
&\frac{\partial\eta^1}{\partial h} = \cos\varphi'\cos\lambda, \quad \frac{\partial\eta^2}{\partial h} = \cos\varphi'\sin\lambda, \quad \frac{\partial\eta^3}{\partial h} = \sin\varphi',\\
&\frac{\partial\eta^1}{\partial\lambda} = -\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\cos\varphi'\sin\lambda,\\
&\frac{\partial\eta^2}{\partial\lambda} = \left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\cos\varphi'\cos\lambda, \quad \frac{\partial\eta^3}{\partial\lambda} = 0,\\
&\frac{\partial\eta^1}{\partial\varphi'} = -\left(\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right)\sin\varphi'\cos\lambda,\\
&\frac{\partial\eta^2}{\partial\varphi'} = -\left(\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right)\sin\varphi'\sin\lambda,\\
&\frac{\partial\eta^3}{\partial\varphi'} = \left(\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right)\cos\varphi';\\
&\frac{\partial\xi^1}{\partial x^k} = \frac{\partial\eta^1}{\partial x^k}\cos ut - \frac{\partial\eta^2}{\partial x^k}\sin ut,\\
&\frac{\partial\xi^2}{\partial x^k} = \frac{\partial\eta^1}{\partial x^k}\sin ut + \frac{\partial\eta^2}{\partial x^k}\cos ut, \quad \frac{\partial\xi^3}{\partial x^k} = \frac{\partial\eta^3}{\partial x^k}.
\end{aligned}\right| \tag{3.312}$$

The Jacobian determinant is

$$J=\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\left(\frac{a}{1-e^2\sin^2\varphi'}+h\right)\cos\varphi',$$

and the reference grid is degenerate on the η^3 axis, where $\varphi' = \pm\pi/2$.

The components a_{ss} of the metric tensor A of the space defined by the coordinates h, λ and φ' are calculated according to (3.312) as follows:

$$\left.\begin{aligned} &a_{11}=0,\quad a_{22}=\left[\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right]^2\cos^2\varphi',\\ &a_{33}=\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]^2. \end{aligned}\right\} \tag{3.313}$$

From (3.313) and (3.174) we obtain the Lame coefficients:

$$\left.\begin{aligned} &h_1=1,\quad h_2=\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\cos\varphi',\\ &h_3=\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h. \end{aligned}\right\} \tag{3.314}$$

We may now find the non-zero Christoffel symbols from formulas (3.175):

$$\left.\begin{aligned}
&\Gamma_{22,1}=\Gamma_{22}^{1}=-\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\cos^2\varphi',\\
&\Gamma_{33,1}=\Gamma_{33}^{1}=-\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}-h,\\
&\Gamma_{12,2}=\Gamma_{21,2}=\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\cos^2\varphi',\\
&\Gamma_{12}^{2}=\Gamma_{21}^{2}=\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)^{-1},\\
&\Gamma_{23,2}=\Gamma_{32,2}=-\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\times\\
&\qquad\times\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\sin\varphi'\cos\varphi',\\
&\Gamma_{23}^{2}=\Gamma_{32}^{2}=-\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)^{-1}\times\\
&\qquad\times\left[h+\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}\right]\operatorname{tg}\varphi',\\
&\Gamma_{31,3}=\Gamma_{13,3}=\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h,\\
&\Gamma_{31}^{3}=\Gamma_{13}^{3}=\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]^{-1},\\
&\Gamma_{22,3}=\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\times\\
&\qquad\times\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\sin\varphi'\cos\varphi',\\
&\Gamma_{22}^{3}=\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]^{-1}\times\\
&\qquad\times\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\sin\varphi'\cos\varphi',\\
&\Gamma_{33,3}=\frac{e^2 3a(1-e^2)\sin\varphi'\cos\varphi'}{(1-e^2\sin^2\varphi')^{5/2}}\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right],\\
&\Gamma_{33}^{3}=\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]^{-1}\frac{e^2 3a(1-e^2)\sin\varphi'\cos\varphi'}{(1-e^2\sin^2\varphi')^{5/2}}
\end{aligned}\right\} \tag{3.315}$$

Finally, using formulas (3.232), (3.177) and (3.178) we find the symbols $\Gamma_{00,s}$, Γ_{00}^{s}, $\Gamma_{0k,s}$ and Γ_{0k}^{s}. The following of them are non-zero:

$$
\left.\begin{aligned}
&\Gamma_{00,1}=\Gamma_{00}^{1}=-u^2\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\cos^2\varphi',\\
&\Gamma_{00,3}=u^2\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\times\\
&\qquad\times\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\sin\varphi'\cos\varphi',\\
&\Gamma_{00}^{3}=u^2\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]^{-1}\times\\
&\qquad\times\left(h+\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}\right)\sin\varphi'\cos\varphi',\\
&\Gamma_{01,2}=-\Gamma_{02,1}=u\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\cos^2\varphi',\\
&\Gamma_{02,3}=-\Gamma_{03,2}=u\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\times\\
&\qquad\times\left[h+\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}\right]\sin\varphi'\cos\varphi',\\
&\Gamma_{02}^{1}=-u\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\cos^2\varphi',\\
&\Gamma_{01}^{2}=u\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)^{-1},\\
&\Gamma_{02}^{3}=u\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]^{-1}\times\\
&\qquad\times\left(h+\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}\right)\sin\varphi'\cos\varphi',\\
&\Gamma_{03}^{2}=-u\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)^{-1}\times\\
&\qquad\times\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\operatorname{tg}\varphi'.
\end{aligned}\right| \tag{3.316}
$$

It is evident that, if in expressions (3.314), (3.315) and (3.316) we set

$$e=0,\quad \varphi'=\varphi,\quad a+h=r,$$

they will coincide with expressions (3.251), (3.252) and (3.267).

Equations (3.210) -- (3.213) contain the Lame coefficients, the Christoffel symbols and the symbols Γ_{00}^{s} and Γ_{0k}^{s}. All of these have been found. Substituting their expressions (3.314), (3.315) and (3.316) into the first three equations (3.210) we arrive at the following relations:

$$
\begin{aligned}
\dot h=\int_0^t\Bigg\{&n_1+\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\dot\varphi'^2+\\
&+\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)(\dot\lambda+u)^2\cos^2\varphi'+\\
&+\operatorname{grad}^r V\eta_0^1\Bigg\}dt+\dot h(0).
\end{aligned} \tag{3.317}
$$

$$
\begin{aligned}
&\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)\dot{\lambda}\cos\varphi' = \\
&\quad = \int_0^t \Big\{ v_2 - \dot{h}(\dot{\lambda}+2u)\cos\varphi' + \\
&\quad + \left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\dot{\varphi}'(\dot{\lambda}+2u)\sin\varphi' + \\
&\quad + h_2\,\mathrm{grad}^l V\,\eta_l^2 \Big\}\,dt + \\
&\quad + \left[\frac{a}{\sqrt{1-e^2\sin^2\varphi'(0)}}+h(0)\right]\dot{\lambda}(0)\cos\varphi(0). \\
&\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\dot{\varphi}' = \\
&\quad = \int_0^t \Big\{ n_3 - \dot{h}\dot{\varphi}' - \left(h+\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}\right)\times \\
&\quad \times(\dot{\lambda}+u)^2\sin\varphi'\cos\varphi' + h_3\,\mathrm{grad}^l V\,\eta_l^3 \Big\}\,dt + \\
&\qquad + \left\{\frac{a(1-e^2)}{[1-e^2\sin^2\varphi'(0)]^{3/2}}+h(0)\right\}\dot{\varphi}'(0).
\end{aligned}
\tag{3.317}
$$

The Christoffel symbol Γ_{33}^{3} does not enter into the third relation (3.317), since in equations (3.210) the summation is carried out only over all k different from s.

To find η_ℓ^k for the case under consideration, there is no need to use equations (3.211), since from the relation

$$\eta_l^k = \eta_l \cdot r^k = \frac{\eta_l \cdot r_k}{h_k^2}$$

and from relations (3.314) it directly follows that

$$\eta_l^1 = \frac{\partial \eta^l}{\partial h}, \quad \eta_l^2 = \frac{1}{h_2^2}\frac{\partial \eta^l}{\partial \lambda}, \quad \eta_l^3 = \frac{1}{h_3^2}\frac{\partial \eta^l}{\partial \varphi'}. \tag{3.318}$$

We will consider that the vector

$$g = \mathrm{grad}\, V \tag{3.319}$$

of the strength of the earth's gravitational field lies in the plane of the meridian. Then in the second equation (3.317) the sum

$$h_2\,\mathrm{grad}^l V\,\eta_l^2 = \frac{1}{h_2}\,\mathrm{grad}^l V\,\frac{\partial \eta^l}{\partial \lambda} = 0, \tag{3.320}$$

since this sum is the projection of the vector $\vec{g}$ on the direction $\vec{r}_2$, normal to the plane of the meridian.

From the condition that the vector $\vec{g}$ lie in the plane of the meridian, the following relations likewise follow:

$$\operatorname{grad}^1 V = g_0^\eta \cos\lambda, \quad \operatorname{grad}^2 V = g_0^\eta \sin\lambda, \quad \operatorname{grad}^3 V = g_3^\eta. \tag{3.321}$$

where g_0^η and g_3^η are functions of φ' and h.

Using relations (3.318), we find:

$$\left.\begin{aligned} \operatorname{grad}' V\eta_i^1 &= g_0^\eta \cos\varphi' + g_3^\eta \sin\varphi', \\ h_3 \operatorname{grad}' \eta_i^3 &= -g_0^\eta \sin\varphi' + g_3^\eta \cos\varphi'. \end{aligned}\right\} \tag{3.322}$$

By analogy with relations (3.266), we introduce the projections $\tilde{g}_0^1$ and $\tilde{g}_0^3$ of the vector $\vec{g}$ = gradV on the direction of the normal to the Clairaut ellipsoid and the direction of the tangent to the geographic meridian. Then, clearly,

$$\left.\begin{aligned} \tilde{g}_0^1 &= g_0^\eta \cos\varphi' + g_3^\eta \sin\varphi', \\ \tilde{g}_0^3 &= -g_0^\eta \sin\varphi' + g_3^\eta \cos\varphi'. \end{aligned}\right\} \tag{3.323}$$

Here $\tilde{g}_0^1$ and $\tilde{g}_0^3$ are functions of h and φ. According to the definitions given in §2.2,

$$\tilde{g}_0^1(h, \varphi') = F_{x_1}(h, \varphi'), \quad \tilde{g}_0^3(h, \varphi') = F_{y_1}(h, \varphi').$$

Since the surface of the Clairaut ellipsoid is a reference surface i.e., a surface of constant gravitational force potential, in the integrand of the third equation (3.317) the sum

$$\tilde{g}_0^3(h = 0, \varphi') - \frac{au^2 \sin\varphi' \cos\varphi'}{\sqrt{1 - e^2 \sin^2\varphi'}} = 0. \tag{3.324}$$

Relations (3.320), (3.322), (3.323) and (3.324) simplify the integrand of equations (3.317). Taking into account these simplifications and adding to equations (3.317) the formulas for h, λ, and φ', we arrive at the following system of equations:

$$
\left.
\begin{aligned}
\dot{h} &= \int_0^t \Big\{ n_1 + \left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right](\dot{\varphi}')^2 + \\
&\quad + \left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\times \\
&\quad \times(\dot{\lambda} + u)^2\cos^2\varphi' + \tilde{g}_0^1(\varphi', h)\Big\}\,dt + \dot{h}(0). \\
&\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\dot{\lambda}\cos\varphi' = \\
&= \int_0^t \Big\{ n_2 - \dot{h}(\dot{\lambda} + 2u)\cos\varphi' + \dot{\varphi}'(\dot{\lambda} + 2u)\times \\
&\quad \times\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right]\sin\varphi'\Big\}\,dt + \\
&\quad + \left[\frac{a}{\sqrt{1-e^2\sin^2\varphi'(0)}} + h(0)\right]\dot{\lambda}(0)\cos\varphi'(0). \\
&\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right]\dot{\varphi}' = \\
&= \int_0^t \Big\{ n_3 - \dot{h}\dot{\varphi}' - \left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\times \\
&\quad \times\dot{\lambda}(\dot{\lambda} + 2u)\sin\varphi'\cos\varphi' - hu^2\sin\varphi'\cos\varphi' + \\
&\quad + \tilde{g}_0^3(h, \varphi') - \tilde{g}_0^3(0, \varphi')\Big\}\,dt + \\
&\qquad + \left\{\frac{a(1-e^2)}{[1-e^2\sin^2\varphi'(0)]^{3/2}} + h(0)\right\}\dot{\varphi}'(0).
\end{aligned}
\right|
$$

$$
\left.
\begin{aligned}
h &= \int_0^t \dot{h}\,dt + h(0). \\
\lambda &= \int_0^t \frac{1}{\cos\varphi'}\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)^{-1}\times \\
&\quad \times\left[\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)\dot{\lambda}\cos\varphi'\right]dt + \lambda(0). \\
\varphi' &= \int_0^t \left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right]^{-1}\times \\
&\qquad \times\left\{\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right]\dot{\varphi}'\right\}dt + \varphi(0)
\end{aligned}
\right|
\qquad (3.325)
$$

The system of equations (3.325) constitutes a portion of the ideal equation of an inertial system operating in a geographic reference grid. This portion of the equations deals with the determination of $\dot{h}$, $\dot{\lambda}$, $\dot{\varphi}$, h, λ and φ from the initial values of these coordinates and the readings of the newtonometers n_1, n_2, n_3. Newtonometer n_1 (its axis of sensitivity) is situated in the direction of the normal $\vec{r}_1$ to the reference ellipsoid, while newtonometers n_2 and n_3 are situated in a plane perpendicular to $\vec{r}_1$, i.e., in a plane parallel to the plane of the geographic horizon, the axis of sensitivity of newtonometer n_3 coinciding with the direction $\vec{r}_3$ lying in the plane of the meridian, and the axis of sensitivity of newtonometer n_2 coinciding with direction $\vec{r}_2$, normal to this plane. The vectors $\vec{r}_2$ and $\vec{r}_3$ point, clearly, in the direction of increasing λ and φ'.

To determine the direction cosines of the vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ relative to the ξ^1, ξ^2, ξ^3 axes of the stabilized platform, relations (3.312) should be used. From them we immediately obtain the direction cosines between the vectors $\vec{r}_s$ and the η^1, η^2, η^3 axes:

	η_1	η_2	η_3
r_1	$\cos\varphi'\cos\lambda$	$\cos\varphi'\sin\lambda$	$\sin\varphi'$
r_2	$-\sin\lambda$	$\cos\lambda$	0
r_3	$-\sin\varphi'\cos\lambda$	$-\sin\varphi'\sin\lambda$	$\cos\varphi'$.

These direction cosines, together with the direction cosines between the η^1, η^2, η^3 and ξ^1, ξ^2, ξ^3 axes

	η^1	η^2	η^3
ξ^1	$\cos ut$	$-\sin ut$	0
ξ^2	$\sin ut$	$\cos ut$	0
ξ^3	0	0	1

fully determine the position of the vectors $\vec{r}_s$ relative to the ξ^1, ξ^2, ξ^3 axes.

The rotation angles α, β, and γ of the gimbal rings of the stabilized platform enable us to construct table (3.66) of the direction cosines between the axes of the trihedron XYZ bound to the object and the x, y, z (ξ^1, ξ^2, ξ^3) axes of the stabilized platform. Table (3.66), together with the tables of the direction cosines between the η^1, η^2, η^3 and ξ^1, ξ^2, ξ^3 axes, and between the η_1, η_2, η_3 axes and vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, enable us to determine the orientation of the object relative to the plane of the geographic horizon and the points of the compass.

If for the case under consideration the functional diagram is a maneuverable gyroplatform, the controlling moments M^4_{1x}, M^5_{1y}, and M^6_{1x}, are formed according to formulas (3.209). The quantities $\omega_{(1)}$, $\omega_{(2)}$, $\omega_{(3)}$ required here are easily found from expressions (3.205), (3.315), (3.316) and (3.314):

$$\left.\begin{aligned} \omega_{(1)} = (\dot{\lambda} + u)\sin\varphi', \quad \omega_{(2)} = -\dot{\varphi}', \\ \omega_{(3)} = (\dot{\lambda} + u)\cos\varphi'. \end{aligned}\right\} \tag{3.326}$$

We note that equations (3.317) may also be represented in a form analogous to equations (3.289) and, further, equations (3.82).

According to the well-known Dupin theorem, the surfaces of equal values of the coordinates of a triple orthogonal system intersect along lines of curvature. We will use r_2 and r_3 to designate the radii of curvature of the normal sections of the surface h = const passing through the vectors $\vec{r}_2$ and $\vec{r}_3$, respectively. It then follows from relations (3.314) that

$$r_3 = h_3 = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h. \tag{3.327}$$

Since r_2 is the radius of curvature of the parallel of the surface of rotation h = const, according to Meusnier's theorem[9]

$$r_2 = \frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h. \tag{3.328}$$

The projections $v_{(2)}$ and $v_{(3)}$ of the absolute velocity of the origin of trihedron $\vec{r}_1\vec{r}_2\vec{r}_3$ may now be expressed in terms of $\omega_{(3)}$ and $\omega_{(2)}$, respectively as follows:

$$v_{(2)} = \omega_{(3)} r_2, \quad v_{(3)} = -\omega_{(2)} r_3. \tag{3.329}$$

Before substituting these relations into equations (3.317), let us transform the second of these equations. We may form the total derivative from a portion of the integrand of the second equation (3.317) as follows:

$$-\cos\varphi'\dot{h}u + \left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}} + h\right]\dot{\varphi}'u\sin\varphi' =$$
$$= -\frac{d}{dt}\left[\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}} + h\right)u\cos\varphi'\right]. \tag{3.330}$$

Integrating this portion, we reduce the second equation (3.317) to the form:

$$\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)(\dot\lambda+u)\cos\varphi'=$$
$$=\int_0^t\Big\{n_2-\dot h(\dot\lambda+u)\cos\varphi'+$$
$$+\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]\dot\varphi'(\dot\lambda+u)\sin\varphi'+h_2\,\mathrm{grad}\cdot V\eta_i^2\Big\}dt+$$
$$+\left[\frac{a}{\sqrt{1-e^2\sin^2\varphi'(0)}}+h(0)\right]\dot\lambda(0)\cos\varphi'(0). \tag{3.331}$$

We now substitute (3.326), (3.327), (3.328) and (3.329) in equation (3.331) and the first and third equations (3.317). Taking into account equalities (3.323) and (3.320), we obtain:

$$\left.\begin{aligned}
\dot h&=\int_0^t[n_1+v_{(2)}\omega_{(3)}-v_{(3)}\omega_{(2)}+\tilde g_0^1]\,dt+\dot h(0).\\
v_{(2)}&=\int_0^t[n_2+v_{(3)}\omega_{(1)}-\dot h\omega_{(3)}]\,dt+v_{(2)}(0).\\
v_{(3)}&=\int_0^t[n_3+\dot h\omega_{(2)}-v_{(2)}\omega_{(1)}+\tilde g_0^3]\,dt+v_{(3)}(0).
\end{aligned}\right| \tag{3.332}$$

Thus, the ideal equations of an inertial system operating in geographic coordinates, when the basis of the system is a maneuverable gyroplatform, may be represented in the following form:

$$\left.\begin{aligned}
\dot h&=\int_0^t[n_z+v_x\omega_y-v_y\omega_x+\tilde g_0^1(h,\varphi')]\,dt+\dot h(0).\\
v_x&=\int_0^t(n_x+v_y\omega_z-\dot h\omega_y)\,dt+v_x(0).\\
v_y&=\int_0^t[n_y+\dot h\omega_x-v_x\omega_z+\tilde g_0^3(h,\varphi')]\,dt+v_y(0).\\
\omega_y&=\frac{v_x}{r_2},\quad \omega_x=-\frac{v_y}{r_3}.\\
\varphi'&=-\int_0^t\omega_x\,dt+\varphi'(0).\\
\lambda&=\int_0^t\left(\frac{\omega_y}{\cos\varphi'}-u\right)dt+\lambda(0).\\
h&=\int_0^t\dot h\,dt+h(0).
\end{aligned}\right|$$

$$\left.\begin{aligned}
&r_2=\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h,\quad r_3=\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h.\\
&\omega_z=\omega_y\,\mathrm{tg}\,\varphi'.\\
&M_{1y}^{s}=-H\omega_x,\quad M_{1x}^{s}=H\omega_y,\quad M_{1z}^{s}=H\omega_z.
\end{aligned}\right\} \tag{3.333}$$

Equations (3.333) were obtained from relations (3.332), (3.329), (3.326) and (3.209). The x, y, z axes of the maneuverable platform are superposed on the directions $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_1$, the indices (1), (2), (3) in formulas (3.333) being replaced by x, y, z.

As for the preceding cases, if the basis of the system is not an all-moving but a free gyrostabilized platform, the last three relations (3.333) drop out. The appropriate tables of direction cosines should be used in their place to determine the required orientation of the newtonometers.

3.3.6. <u>An example of non-orthogonal curvilinear coordinates</u>. In conclusion let us consider an example of non-orthogonal coordinates. Let the coordinates defining the position of the point O in the basic Cartesian system be the distance r of the point O from the center of the earth and the angles σ_1 and σ_2 which form the vector $\vec{r}$ with the ξ^1 and ξ^2 axes (Figure 3.5). Then

$$\left.\begin{aligned} &\xi^1 = r\cos\sigma_1, \quad \xi^2 = r\cos\sigma_2, \\ &\xi^3 = r\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}. \end{aligned}\right\} \tag{3.334}$$

This is a stationary spherical, but not orthogonal reference grid.

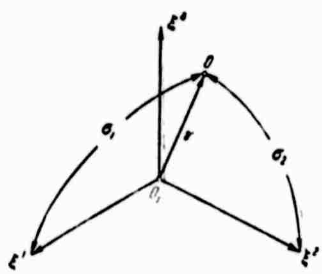

Figure 3.5

From (3.334) it follows that:

$$
\left.
\begin{aligned}
&\frac{\partial\xi^1}{\partial r}=\cos\sigma_1,\quad \frac{\partial\xi^2}{\partial r}=\cos\sigma_2,\\
&\frac{\partial\xi^3}{\partial r}=\sqrt{\sin^2\sigma_1-\cos^2\sigma_2},\\
&\frac{\partial\xi^1}{\partial\sigma_1}=-r\sin\sigma_1,\quad \frac{\partial\xi^2}{\partial\sigma_1}=0,\\
&\frac{\partial\xi^3}{\partial\sigma_1}=\frac{r\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}},\\
&\frac{\partial\xi^1}{\partial\sigma_2}=0,\quad \frac{\partial\xi^2}{\partial\sigma_2}=-r\sin\sigma_2,\\
&\frac{\partial\xi^3}{\partial\sigma_2}=\frac{r\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}},\\
&J=\frac{r^2\sin\sigma_1\sin\sigma_2}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}}.
\end{aligned}
\right|
\qquad (3.335)
$$

The reference grid degenerates on the $O_1\xi^1\xi^2$ plane, where $J = 0$.

Let us assume that

$$x^1=r,\quad x^2=\sigma_1,\quad x^3=\sigma_2. \qquad (3.336)$$

The covariant components of the metric tensor will then be:

$$
\left.
\begin{aligned}
&a_{11}=1,\quad a_{12}=a_{21}=0,\quad a_{13}=a_{31}=0,\\
&a_{22}=a_{33}=\frac{r^2\sin^2\sigma_1\sin^2\sigma_2}{\sin^2\sigma_1-\cos^2\sigma_2},\\
&a_{23}=a_{32}=\frac{r^2\sin\sigma_1\cos\sigma_1\sin\sigma_2\cos\sigma_2}{\sin^2\sigma_1-\cos^2\sigma_2}.
\end{aligned}
\right|
\qquad (3.337)
$$

In order to use formulas (3.129), expressing the Christoffel symbols of the first kind in terms of the derivatives of the covariant components of the metric tensor, we write out the derivatives $\frac{\partial a_{st}}{\partial x^k}$. The following derivaties are non-zero:

$$
\left.
\begin{aligned}
&\frac{\partial a_{22}}{\partial r}=\frac{\partial a_{33}}{\partial r}=\frac{2r\sin^2\sigma_1\sin^2\sigma_2}{\sin^2\sigma_1-\cos^2\sigma_2},\\
&\frac{\partial a_{23}}{\partial r}=\frac{\partial a_{32}}{\partial r}=\frac{2r\sin\sigma_1\cos\sigma_1\sin\sigma_2\cos\sigma_2}{\sin^2\sigma_1-\cos^2\sigma_2},\\
&\frac{\partial a_{22}}{\partial\sigma_1}=\frac{\partial a_{33}}{\partial\sigma_1}=-\frac{2r^2\sin\sigma_1\cos\sigma_1\sin^2\sigma_2\cos^2\sigma_2}{(\sin^2\sigma_1-\cos^2\sigma_2)^2},\\
&\frac{\partial a_{23}}{\partial\sigma_1}=\frac{\partial a_{32}}{\partial\sigma_1}=\\
&\qquad=-\frac{r^2\sin\sigma_2\cos\sigma_2(\cos^2\sigma_1\cos^2\sigma_2+\sin^2\sigma_1\sin^2\sigma_2)}{(\sin^2\sigma_1-\cos^2\sigma_2)^2},\\
&\frac{\partial a_{22}}{\partial\sigma_2}=\frac{\partial a_{33}}{\partial\sigma_2}=-\frac{2r^2\sin^2\sigma_1\cos^2\sigma_1\sin\sigma_2\cos\sigma_2}{(\sin^2\sigma_1-\cos^2\sigma_2)^2},\\
&\frac{\partial a_{23}}{\partial\sigma_2}=\frac{\partial a_{32}}{\partial\sigma_2}=\\
&\qquad=-\frac{r^2\sin\sigma_1\cos\sigma_1(\cos^2\sigma_1\cos^2\sigma_2+\sin^2\sigma_1\sin^2\sigma_2)}{(\sin^2\sigma_1-\cos^2\sigma_2)^2}.
\end{aligned}
\right|
\qquad (3.338)
$$

Taking into account equalities (3.338), from formulas (3.129) we find the non-zero Christoffel symbols of the first kind:

$$\left.\begin{aligned}
&\Gamma_{22,1}=-\frac{1}{2}\frac{\partial a_{22}}{\partial r}, \quad \Gamma_{23,1}=\Gamma_{32,1}=-\frac{1}{2}\frac{\partial a_{23}}{\partial r},\\
&\Gamma_{33,1}=-\frac{1}{2}\frac{\partial a_{33}}{\partial r}, \quad \Gamma_{12,2}=\Gamma_{21,2}=\frac{1}{2}\frac{\partial a_{22}}{\partial r},\\
&\Gamma_{13,2}=\Gamma_{31,2}=\frac{1}{2}\frac{\partial a_{23}}{\partial r}, \quad \Gamma_{22,2}=\frac{1}{2}\frac{\partial a_{22}}{\partial \sigma_1},\\
&\Gamma_{23,2}=\Gamma_{32,2}=\frac{1}{2}\frac{\partial a_{22}}{\partial \sigma_2}, \quad \Gamma_{33,2}=\frac{\partial a_{23}}{\partial \sigma_2}-\frac{1}{2}\frac{\partial a_{33}}{\partial \sigma_1},\\
&\Gamma_{12,3}=\Gamma_{21,3}=\frac{1}{2}\frac{\partial a_{23}}{\partial r}, \quad \Gamma_{13,3}=\Gamma_{31,3}=\frac{1}{2}\frac{\partial a_{33}}{\partial r},\\
&\Gamma_{22,3}=\frac{\partial a_{23}}{\partial \sigma_1}-\frac{1}{2}\frac{\partial a_{22}}{\partial \sigma_2},\\
&\Gamma_{23,3}=\Gamma_{32,3}=\frac{1}{2}\frac{\partial a_{33}}{\partial \sigma_1}, \quad \Gamma_{33,3}=\frac{1}{2}\frac{\partial a_{33}}{\partial \sigma_2}.
\end{aligned}\right\} \tag{3.339}$$

Thus, to within the constant factor 1/2, expression (3.338) contains all of the Christoffel symbols of the first kind except the symbols $\Gamma_{22,3}$ and $\Gamma_{33,2}$, which, according to relations (3.338) and (3.339), are:

$$\left.\begin{aligned}
\Gamma_{33,2}&=\frac{r^2\sin\sigma_1\cos\sigma_1}{\sin^2\sigma_1-\cos^2\sigma_2}(\cos^2\sigma_1\cos^2\sigma_2-\sin^2\sigma_1\sin^2\sigma_2+\\
&\qquad+\sin^2\sigma_2\cos^2\sigma_2),\\
\Gamma_{22,3}&=\frac{r^2\sin\sigma_2\cos\sigma_2}{\sin^2\sigma_1-\cos^2\sigma_2}(\cos^2\sigma_1\cos^2\sigma_2-\sin^2\sigma_1\sin^2\sigma_2+\\
&\qquad+\sin^2\sigma_1\cos^2\sigma_1).
\end{aligned}\right\} \tag{3.340}$$

To find the Christoffel symbols of the second kind, we must use relations (3.126), the right sides of which contain, in addition to the Christoffel symbols of the first kind, the contravariant components $a^{\ell t}$ of the metric tensor. To find the latter we may use the fact that the matrix composed of the contravariant components $a^{\ell t}$ of the metric tensor is the inverse of the matrix of the covariant components a_{sk}.

Forming the inverse of the matrix $\|a_{sk}\|$, we find:

$$\left.\begin{aligned}
&a^{11}=1, \quad a^{22}=a^{33}=\frac{1}{r^2},\\
&a^{23}=a^{32}=-\frac{1}{r^2}\operatorname{ctg}\sigma_1\operatorname{ctg}\sigma_2.
\end{aligned}\right\} \tag{3.341}$$

The remaining components $a^{\ell t}$ are equal to 0.

From equalities (3.126), (3.341), (3.340), (3.339) and (3.338), we now obtain explicit expressions for the non-zero Christoffel symbols of the second kind Γ_{sk}^{m}:

$$\left.\begin{aligned}
&\Gamma_{22}^{1} = -\frac{r \sin^2\sigma_1 \sin^2\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
&\Gamma_{23}^{1} = \Gamma_{32}^{1} = -\frac{r \sin\sigma_1 \cos\sigma_1 \sin\sigma_2 \cos\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
&\Gamma_{33}^{1} = -\frac{r \sin^2\sigma_1 \sin^2\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
&\Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{1}{r}, \quad \Gamma_{22}^{2} = -\frac{\cos^2\sigma_1 \cos^2\sigma_2}{\sin\sigma_1 (\sin^2\sigma_1 - \cos^2\sigma_2)}, \\
&\Gamma_{23}^{2} = \Gamma_{32}^{2} = -\frac{\cos^2\sigma_1 \sin\sigma_2 \cos\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2},
\end{aligned}\right|$$

$$\left.\begin{aligned}
&\Gamma_{33}^{2} = -\frac{\sin\sigma_1 \cos\sigma_1 \sin^2\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
&\Gamma_{22}^{3} = -\frac{\sin^2\sigma_1 \sin\sigma_2 \cos\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
&\Gamma_{23}^{3} = \Gamma_{32}^{3} = -\frac{\cos^2\sigma_2 \sin\sigma_1 \cos\sigma_1}{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
&\Gamma_{33}^{3} = \frac{\cos^2\sigma_1 \cos^2\sigma_2}{\sin\sigma_2 (\sin^2\sigma_1 - \cos^2\sigma_2)}.
\end{aligned}\right|$$

(3.342)

Since the coordinates r, σ_1, σ_2 are stationary,

$$\Gamma_{0n}^{i} = \Gamma_{0i}^{i} = 0. \tag{3.343}$$

The reference grid under consideration is not orthogonal. Therefore to obtain the ideal equations we will use formulas (3.172), (3.163) or (3.164), as well as the table of direction cosines (3.173).

Substituting expressions (3.343), (3.342) and (3.341) into relations (3.172), we find:

$$\left.\begin{aligned}
&\dot{r} = \int_0^t \Big[n_1 + \frac{r \sin^2\sigma_1 \sin^2\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2}(\dot{\sigma}_1^2 + \dot{\sigma}_2^2 + \\
&+ 2\dot{\sigma}_1\dot{\sigma}_2 \operatorname{ctg}\sigma_1 \operatorname{ctg}\sigma_2) + \operatorname{grad}^1 V \eta_i^1\Big] dt + \dot{r}(0), \\
&r\dot{\sigma}_1 = \int_0^t \Big[n_2 - 2\dot{r}\dot{\sigma}_1 + \frac{\operatorname{ctg}\sigma_1}{\sin^2\sigma_1 - \cos^2\sigma_2}(\dot{\sigma}_1 \cos\sigma_1 \cos\sigma_2 + \\
&+ \dot{\sigma}_2 \sin\sigma_1 \sin\sigma_2)^2 + r \operatorname{grad}^1 V \eta_i^2\Big] dt + r(0)\dot{\sigma}_1(0), \\
&r\dot{\sigma}_2 = \int_0^t \Big[n_3 - 2\dot{r}\dot{\sigma}_2 + \frac{\operatorname{ctg}\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2}(\dot{\sigma}_1 \sin\sigma_1 \sin\sigma_2 + \\
&+ \dot{\sigma}_2 \cos\sigma_1 \cos\sigma_2)^2 + r \operatorname{grad}^1 V \eta_i^3\Big] dt + r(0)\dot{\sigma}_2(0), \\
&r = \int_0^t \dot{r}\, dt + r(0), \quad \sigma_1 = \int_0^t \frac{1}{r}(r\dot{\sigma}_1)\, dt + \sigma_1(0), \\
&\sigma_2 = \int_0^t \frac{1}{r}(r\dot{\sigma}_2)\, dt + \sigma_2(0).
\end{aligned}\right|$$

(3.444)

From formulas (3.163) we obtain:

$$
\left.
\begin{aligned}
\eta_l^1 = & -\int_0^t \Big\{ \frac{r \sin\sigma_1 \sin\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2} \big[\eta_l^2(-\dot{\sigma}_1 \sin\sigma_1 \sin\sigma_2 - \\
& -\dot{\sigma}_2 \cos\sigma_1 \cos\sigma_2) + \eta_l^3(-\dot{\sigma}_2 \sin\sigma_1 \sin\sigma_2 - \dot{\sigma}_1 \cos\sigma_1 \cos\sigma_2)\big] + \\
& + \frac{r^2 u \sin\sigma_1 \sin\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}} (\eta_l^2\eta_3^1 - \eta_l^3\eta_3^2)\Big\} dt + \eta_l^1(0). \\
\eta_l^2 = & -\int_0^t \Big\{ \eta_l^1 \frac{\dot{\sigma}_1}{r} + \frac{\cos\sigma_1 \cos\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2} \times \\
& \times \big[\eta_l^2(-\dot{\sigma}_1 \cot\sigma_1 \cos\sigma_2 - \dot{\sigma}_2 \cos\sigma_1 \sin\sigma_2) + \\
& + \eta_l^3(-\dot{\sigma}_2 \cot\sigma_2 \cos\sigma_1 - \dot{\sigma}_1 \sin\sigma_2 \cos\sigma_1)\big] + \\
& + \eta_l^2 \frac{\dot{r}}{r} + \frac{u}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}} \big[\cos\sigma_1 \cos\sigma_2 (\eta_l^2\eta_3^1 - \\
& - \eta_l^1\eta_3^2) + \sin\sigma_1 \sin\sigma_2 (\eta_l^3\eta_3^1 - \eta_l^1\eta_3^3)\big]\Big\} dt + \eta_l^2(0). \\
\eta_l^3 = & -\int_0^t \Big\{ \eta_l^1 \frac{\dot{\sigma}_2}{r} + \frac{\cos\sigma_1 \cos\sigma_2}{\sin^2\sigma_1 - \cos^2\sigma_2} \times \\
& \times \big[\eta_l^3(-\dot{\sigma}_2 \cot\sigma_2 \cos\sigma_1 - \dot{\sigma}_1 \cos\sigma_2 \sin\sigma_1) + \\
& + \eta_l^2(-\dot{\sigma}_1 \cot\sigma_1 \cos\sigma_2 - \dot{\sigma}_2 \sin\sigma_1 \cos\sigma_2)\big] + \\
& + \eta_l^3 \frac{\dot{r}}{r} + \frac{u}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}} \big[\cos\sigma_1 \cos\sigma_2 (\eta_l^3\eta_3^1 - \\
& - \eta_l^1\eta_3^3) + \sin\sigma_1 \sin\sigma_2 (\eta_l^2\eta_3^1 - \eta_l^1\eta_3^2)\big]\Big\} dt + \eta_l^3(0).
\end{aligned}
\right|
\tag{3.345}
$$

Equations (3.345) are satisfied for constant u by the following values of η_ℓ^k:

$$
\left.
\begin{aligned}
\eta_1^1 &= \cos\sigma_1 \cos ut + \cos\sigma_2 \sin ut, \\
\eta_1^2 &= \frac{1}{r}(-\sin\sigma_1 \cos ut + \cot\sigma_1 \cos\sigma_2 \sin ut), \\
\eta_1^3 &= \frac{1}{r}(\cos\sigma_1 \cot\sigma_2 \cos ut - \sin\sigma_2 \sin ut).
\end{aligned}
\right\}
$$

$$
\left.
\begin{aligned}
\eta_2^1 &= -\cos\sigma_1 \sin ut + \cos\sigma_2 \cos ut, \\
\eta_2^2 &= \frac{1}{r}(\sin\sigma_1 \sin ut + \cot\sigma_1 \cos\sigma_2 \cos ut), \\
\eta_2^3 &= \frac{1}{r}(-\cos\sigma_1 \cot\sigma_2 \sin ut - \sin\sigma_2 \cos ut), \\
\eta_3^1 &= \sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
\eta_3^2 &= \frac{\cot\sigma_1}{r}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
\eta_3^3 &= \frac{\cot\sigma_2}{r}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}.
\end{aligned}
\right|
\tag{3.346}
$$

The quantities η_ℓ^k may also be obtained from the equalities

$$
\eta_l^k = \eta_l \cdot r^k. \tag{3.347}
$$

in which $\vec{\eta}_\ell$ are expressed in terms of $\vec{\xi}_s$ by means of the formulas

$$\left.\begin{aligned}\eta_1 &= \xi_1\cos ut + \xi_2\sin ut,\\ \eta_2 &= -\xi_1\sin ut + \xi_2\cos ut, \quad \eta_3 = \xi_3.\end{aligned}\right\} \qquad (3.348)$$

In order to use equalities (3.347), $\vec{r}^k$ must be known. Since

$$r^k = r_s a^{sk}, \qquad (3.349)$$

by taking into account expressions (3.341) for the contravariant components of the metric tensor, we obtain:

$$\left.\begin{aligned}&r^1 = r_1, \quad r^2 = \frac{1}{r^2}(r_2 - r_3\cot\sigma_1\cot\sigma_2),\\ &r^3 = \frac{1}{r^2}(-r_2\cot\sigma_1\cot\sigma_2 + r_3).\end{aligned}\right\} \qquad (3.350)$$

At the same time, according to relations (3.334), (3.335) and (3.336),

$$\left.\begin{aligned}r_1 &= \xi_1\cos\sigma_1 + \xi_2\cos\sigma_2 + \xi_3\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2},\\ r_2 &= -\xi_1 r\sin\sigma_1 + \xi_3\frac{r\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}},\\ r_3 &= -\xi_2 r\sin\sigma_2 + \xi_3\frac{r\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}.\end{aligned}\right\} \qquad (3.351)$$

Substituting $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ into formulas (3.350) and using equalities (3.348) and (3.347), we obtain the same values of η_ℓ^k as those obtained using formulas (3.346).

Turning to the second group of equations (3.163), we obtain:

$$\eta^k = \eta_k^1 r,$$

or, considering formulas (3.346),

$$\left.\begin{aligned}\eta^1 &= r(\cos\sigma_1\cos ut + \cos\sigma_2\sin ut),\\ \eta^2 &= r(-\cos\sigma_1\sin ut + \cos\sigma_2\cos ut),\\ \eta^3 &= r\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}.\end{aligned}\right\} \qquad (3.352)$$

The quantities η^1, η^2 and η^3 are required for the formation of the force function V of the gravitational field, which is assumed to be a function of these coordinates. Of course, the sums $\mathrm{grad}^\varphi V\eta_\ell^s$ may be transformed, as in the preceding cases, so as to contain only g_0^η and g_3^η or g^1 and g^3, defined by equalities (3.266). The latitude φ is required for the formation of g_0^η and g_3^η or g^1 and g^3. It may be obtained from a comparison of relations (3.352) with the equalities

$$\eta^1 = r\cos\varphi\cos\lambda, \quad \eta^2 = r\cos\varphi\sin\lambda, \quad \eta^3 = r\sin\varphi. \qquad (3.353)$$

To complete the compilation of the ideal equations of the system in question, only the direction cosines between the newtonometers axes $\vec{e}_s$ and the axes of the stabilized platform, i.e., the ξ^1, ξ^2, ξ^3 axes remain to be found.

From table (3.173) and expressions (3.351) and (3.341) we find the following direction cosines:

$$\begin{array}{cccc} & \xi^1 & \xi^2 & \xi^3 \\ e_1 & \cos\sigma_1 & \cos\sigma_2 & \sqrt{\sin^2\sigma_1 - \cos^2\sigma_2} \\ e_2 & -\sin\sigma_1 & \operatorname{ctg}\sigma_1\cos\sigma_2 & \operatorname{ctg}\sigma_1\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2} \\ e_3 & \cos\sigma_1\operatorname{ctg}\sigma_2 & -\sin\sigma_2 & \operatorname{ctg}\sigma_2\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}. \end{array} \qquad (3.353a)$$

§3.4. Ideal Equations of Inertial Systems Not Containing Gyroscopic Gauges of Absolute Angular Velocity [absolute angular rate meter][10]

3.4.1. General considerations. Until now we have assumed that the inertial systems which we have been considering have contained gyroscopic sensing elements as well as newtonometers. The gyroscopic elements were used to effect the required orientation of the directions of the axes of sensitivity of the three newtonometers in the main Cartesian coordinate system, i.e., basically, in some inertial reference system. Specifically, the gyroscopic gauge of absolute angular velocity was used to determine the projections of the absolute angular velocity of the platform of the gauge on its axes, which made possible the integration of the fundamental equation of inertial navigation for the case in which the axes of sensitivity of the newtonomters are rigidly bound to the platform.

It was also assumed that the sensing masses of the three linear newtonometers are always situated at a single point on the object. The determination of the location of this point in the main Cartesian coordinate system was, therefore, a problem which was solved with the aid of the ideal equations obtained in the preceding sections.

Even the most general considerations indicate, however, that in the design of an inertial navigation system it is in principle possible to dispense with gyroscopic sensing elements[11]. In fact, let us return to the system considered in §3.1. This system (Figure 3.1) consists of the platform of a three-component gauge of absolute angular velocity to which are attached three newtonometers, the sensing masses of which are situated in the center of the platform gimbals on the object. In this case the newtonomter readings are the projections of the fundamental equation of inertial navigation (1.79) or (1.88) on the axes of the Oxyz coordinate system.

The newtonometer readings are functions of the first and second derivatives of the Cartesian coordinates x, y, z of the point O in the O_1xyz coordinate system, the projections and derivatives of the projections of the absolute angular velocity $\vec{\omega}$ of the trihedron Oxyz on its axes, as well as the projections of the vector $\vec{g}$ of the strength of the earth's gravitational field on the x, y, z axes. To integrate the fundamental equation of inertial navigation, i.e., to obtain the ideal equations (3.59) -- (3.65) and the formulas deriving from them, it was assumed that the vector $\vec{g}$ is a known function of the radius vector $\vec{r}$ in the earth body-axis system. In integrating the fundamental equation and in converting from the coordinates x, y, z to the coordinates ξ_*, η_*, ζ_*, we use the readings m_x, m_y, m_z of the gauge of absolute angular velocity, which are equal to ω_x, ω_y, ω_z, respectively, for errorless operation of this instrument.

Let us now assume that, on the platform or, equivalently, in the trihedron Oxyz, newtonometers are rigidly attached not only at the point O, but at several other points O^i. It is evident that the newtonometer readings at these points will differ from the newtonometer readings at point O even when their axes of sensitivity are identically oriented.

There are two causes for differences in the newtonometer readings (assuming that trihedron Oxyz is rigid): non-uniformity of the gravitational field and rotation of trihedron Oxyz in inertial space.
If we consider, as previously, the gravitational field to be a known function of a point in space, then the difference in the newtonometer readings at points O^i and O, caused by non-uniformity in the gravitational field, may be calculated as a function of the coördinates determined by the inertial system. Thus, by comparing the newtonometer readings at the points O^i and O, it is possible to obtain information regarding the angular velocity of trihedron Oxyz, whence derives the theoretical possibility of dispensing with gyroscopic sensing elements in the design of inertial systems.

3.4.2. Information contained in the readings of a group of mutually displaced newtonometers. Let us assume that an inertial system contains a platform which is either rigidly attached to the object or is gimballed. As previously, let us attached to this platform a right orthogonal trihedron Oxyz (Figure 3.6). We will locate

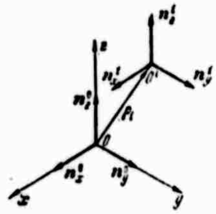

Figure 3.6

the newtonometers on the platform in the following manner. We will place three newtonometers, the readings of which we designate as n_x^0, n_y^0, n_z^0, at the origin O of Oxyz, directing their axes of sensitivity (the unit vectors of which in Figure 3.6 are also designated as n_x^0, n_y^0, n_z^0) along the x, y, z axes in precisely the same way as in the system (Figure 3.1) considered in §3.1. We select in trihedron Oxyz several points O^i and place at each of these points three newtonometers oriented similarly to newtonometers n_x^0, n_y^0, n_z^0. The newtonometer readings at the point O^i will be designated by n_x^i, n_y^i, n_z^i.

As before, we will consider the task of the inertial navigation system to be the determination of the coordinates of the point O^i at which sensing masses of the newtonomters n_x^0, n_y^0, n_z^0 are located.

Let $\vec{\rho}_i$ be the radius vector of the point O^i relative to the point O, and let $\vec{r}$ be the radius vector of the point O relative to the center of the earth O_1. The radius vector of O^i relative to point O_1 will then be

$$r_i = r + \rho_i. \tag{3.354}$$

Since the position of point O^i in the Oxyz coordinate system is assumed to be constant,

$$\rho_i = \rho_{ix} x + \rho_{iy} y + \rho_{iz} z. \tag{3.355}$$

where ρ_{ix}, ρ_{iy}, ρ_{iz} are constants.

In accordance with (1.88),

$$n^0 = \frac{d^2 r}{dt^2} - g(r), \quad n^i = \frac{d^2 r_i}{dt^2} - g(r_i). \tag{3.356}$$

where

$$\left.\begin{aligned} n^0 &= n_x^0 x + n_y^0 y + n_z^0 z, \\ n^i &= n_x^i x + n_y^i y + n_z^i z. \end{aligned}\right\} \tag{3.357}$$

Let us subtract the vector $\vec{n}^0$ from the vector $\vec{n}^i$, and denote this difference by $\vec{n}^{0i}$. From relations (3.356) and (3.354) we obtain:

$$n^{0i} = n^i - n^0 = \frac{d^2 \rho_i}{dt^2} - g(r + \rho_i) + g(r). \tag{3.358}$$

Using relations (3.3), we find that

$$\frac{d^2 \rho_i}{dt^2} = \ddot{\rho}_i + 2\omega \times \dot{\rho}_i + \dot{\omega} \times \rho_i + \omega \times (\omega \times \rho_i). \tag{3.359}$$

where $\vec{\omega}$ is the absolute angular velocity of the trihedron Oxyz, and the dots, as before, designate the local derivatives of the vectors in the Oxyz coordinate system, or, equivalently, in the O_1xyz coordinate system, since these coordinate systems are identically oriented.

It follows from (3.355) that

$$\ddot{\rho}_i = \dot{\rho}_i = 0, \tag{3.360}$$

since in the Oxyz coordinate system the vector ρ_i is constant. Thus,

$$\frac{d^2\rho_i}{dt^2} = \dot{\omega} \times \rho_i + \omega \times (\omega \times \rho_i). \tag{3.361}$$

We introduce the notation

$$a^i = g(r) - g(r + \rho_i). \tag{3.362}$$

From expressions (3.358), (3.361) and (3.362) we obtain:

$$n^{0i} = \dot{\omega} \times \rho_i + \omega \times (\omega \times \rho_i) + a^i \tag{3.363}$$

Projecting the vector defined by equality (3.363) on the x, y, z axes, we find the expressions for the difference between the quantities n_x^i, n_y^i, n_z^i and n_x^0, n_y^0, n_z^0 measured by the newtonomters:

$$\left.\begin{aligned}
n_x^{0i} &= n_x^i - n_x^0 = \dot{\omega}_y \rho_{iz} - \dot{\omega}_z \rho_{iy} + \\
&\quad + \omega_x (\omega_y \rho_{iy} + \omega_z \rho_{iz}) - \rho_{ix} (\omega_y^2 + \omega_z^2) + a_x^i, \\
n_y^{0i} &= n_y^i - n_y^0 = \dot{\omega}_z \rho_{ix} - \dot{\omega}_x \rho_{iz} + \\
&\quad + \omega_y (\omega_x \rho_{ix} + \omega_z \rho_{iz}) - \rho_{iy} (\omega_z^2 + \omega_x^2) + a_y^i, \\
n_z^{0i} &= n_z^i - n_z^0 = \dot{\omega}_x \rho_{iy} - \dot{\omega}_y \rho_{ix} + \\
&\quad + \omega_z (\omega_x \rho_{ix} + \omega_y \rho_{iy}) - \rho_{iz} (\omega_x^2 + \omega_y^2) + a_z^i
\end{aligned}\right| \tag{3.364}$$

Equalities (3.364) contain the newtonometer readings n_x^{0i}, n_y^{0i}, n_z^{0i} and ρ_{ix}, ρ_{iy}, ρ_{iz}, which are known quantities given that the relative locations of the newtonometers are known, the projections ω_x, ω_y, ω_z of the absolute angular velocity of trihedron Oxyz on its axes, which are being sought, and the quantities a_x^i, a_y^i, a_z^i.

To find ω_x, ω_y, and ω_z from equalities (3.364), we must first either use the latter equalities to find relations which do not contain a_x^i, a_y^i, a_z^i, or express a_x^i, a_y^i, a_z^i in terms of parameters determined by the inertial system, for example, in terms of the coordinates x, y, z of the point O in the O_1xyz coordinate system. The expression of a_x^i, a_y^i, a_z^i in terms of the coordinates x, y, z gives rise to certain difficulties. These difficulties stem from the fact that the earth's gravitational field is determined in the $O_1\xi\eta\zeta$ coordinate system, which is a rigid earth body-axis system. In this coordinate system the projections g_ξ, g_η, g_ζ of the strength vector $\vec{g}$ of the gravitational field on the coordinate axes are functions of the coordinates ξ, η, ζ. The O_1xyz coordinate system, however, rotates relative to the $O_1\xi\eta\zeta$ coordinate system. The projections of the strength vector $\vec{g}$ of the gravitational field at the points O and O^i on the x, y, z axes will be functions not only of the coordinates x, y, z of these points in the O_1xyz coordinate system, but of the parameters defining the orientation of the O_1xyz coordinate system relative to the $O_1\xi\eta\zeta$ coordinate system, for example, the direction cosines between the corresponding axes. The latter are defined in terms of ω_x, ω_y, and ω_z and the earth rate from relations (3.61) and (3.64) or the equivalent relations (3.31) -- (3.33), (3.60) and (3.41). To find a_x^i, a_y^i, a_z^i, therefore, it is necessary to use these relations, as well as relations (3.62) and (3.63), relating the coordinates ξ_*, η_*, ζ_* and ξ, η, ζ with the coordinates x, y, z.

As before, we will use V to denote the force function of the earth's gravitational field:

$$V = V(\xi, \eta, \zeta) \qquad (3.365)$$

The projections g_ξ, g_η, g_ζ of the strength vector g of the gravitational field on the ξ, η, ζ axes will then be:

$$g_\xi = \frac{\partial V}{\partial \xi}, \quad g_\eta = \frac{\partial V}{\partial \eta}, \quad g_\zeta = \frac{\partial V}{\partial \zeta}. \qquad (3.366)$$

Substituting into the function V (ξ, η, ζ) the expressions (3.63) for the coordinates ξ, η, ζ in terms of the coordinates x, y, z and the direction cosines $\beta_{ij}(t)$, we now obtain:

$$V = V(x, y, z, t). \qquad (3.367)$$

Time enters explicitly into relations (3.367) in terms of $\beta_{ij}(t)$.

The function V(x, y, z, t) is the force function of the earth's gravitational field in the O_1xyz coordinate system. Indeed, from relations (3.367) and (3.63) we have:

$$\left.\begin{aligned}\frac{\partial V}{\partial x}&=\frac{\partial V}{\partial\xi}\beta_{11}+\frac{\partial V}{\partial\eta}\beta_{21}+\frac{\partial V}{\partial\zeta}\beta_{31},\\ \frac{\partial V}{\partial y}&=\frac{\partial V}{\partial\xi}\beta_{12}+\frac{\partial V}{\partial\eta}\beta_{22}+\frac{\partial V}{\partial\zeta}\beta_{32},\\ \frac{\partial V}{\partial z}&=\frac{\partial V}{\partial\xi}\beta_{13}+\frac{\partial V}{\partial\eta}\beta_{23}+\frac{\partial V}{\partial\zeta}\beta_{33},\end{aligned}\right| \tag{3.368}$$

i.e.,

$$\left.\begin{aligned}g_x&=g_\xi\beta_{11}+g_\eta\beta_{21}+g_\zeta\beta_{31},\\ g_y&=g_\xi\beta_{12}+g_\eta\beta_{22}+g_\zeta\beta_{32},\\ g_z&=g_\xi\beta_{13}+g_\eta\beta_{23}+g_\zeta\beta_{33}.\end{aligned}\right| \tag{3.369}$$

Differentiating equalities (3.368) with respect to x, y and z, we find:

$$\left.\begin{aligned}\frac{\partial^2 V}{\partial x^2}&=\frac{\partial^2 V}{\partial\xi^2}\beta_{11}^2+\frac{\partial^2 V}{\partial\eta^2}\beta_{21}^2+\frac{\partial^2 V}{\partial\zeta^2}\beta_{31}^2+\\ &\quad+2\left(\frac{\partial^2 V}{\partial\xi\,\partial\eta}\beta_{11}\beta_{21}+\frac{\partial^2 V}{\partial\xi\,\partial\zeta}\beta_{11}\beta_{31}+\frac{\partial^2 V}{\partial\eta\,\partial\zeta}\beta_{21}\beta_{31}\right),\\ \frac{\partial^2 V}{\partial y^2}&=\frac{\partial^2 V}{\partial\xi^2}\beta_{12}^2+\frac{\partial^2 V}{\partial\eta^2}\beta_{22}^2+\frac{\partial^2 V}{\partial\zeta^2}\beta_{32}^2+\\ &\quad+2\left(\frac{\partial^2 V}{\partial\xi\,\partial\eta}\beta_{12}\beta_{22}+\frac{\partial^2 V}{\partial\xi\,\partial\zeta}\beta_{12}\beta_{32}+\frac{\partial^2 V}{\partial\eta\,\partial\zeta}\beta_{22}\beta_{32}\right),\\ \frac{\partial^2 V}{\partial z^2}&=\frac{\partial^2 V}{\partial\xi^2}\beta_{13}^2+\frac{\partial^2 V}{\partial\eta^2}\beta_{23}^2+\frac{\partial^2 V}{\partial\zeta^2}\beta_{33}^2+\\ &\quad+2\left(\frac{\partial^2 V}{\partial\xi\,\partial\eta}\beta_{13}\beta_{23}+\frac{\partial^2 V}{\partial\xi\,\partial\zeta}\beta_{13}\beta_{33}+\frac{\partial^2 V}{\partial\eta\,\partial\zeta}\beta_{23}\beta_{33}\right).\end{aligned}\right| \tag{3.370}$$

Now, taking into account the fact that the O_1xyz and $O_1\xi\eta\zeta$ coordinate systems are orthogonal, the validity of the following equality is easily demonstrated:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial\xi^2}+\frac{\partial^2 V}{\partial\zeta^2}+\frac{\partial^2 V}{\partial\eta^2}. \tag{3.371}$$

But since the function $V(\xi, \eta, \zeta)$ is the gravitational potential, it satisfied Laplace's equation in the $O_1\xi\eta\zeta$ coordinate system.[12] Therefore:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0. \tag{3.372}$$

As a result, the strength of the earth's gravitational field in the O_1xyz coordinate system will be the gradient of the function $V(x, y, z, t)$:

$$g = \operatorname{grad} V(x, y, z, t) \tag{3.373}$$

Let us return to relation (3.362) and find explicit expressions or a_x^i, a_y^i, and a_z^i.

From relations (3.362) and equalities (3.373) and (3.355), it follows that

$$-\operatorname{grad} V(x+\rho_{ix}, y+\rho_{iy}, z+\rho_{iz}, t). \tag{3.374}$$

Assuming that ρ_x, ρ_y, $\dot{\rho}_z$ are small, we obtain:

$$\left.\begin{aligned}
a_x^i &= -\frac{\partial^2 V}{\partial x^2}\rho_{ix} - \frac{\partial^2 V}{\partial x\,\partial y}\rho_{iy} - \frac{\partial^2 V}{\partial x\,\partial z}\rho_{iz},\\
a_y^i &= -\frac{\partial^2 V}{\partial x\,\partial y}\rho_{ix} - \frac{\partial^2 V}{\partial y^2}\rho_{iy} - \frac{\partial^2 V}{\partial y\,\partial z}\rho_{iz},\\
a_z^i &= -\frac{\partial^2 V}{\partial x\,\partial z}\rho_{ix} - \frac{\partial^2 V}{\partial y\,\partial z}\rho_{iy} - \frac{\partial^2 V}{\partial z^2}\rho_{iz}.
\end{aligned}\right\} \tag{3.375}$$

In relations (3.375) the second derivatives of V are taken at the point O, i.e., at the point with the coordinates x, y, z. The second partial derivatives of V with respect to x, y and z are determined by equalities (3.370). The mixed derivatives are found from expressions (3.368) and (3.363). They are:

$$\left.\begin{aligned}
\frac{\partial^2 V}{\partial x\,\partial y} &= \frac{\partial^2 V}{\partial \xi^2}\beta_{11}\beta_{12} + \frac{\partial^2 V}{\partial \eta^2}\beta_{21}\beta_{22} + \frac{\partial^2 V}{\partial \zeta^2}\beta_{31}\beta_{32} + \\
&+ \frac{\partial^2 V}{\partial \xi\,\partial \eta}(\beta_{11}\beta_{22} + \beta_{21}\beta_{12}) + \frac{\partial^2 V}{\partial \xi\,\partial \zeta}(\beta_{11}\beta_{32} + \beta_{31}\beta_{12}) + \\
&+ \frac{\partial^2 V}{\partial \eta\,\partial \zeta}(\beta_{21}\beta_{32} + \beta_{31}\beta_{22}).
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{\partial^2 V}{\partial x\,\partial z} &= \frac{\partial^2 V}{\partial \xi^2}\beta_{11}\beta_{13} + \frac{\partial^2 V}{\partial \eta^2}\beta_{21}\beta_{23} + \frac{\partial^2 V}{\partial \zeta^2}\beta_{31}\beta_{33} + \\
&+ \frac{\partial^2 V}{\partial \xi\,\partial \eta}(\beta_{11}\beta_{23} + \beta_{21}\beta_{13}) + \frac{\partial^2 V}{\partial \xi\,\partial \zeta}(\beta_{11}\beta_{33} + \beta_{31}\beta_{13}) + \\
&+ \frac{\partial^2 V}{\partial \eta\,\partial \zeta}(\beta_{21}\beta_{33} + \beta_{31}\beta_{23}).\\
\frac{\partial^2 V}{\partial y\,\partial z} &= \frac{\partial^2 V}{\partial \xi^2}\beta_{12}\beta_{13} + \frac{\partial^2 V}{\partial \eta^2}\beta_{22}\beta_{23} + \frac{\partial^2 V}{\partial \zeta^2}\beta_{32}\beta_{33} + \\
&+ \frac{\partial^2 V}{\partial \xi\,\partial \eta}(\beta_{12}\beta_{23} + \beta_{22}\beta_{13}) + \frac{\partial^2 V}{\partial \xi\,\partial \zeta}(\beta_{12}\beta_{33} + \beta_{32}\beta_{13}) + \\
&+ \frac{\partial^2 V}{\partial \eta\,\partial \zeta}(\beta_{22}\beta_{33} + \beta_{32}\beta_{23}).
\end{aligned}\right\} \tag{3.376}$$

Relations (3.364) and (3.375) define the quantities n_x^{0i}, n_y^{0i}, n_z^{0i} for arbitrary selection of the point O^i.

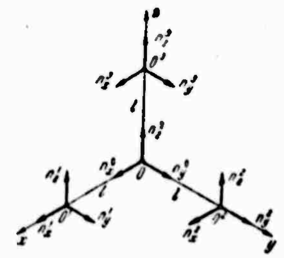

Figure 3.7

It is convenient to take as the points O^i the points O^1, O^2, and O^3, situated on the x, y and z axes, respectively, at equal distances from the origin O of the Oxyz coordinate system (Figure 3.7); as will be seen below, this does not cause the analysis to lose generality.

Then

$$\left.\begin{aligned} &\rho_{1x} = \rho_{2y} = \rho_{3z} = l, \\ &\rho_{1y} = \rho_{1z} = \rho_{2x} = \\ &\quad = \rho_{2z} = \rho_{3x} = \\ &\quad\quad = \rho_{3y} = 0. \end{aligned}\right\} \tag{3.377}$$

Substituting equalities (3.377) into (3.375) and (3.64), we arrive at the following expressions:

$$\left.\begin{aligned} n_x^{01} &= -l\left(\omega_y^2 + \omega_z^2 + \frac{\partial^2 V}{\partial x^2}\right), \\ n_y^{01} &= l\left(\dot{\omega}_z + \omega_x\omega_y - \frac{\partial^2 V}{\partial x\,\partial y}\right), \\ n_z^{01} &= l\left(-\dot{\omega}_y + \omega_x\omega_z - \frac{\partial^2 V}{\partial x\,\partial z}\right), \end{aligned}\right|$$

$$\left.\begin{aligned} n_x^{02} &= l\left(-\dot{\omega}_z + \omega_x\omega_y - \frac{\partial^2 V}{\partial x\,\partial y}\right), \\ n_y^{02} &= -l\left(\omega_x^2 + \omega_z^2 + \frac{\partial^2 V}{\partial y^2}\right), \\ n_z^{02} &= l\left(\dot{\omega}_x + \omega_y\omega_z - \frac{\partial^2 V}{\partial y\,\partial z}\right), \\ n_x^{03} &= l\left(\dot{\omega}_y + \omega_x\omega_z - \frac{\partial^2 V}{\partial x\,\partial z}\right), \\ n_y^{03} &= l\left(-\dot{\omega}_x + \omega_y\omega_z - \frac{\partial^2 V}{\partial y\,\partial z}\right), \\ n_z^{03} &= -l\left(\omega_x^2 + \omega_y^2 + \frac{\partial^2 V}{\partial z^2}\right). \end{aligned}\right| \tag{3.378}$$

It can be shown that equalities (3.378), together with the first equality (3.356), exhaust the information obtainable from newtonometers rigidly situated in the Oxyz coordinate system near its origin: they exhaust it in the sense that the mounting of extra newtonomters in addition to the twelve situated at points O, O^1, O^2, and O^3, add nothing to the information already available .

Indeed, let an additional newtonometer be situated at some P with coordinates ρ_x, ρ_y and ρ_z, and let the direction of its axis of sensitivity $\vec{e}$ form constant angles with the x, y, z axes, the cosines of which are γ_1, γ_2 and γ_3. Then from equalities (3.356), (3.364) and (3.375) we see that the reading of newtonometer n_p will be:

$$\begin{aligned} n_p = {} & n_x^0\gamma_1 + n_y^0\gamma_2 + n_z^0\gamma_3 + \Big[(\dot\omega_y\rho_z - \dot\omega_z\rho_y) + \\ & + \omega_x(\omega_y\rho_y + \omega_z\rho_z) - \rho_x(\omega_y^2 + \omega_z^2) - \\ & - \frac{\partial^2 V}{\partial x^2}\rho_x - \frac{\partial^2 V}{\partial x\,\partial y}\rho_y - \frac{\partial^2 V}{\partial x\,\partial z}\rho_z\Big]\gamma_1 + \\ & + \Big[(\dot\omega_z\rho_x - \dot\omega_x\rho_z) + \omega_y(\omega_x\rho_x + \omega_z\rho_z) - \\ & - \rho_y(\omega_x^2 + \omega_z^2) - \frac{\partial^2 V}{\partial x\,\partial y}\rho_x - \frac{\partial^2 V}{\partial y^2}\rho_y - \frac{\partial^2 V}{\partial y\,\partial z}\rho_z\Big]\gamma_2 + \\ & + \Big[(\dot\omega_x\rho_y - \dot\omega_y\rho_x) + \omega_z(\omega_x\rho_x + \omega_y\rho_y) - \rho_z(\omega_x^2 + \omega_y^2) - \\ & - \frac{\partial^2 V}{\partial x\,\partial z}\rho_x - \frac{\partial^2 V}{\partial y\,\partial z}\rho_y - \frac{\partial^2 V}{\partial z^2}\rho_z\Big]\gamma_3. \end{aligned} \tag{3.379}$$

Grouping the terms on the right side of equality (3.379) reduces it to the form

$$\begin{aligned} n_p = {} & n_x^0\gamma_1 + n_y^0\gamma_2 + n_z^0\gamma_3 + \\ & + \rho_x\Big[-\Big(\omega_y^2 + \omega_z^2 + \frac{\partial^2 V}{\partial x^2}\Big)\gamma_1 + \Big(\dot\omega_z + \omega_x\omega_y - \frac{\partial^2 V}{\partial x\,\partial y}\Big)\gamma_2 + \\ & + \Big(-\dot\omega_y + \omega_x\omega_z - \frac{\partial^2 V}{\partial x\,\partial z}\Big)\gamma_3\Big] + \\ & + \rho_y\Big[\Big(-\dot\omega_z + \omega_x\omega_y - \frac{\partial^2 V}{\partial x\,\partial y}\Big)\gamma_1 - \Big(\omega_x^2 + \omega_z^2 + \frac{\partial^2 V}{\partial y^2}\Big)\gamma_2 + \\ & + \Big(\dot\omega_x + \omega_y\omega_z - \frac{\partial^2 V}{\partial y\,\partial z}\Big)\gamma_3\Big] + \rho_z\Big[\Big(\dot\omega_y + \omega_x\omega_z - \frac{\partial^2 V}{\partial x\,\partial z}\Big)\gamma_1 + \\ & + \Big(-\dot\omega_x + \omega_y\omega_z - \frac{\partial^2 V}{\partial y\,\partial z}\Big)\gamma_2 - \Big(\omega_x^2 + \omega_y^2 + \frac{\partial^2 V}{\partial z^2}\Big)\gamma_3\Big]. \end{aligned} \tag{3.380}$$

Comparing equality (3.380) with relations (3.378), we see that the expressions in parentheses on the right side of equality (3.380) are the right sides of relations (3.378), divided by ℓ. Therefore equality (3.380) may be represented in the form:

$$n_p = n_x^0\gamma_1 + n_y^0\gamma_2 + n_z^0\gamma_3 + \frac{\rho_x}{l}\left(n_x^{01}\gamma_1 + n_y^{01}\gamma_2 + n_z^{01}\gamma_3\right) + \frac{\rho_y}{l}\left(n_x^{02}\gamma_1 + n_y^{02}\gamma_2 + n_z^{02}\gamma_3\right) + \frac{\rho_z}{l}\left(n_x^{03}\gamma_1 + n_y^{03}\gamma_2 + n_z^{03}\gamma_3\right) \tag{3.381}$$

Thus, the reading n_p of the newtonometer is a linear combination of the readings of the twelve newtonomters n_x^0, n_y^0, n_z^0, n_x^1, n_y^1, n_z^1, n_x^2, n_y^2, n_z^2, n_x^3, n_y^3, n_z^3 situated at points O, O^1, O^2, and O^3. This demonstrates that addition of newtonometers to those situated at points O, O^1, O^2 and O^3 does not increase the volume of information contained in the newtonometer readings. Of course, this demonstration is valid only under the assumption that ℓ, ρ_x, ρ_y and ρ_z are sufficiently small such that, in the Taylor series expansion of $\partial V/\partial x$, $\partial V/\partial y$, $\partial V/\partial z$ in the neighborhood of point O, only linear terms in ℓ, ρ_x, ρ_y and ρ_z need be considered.

3.4.3. The ideal equations of an inertial system with only newtonometers as its sensing elements. Let us consider the basic alternatives for compiling the ideal equations of systems in which only newtonometers are used as the sensing elements. To do this we will represent equalities (3.378) in a somewhat different form.

Forming the appropriate linear combinations of equalities (3.378), we obtain the following relations:

$$\left.\begin{aligned} &n_x^{01} + n_y^{02} - n_z^{03} + l\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial z^2}\right) = -2l\omega_z^2, \\ &-n_x^{01} + n_y^{02} + n_z^{03} + \\ &\qquad + l\left(-\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) = -2l\omega_x^2, \\ &n_x^{01} - n_y^{02} + n_z^{03} + l\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) = -2l\omega_y^2; \end{aligned}\right\} \tag{3.382}$$

$$\left.\begin{aligned} &n_y^{01} + n_x^{02} + 2l\frac{\partial^2 V}{\partial x\,\partial y} = 2l\omega_x\omega_y, \\ &n_z^{01} + n_x^{03} + 2l\frac{\partial^2 V}{\partial x\,\partial z} = 2l\omega_x\omega_z, \\ &n_z^{02} + n_y^{03} + 2l\frac{\partial^2 V}{\partial y\,\partial z} = 2l\omega_y\omega_z; \end{aligned}\right\} \tag{3.383}$$

$$
\left.\begin{aligned}
n_y^{01} - n_x^{02} &= 2I\dot{\omega}_z, \\
-n_z^{01} + n_x^{03} &= 2I\dot{\omega}_y, \\
n_z^{02} - n_y^{03} &= 2I\dot{\omega}_x;
\end{aligned}\right\} \tag{3.384}
$$

The nine equalities (3.382), (3.383) and (3.384) are linearly independent and therefore fully equivalent to equalities (3.378) Their linear independence follows from the fact that the determinant of the coefficients for n_x^{0s}, n_y^{0s}, n_z^{0s} ($s = 1, 2, 3$) is non-zero:

$$
\Delta = \begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{vmatrix} = 16 \neq 0. \tag{3.385}
$$

Equalities (3.382) may be simplified on the basis of the fact that it follows from the Laplace equation (3.372) that

$$
\left.\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial z^2} &= -2\frac{\partial^2 V}{\partial z^2}, \\
\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= -2\frac{\partial^2 V}{\partial y^2}, \\
-\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= -2\frac{\partial^2 V}{\partial x^2}.
\end{aligned}\right\} \tag{3.386}
$$

Let us substitute these values into relations (3.382) and complete equalities (3.382), (3.383) and (3.384) by adding to them the equations for n_x^0, n_y^0, and n_z^0 deriving from formulas (3.356), (3.3) and (3.373). We then obtain four groups of equations:

$$
\left.\begin{aligned}
\dot{v}_x &= n_x^0 - \omega_y v_z + \omega_z v_y + \frac{\partial V}{\partial x}, \\
\dot{v}_y &= n_y^0 - \omega_z v_x + \omega_x v_z + \frac{\partial V}{\partial y}, \\
\dot{v}_z &= n_z^0 - \omega_x v_y + \omega_y v_x + \frac{\partial V}{\partial z}, \\
\dot{x} &= v_x - \omega_y z + \omega_z y, \\
\dot{y} &= v_y - \omega_z x + \omega_x z, \\
\dot{z} &= v_z - \omega_x y + \omega_y x;
\end{aligned}\right\} \tag{3.387}
$$

$$
\left.\begin{aligned}
-2l\omega_x^2 &= -n_x^{01} + n_y^{02} + n_z^{03} - 2l\frac{\partial^2 V}{\partial x^2},\\
-2l\omega_y^2 &= n_x^{01} - n_y^{02} + n_z^{03} - 2l\frac{\partial^2 V}{\partial y^2},\\
-2l\omega_z^2 &= n_x^{01} + n_y^{02} - n_z^{03} - 2l\frac{\partial^2 V}{\partial z^2};
\end{aligned}\right| \tag{3.388}
$$

$$
\left.\begin{aligned}
2l\omega_x\omega_y &= n_y^{01} + n_x^{02} + 2l\frac{\partial^2 V}{\partial x\,\partial y},\\
2l\omega_x\omega_z &= n_z^{01} + n_x^{03} + 2l\frac{\partial^2 V}{\partial x\,\partial z},\\
2l\omega_y\omega_z &= n_z^{02} + n_y^{03} + 2l\frac{\partial^2 V}{\partial y\,\partial z};
\end{aligned}\right| \tag{3.389}
$$

$$
\left.\begin{aligned}
2l\dot{\omega}_x &= n_z^{02} - n_y^{03},\\
2l\dot{\omega}_y &= -n_z^{01} + n_x^{03},\\
2l\dot{\omega}_z &= n_y^{01} - n_x^{02}.
\end{aligned}\right| \tag{3.390}
$$

The first group of equations, i.e., equations (3.387), are the same as equations (3.59). They enable us to use n_x^0, n_y^0, n_z^0, ω_x, ω_y, ω_z to determine the Cartesian coordinates x, y, z of the point O if the force function V of the earth's gravitational field, the initial values of the coordinates x(0), y(0), z(0) and the initial values of their derivatives $\dot{x}(0)$, $\dot{y}(0)$, $\dot{z}(0)$ are known.

The second and third groups of equations relate the projections of the absolute angular velocity ω_x, ω_y and ω_z to the characteristics of the gravitational field and the newtonometer readings. Equations (3.388) contain, in addition n_x^0, n_y^0, and n_z^0, the readings n_x^1, n_y^2, and n_z^3 of only three newtonometers, while equations (3.389) contain the readings n_y^1, n_x^2, n_z^1, n_x^3, n_z^2 and n_y^3 of six newtonometers. Equations (3.388) and (3.389) are second order algebraic equations.

The expressions ω_x^2, ω_y^2, and ω_z^2 in terms of n_x^{01}, n_y^{02}, and n_z^{03} and the second derivatives of the force function V are evident from relations (3.388). In order to find them, we have only to divide relations (3.388) by $2l$.

It is also easy to find expressions for ω_x^2, ω_y^2 and ω_z^2 from equalities (3.389). They have the form:

$$
\left.
\begin{aligned}
\omega_x^2 &= \frac{\left(n_y^{01} + n_x^{02} + 2l\,\frac{\partial^2 V}{\partial x\,\partial y}\right)\left(n_z^{01} + n_x^{03} + 2l\,\frac{\partial^2 V}{\partial x\,\partial z}\right)}{2l\left(n_z^{02} + n_y^{03} + 2l\,\frac{\partial^2 V}{\partial y\,\partial z}\right)},\\
\omega_y^2 &= \frac{\left(n_y^{01} + n_x^{02} + 2l\,\frac{\partial^2 V}{\partial x\,\partial y}\right)\left(n_z^{02} + n_y^{03} + 2l\,\frac{\partial^2 V}{\partial y\,\partial z}\right)}{2l\left(n_z^{01} + n_x^{03} + 2l\,\frac{\partial^2 V}{\partial x\,\partial z}\right)},\\
\omega_z^2 &= \frac{\left(n_z^{02} + n_y^{03} + 2l\,\frac{\partial^2 V}{\partial y\,\partial z}\right)\left(n_z^{01} + n_x^{03} + 2l\,\frac{\partial^2 V}{\partial x\,\partial z}\right)}{2l\left(n_y^{01} + n_x^{02} + 2l\,\frac{\partial^2 V}{\partial x\,\partial y}\right)}.
\end{aligned}
\right| \tag{3.391}
$$

The fourth group of equations, equations (3.390), relate ω_x, ω_y and ω_z to the readings of six additional newtonometers. These are the same six newtonometers the readings of which are contained in equations (3.389). Equations (3.390), as distinct from relations (3.388) and (3.389), are differential equations. A noteworthy characteristic of equations (3.390) is the fact that they are linear and do not contain any characteristics of the gravitational field. It follows from equations (3.390) that:

$$
\left.
\begin{aligned}
\omega_x &= \frac{1}{2l}\int_0^t \left(n_z^{02} - n_y^{03}\right) dt + \omega_x(0),\\
\omega_y &= \frac{1}{2l}\int_0^t \left(-n_z^{01} + n_x^{03}\right) dt + \omega_y(0),\\
\omega_z &= \frac{1}{2l}\int_0^t \left(n_y^{01} - n_x^{02}\right) dt + \omega_z(0).
\end{aligned}
\right\} \tag{3.392}
$$

Like equations (3.390), the systems of equations (3.388) and (3.389) contain, if the characteristics of the gravitational field are known, three unknowns: ω_x, ω_y and ω_z. However, only the system of equations (3.390) permits complete determination of ω_x, ω_y and ω_z by means of formulas (3.392). The quadratic equations (3.388) permit determination only of the moduli $|\omega_x|$, $|\omega_y|$, and $|\omega_z|$, but do not permit determination of the signs of these quantities. It is evident that knowing the signs of ω_x, ω_y and ω_z at the initial moment of time is, in general, insufficient to determine the signs of these quantities subsequently. Equations (3.389) also reduce to quadratic equations and therefore do not permit determination of the signs of

ω_x, ω_y and ω_z. Indeed, if the signs of these quantities change, the left sides of equations (3.389) do not change sign.

Thus, use of equations (3.388) to find ω_x, ω_y and ω_z requires the simultaneous use of all three equations (3.390) in order to find the signs of the projections. But since the signs may be found using equations (3.390) only by fully determining ω_x, ω_y and ω_z from these equations, equations (3.388) become superfluous.

Consequently, to find ω_x, ω_y and ω_z we may either use the three equations (3.390), or some combination of equations (3.389) and (3.390).[13] From the point of view of simplicity the first approach appears to be the most appropriate. The second method has several variants. Thus, it is possible to use one of equations (3.390) and two of equations (3.389), for example, the first equatior of (3.390) and the first and second equations of (3.389). It is possible, on the other hand, to use two equations from system (3.390) and one from (3.389), for example, the first two equations (3.390) and the second or third equation (3.389). The remaining equations in systems (3.388), (3.389) and (3.390) are superfluous here and may be used only as redundant information.

To summarize, let us enumerate the equations which can constitute the operational algorithm of an inertial system without gyroscopic sensing elements.

These are primarily equations (3.391), the integration of which yields relations (3.59). Included, further, are the equations for determining the projections ω_x, ω_y and ω_z: either equations (3.390) or, equivalently, equations (3.392), or one of the above-mentioned combinations of equations in systems (3.389) and (3.392). The operational algorithm will contain equations (3.60), by means of which the direction cosines α_{ij} between the ξ_*, η_*, ζ_* and x, y, z axes are found, and equations (3.61) and (3.64) for the direction cosines α_{ij} between the ξ, η, ζ, and x, y, z axes, or the equivalent equations (3.31) -- (3.33), (3.41), and (3.60). Finally, the operational algorithm will contain equalities (3.62), relating the coordinates ξ_*, η_*, and ζ_* to the coordinates x, y, z, equalities (3.63), relating the coordinates ξ, η, ζ to the coordinates x, y, z, and also equality (3.365), by means of which the force function of the earth's gravitational field is determined in the earth body-axis coordinate system $O_1\xi\eta\zeta$, and the equalities (3.368), together with the required relations from equalities (3.370) and (3.376), depending on which of the equations from systems (3.389) and (3.392) are chosen to determine the projections ω_x, ω_y and ω_z.

The combinations of relations enumerated above form closed systems of equations. From them may be found the coordinates x, y, z and the velocities $\dot{x}$, $\dot{y}$, $\dot{z}$, the coordinates ξ_*, η_*, ζ_* and ξ, η, ζ, and the projections ω_x, ω_y, ω_z of the absolute angular velocity. Of course, the velocities $\dot{\xi}_*$, $\dot{\eta}_*$, $\dot{\zeta}_*$ and $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ may also be found if necessary. These equations may also be used to determine the direction cosines α_{ij}, α'_{ij} and β_{ij}, characterizing the relative positions of the Oxyz, $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$ coordinate systems. The first of these coordinate systems is bound to the platform on which the newtonometers are mounted, the third is rigidly bound to the earth, and the second is formed by the coordinate axes of the main Cartesian coordinate system. If the platform on which the newtonometers are mounted is itself gimballed, the rotation angles of the gimbal rings determine, in accordance with table (3.66), the orientation of the OXYZ coordinate system attached to the object in relation to the Oxyz coordinate system. Together with the direction cosines α_{ij}, table (3.66) defines the orientation of the object relative to the main Cartesian coordinate system, and, together with β_{ij}, in relation to the earth body-axis coordinate system $O_1\xi\eta\zeta$.

In constructing the operational algorithm based on the relations enumerated above, we place no restrictions on the orientation of the Oxyz coordinate system, leaving it arbitrary. As was the case with the systems using gyroscopic sensing elements analyzed in preceding sections, it is possible to place various requirements on the orientation of the Oxyz trihedron. Thus, it is possible to rigidly attach Oxyz to the object. It is possible to orient it identically to $O_1\xi_*\eta_*\zeta_*$ or $O_1\xi\eta\zeta$.

Finally, it is possible to make its orientation a specific function of time and the coordinates calculated by the inertial system.

Of special interest in this regard is the case in which the gravimetric system is used only to orient the platform in a particular way relative to the gravitational field, for example, such that one of its axes always coincides during unperturbed motion with the direction of the strength vector of the gravitational field. In this case, since the coordinates are not determined, ω_x, ω_y, and ω_z also need not be determined. They can be eliminated from the equations. The quantity to be determined will be the parameters characterizing the deviation of the platform from a given position in relation to the gravitational field.

Up to this point it has been assumed that the gravitational field is known but arbitrary. It is to be expected that, for a specific form of the gravitational field, the ideal equations may prove to be simpler than those derived for the general case. They may also be simpler if the Oxyz coordinate system is oriented in a specific manner in relation to the gravitational field.

If the earth's gravitational field is considered to be spherical, then

$$V = \frac{\mu}{r}. \tag{3.393}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{3.394}$$

and μ is the product of the mass M of the earth and the gravitational constant.

Differentiating equation (3.393), we find:

$$\left.\begin{array}{l}
\frac{\partial V}{\partial x} = -\frac{\mu x}{r^3}, \quad \frac{\partial V}{\partial y} = -\frac{\mu y}{r^3}, \quad \frac{\partial V}{\partial z} = -\frac{\mu z}{r^3}, \\
\frac{\partial^2 V}{\partial x^2} = -\frac{\mu}{r^3}\left(1 - \frac{3x^2}{r^2}\right), \quad \frac{\partial^2 V}{\partial y^2} = -\frac{\mu}{r^3}\left(1 - \frac{3y^2}{r^2}\right), \\
\frac{\partial^2 V}{\partial z^2} = -\frac{\mu}{r^3}\left(1 - \frac{3z^2}{r^2}\right), \\
\frac{\partial^2 V}{\partial x \partial y} = \frac{3\mu xy}{r^5}, \quad \frac{\partial^2 V}{\partial y \partial z} = \frac{3\mu yz}{5}, \quad \frac{\partial^2 V}{\partial x \partial z} = \frac{3\mu xz}{r^5}.
\end{array}\right| \tag{3.395}$$

If the Oz axis of the Oxyz coordinate system is superposed on the radius vector $\vec{r}$, expressions (3.395) simplify, since in this case

$$x = y = 0, \quad z = r. \tag{3.396}$$

It follows from equalities (3.395) and (3.396) that

$$\left.\begin{aligned} &\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial x \partial z} = \frac{\partial^2 V}{\partial y \partial z} = 0, \\ &\frac{\partial V}{\partial z} = -\frac{\mu}{r^2}, \quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = -\frac{\mu}{r^3}, \\ &\frac{\partial^2 V}{\partial z^2} = \frac{2\mu}{r^3}. \end{aligned}\right| \tag{3.397}$$

Turning to equations (3.387), (3.388), (3.389) and (3.390), we note that for the case of a spherical gravitational field, its parameters drop out of the first two equations (3.387) and all three equations (3.389). In this case, as in systems (3.390), only the projections of the absolute angular velocity and the newtonometer readings remain in system (3.389). System (3.389) enables us to determine ω_x^2, ω_y^2, and ω_z^2 algebraically using only the newtonometer readings. Corresponding formulas are derived from relations (3.391), if the mixed derivatives of the force function V are set equal to 0.

3.4.4. <u>Using algebraic equations only. Additional remarks</u>. Let us consider equations (3.388) and (3.389) for the case of a spherical gravitational field in greater detail. We substitute into these equations the derivatives (3.395) of the force function of the earth's gravitational field, after first having introduced the following designations:

$$\begin{gathered} a_x = \frac{1}{2l}\left(n_x^{(1)} - n_y^{(2)} - n_z^{(3)}\right), \quad a_y = \frac{1}{2l}\left(n_y^{(2)} - n_z^{(3)} - n_x^{(1)}\right), \\ a_z = \frac{1}{2l}\left(n_z^{(3)} - n_x^{(1)} - n_y^{(2)}\right), \\ a_{xy} = \frac{1}{2l}\left(n_y^{(1)} + n_x^{(2)}\right), \quad a_{yz} = \frac{1}{2l}\left(n_z^{(2)} + n_y^{(3)}\right), \\ a_{zx} = \frac{1}{2l}\left(n_x^{(3)} + n_z^{(1)}\right), \\ \mu/r^3 = k, \quad x/r = x', \quad y/r = y', \quad z/r = z'. \end{gathered}$$

After substituting we arrive at the equations:

$$\left.\begin{array}{c}\omega_x^2=a_x-k(1-3x'^2),\quad \omega_y^2=a_y-k(1-3y'^2),\\ \omega_z^2=a_z-k(1-3z'^2),\\ \omega_x\omega_y=a_{xy}+3kx'y',\quad \omega_y\omega_z=a_{yz}+3ky'z',\\ \omega_z\omega_x=a_{zx}+3kz'x',\\ x'^2+y'^2+z'^2=1.\end{array}\right| \tag{3.398}$$

Equations (3.398) are algebraic. They contain seven unknowns: x', y', z', ω_x, ω_y, ω_z and k, which under certain conditions may be determined from these equations. One of these conditions is knowledge of the sign of at least one of the projections ω_x, ω_y, ω_z of the absolute angular velocity, and also the sign of at least one of the coordinates x, y, z. This is possible in certain cases. Thus, if the platform is mounted on an artificial satellite of the earth and the direction of its z axis approaches the direction of $\vec{r}$, $z \approx r$, and therefore $z > 0$. If, in addition, the orientation of the y axis is close to that of the normal to the orbit and the velocity of the angular oscillations of the platform around this axis is less than the angular velocity of the satellite's rotation around the earth, then the sign of ω_y becomes known.

The first six equations (3.398) may be written in the form of the tensor equation

$$T^{(1)}-3kT^{(2)}+kT^{(3)}+T^{(4)}=0,$$

where

$$\left.\begin{array}{c}T^{(1)}=\begin{vmatrix}\omega_x^2 & \omega_x\omega_y & \omega_x\omega_z\\ \omega_y\omega_x & \omega_y^2 & \omega_y\omega_z\\ \omega_z\omega_x & \omega_z\omega_y & \omega_z^2\end{vmatrix},\quad T^{(2)}=\begin{vmatrix}x'^2 & x'y' & x'z'\\ y'x' & y'^2 & y'z'\\ z'x' & z'y' & z'^2\end{vmatrix},\\ T^{(3)}=\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix},\quad T^{(4)}=\begin{vmatrix}a_x & a_{xy} & a_{xz}\\ a_{yx} & a_y & a_{yz}\\ a_{zx} & a_{zy} & a_z\end{vmatrix}.\end{array}\right| \tag{3.399}$$

All of these tensors are symmetrical. Tensors $T^{(1)}$ and $T^{(2)}$ are dyadic products of the vectors $\vec{\omega}$ and $\vec{r}\sqrt{3k}/r$ times themselves.[14] Tensor $T^{(3)}$ is unitary. The components of tensor $T^{(4)}$ are the quantities measured by the newtonometers.

As a result of the fact that tensors $T^{(1)}$ and $T^{(2)}$ are the dyadic products $\vec{\omega}\vec{\omega}$ and $\vec{r}\vec{r}3k/r^2$, only the first invariants of these tensors are non-zero. They are obvious:

$$J_1^{(1)} = \omega_x^2 + \omega_y^2 + \omega_z^2 = \omega^2, \quad J_1^{(2)} = x'^2 + y'^2 + z'^2 = 1.$$

Let us examine the invariants of the tensor $kT^{(3)} + T^{(4)}$:

$$\left.\begin{aligned} J_1 &= a_x + a_y + a_z - 3k, \\ J_2 &= -3k^2 + 2k(a_x + a_y + a_z) - a_x a_y - a_x a_z - \\ &\qquad - a_y a_z + a_{zx}^2 + a_{zy}^2 + a_{xy}^2, \\ J_3 &= -k^3 + k^2(a_x + a_y + a_z) + k(-a_x a_y - \\ &\quad - a_x a_z - a_y a_z + a_{zx}^2 + a_{zy}^2 + a_{xy}^2) + 2a_{xy} a_{yz} a_{zx} - \\ &\qquad - a_{zx}^2 a_y - a_{yz}^2 a_x - a_{xy}^2 a_z. \end{aligned}\right\} \tag{3.400}$$

The invariants J_1, J_2, and J_3 do not depend on the orientation of Oxyz. Let us calculate the invariants by appropriately selecting Oxyz. Superposing the z axis on the direction of $\vec{r}$, we will have: $x' = y' = 0$, $z' = 1$. Thus we obtain:

$$J_1 = \omega^2 - 3k, \quad J_2 = -\frac{3k(r \times \omega)^2}{r^2}, \quad J_3 = 0. \tag{3.401}$$

The relation $J_3 = 0$ is a cubic equation in k. This equation has, in general, three distinct real roots $(k_1 < k_2 < k_3)$. One of these is the desired value k_0. The realness of the roots derives from the fact that the roots of the equation are eigenvalues of the symmetric tensor $T^{(4)}$. Comparing expressions (3.400) for J_1, J_2 and J_3, it is evident that

$$J_2 = \frac{\partial J_3}{\partial k}, \quad J_1 = \frac{\partial J_2}{\partial k} = \frac{\partial^2 J_3}{\partial k^2}.$$

According to the second equality (3.401), $\dfrac{\partial J_3(k_0)}{\partial k} = J_2(k_0) > 0$, and therefore the desired quantity is the middle root i.e., $k_0 = k_2$.

The equation $J_3 = 0$ has a double root k_0 for $J_2 = 0$ and a triple root for $J_1 = 0$. In the first case, in accordance with the second equality (3.401), the vector $\vec{\omega}$ of the rate of rotation of Oxyz has the same direction as the vector $\vec{r}$.

In the second case $\omega^2 = 3k$.

For $J_2 = 0$, depending on the sign of $J_1 = \frac{\partial^2 J_3}{\partial k^2} = \omega^2 - 3k$, either the roots k_1 and k_2 or the roots k_2 and k_3 will coincide. It is evident that if $J_1 > 0$, $k_1 = k_2 = k_0$; if $J_1 < 0$, $k_2 = k_3 = k_0$.

After $k = k_0$ is found from the equation $J_3 = 0$, ω_x, ω_y, ω_z, x', y' and z' may also be found. Relations (3.400) and (3.401), along with the equations obtained from the invariants of $T^{(1)} + T^{(3)}$ and $T^{(2)} + T^{(3)}$, may be used for this purpose. The following approach may also be used. Since k_1, k_2 and k_3 are eigenvalues of tensor $T^{(4)}$, equations (3.398) may be reduced to the main axes of this tensor, after which they are easily solved.[15]

It is possible, on the other hand, to find x', y', z', ω_x, ω_y and ω_z by beginning directly with equations (3.398). In this case it is convenient to perform the following change of variables in equations (3.398):

$$\left.\begin{aligned}
&x_1 = \omega_x - x'\sqrt{3k}, \quad x_2 = \omega_y - y'\sqrt{3k},\\
&x_3 = \omega_z - z'\sqrt{3k},\\
&y_1 = \omega_x + x'\sqrt{3k}, \quad y_2 = \omega_y + y'\sqrt{3k},\\
&y_3 = \omega_z + z'\sqrt{3k}.
\end{aligned}\right\} \tag{3.402}$$

As a result we obtain the following equations in place of the first six equations (3.398):

$$\left.\begin{aligned}
&x_1 y_1 = a_x - k, \quad x_2 y_2 = a_y - k, \quad x_3 y_3 = a_z - k;\\
&y_1 x_2 + x_1 y_2 = 2a_{xy}, \quad y_2 x_3 + x_2 y_3 = 2a_{yz},\\
&y_3 x_1 + x_3 y_1 = 2a_{zx}
\end{aligned}\right\} \tag{3.403}$$

Substituting now y_1, y_2 and y_3 from the first three equations (3.403) into the last three, we obtain the quadratic equations

$$\left.\begin{aligned}(a_x-k)\left(\frac{x_2}{x_1}\right)^2-2a_{xy}\frac{x_2}{x_1}+a_y-k=0,\\(a_y-k)\left(\frac{x_3}{x_2}\right)^2-2a_{yz}\frac{x_3}{x_2}+a_z-k=0,\\(a_z-k)\left(\frac{x_1}{x_3}\right)^2-2a_{zx}\frac{x_1}{x_3}+a_x-k=0.\end{aligned}\right\} \qquad (3.404)$$

Since x_i and y_i enter symmetrically into equations (3.403), equations for y_2/y_1, y_3/y_2 and y_1/y_3 coinciding exactly with equations (3.404) are obtained:

$$\left.\begin{aligned}(a_x-k)\left(\frac{y_2}{y_1}\right)^2-2a_{xy}\frac{y_2}{y_1}+a_y-k=0,\\(a_y-k)\left(\frac{y_3}{y_2}\right)^2-2a_{yz}\frac{y_3}{y_2}+a_z-k=0,\\(a_z-k)\left(\frac{y_1}{y_3}\right)^2-2a_{zx}\frac{y_1}{y_3}+a_x-k=0.\end{aligned}\right\} \qquad (3.405)$$

From equations (3.404) we obtain:

$$\left.\begin{aligned}\frac{x_2}{x_1}=\frac{1}{a_x-k}\left[a_{xy}\pm\sqrt{a_{xy}^2-(a_y-k)(a_x-k)}\right],\\\frac{x_3}{x_2}=\frac{1}{a_y-k}\left[a_{yz}\pm\sqrt{a_{yz}^2-(a_y-k)(a_z-k)}\right],\\\frac{x_1}{x_3}=\frac{1}{a_z-k}\left[a_{zx}\pm\sqrt{a_{zx}^2-(a_z-k)(a_x-k)}\right].\end{aligned}\right\} \qquad (3.406)$$

The expressions for y_2/y_1, y_3/y_2 and y_1/y_3 differ from the expressions for x_2/x_1, x_3/x_2 and x_1/x_3 only in that the signs $\mp$ appear in the brackets in front of the root in place of $\pm$. This correspondence between the signs derives from the last three equations (3.403).

We note that the equation for k may be obtained from relations (3.406) by multiplying the left and right sides of these equalities. It can be shown that this approach also leads to the equation $J_3 = 0$ obtained above.

Denoting the right sides of equalities (3.406) by a_{21}, a_{32} and a_{13}, and the right sides of the analogous equalities for y_2/y_1, y_3/y_2 and y_1/y_3 by b_{21}, b_{32} and b_{13}, we obtain two systems of homogeneous linear equations:

$$\left.\begin{aligned}x_2-a_{21}x_1=0,\ x_3-a_{32}x_2=0,\ x_1-a_{13}x_3=0;\\y_2-b_{21}y_1=0,\ y_3-b_{32}y_2=0,\ y_1-b_{13}y_3=0.\end{aligned}\right\} \qquad (3.407)$$

Solving them, we can express x_1 and x_2 in terms of x_3 and y_1 and y_2 in terms of y_3:

$$x_1 = a_{13}x_3, \quad x_2 = \frac{x_3}{a_{33}}, \quad y_1 = b_{13}y_3, \quad y_2 = \frac{y_3}{b_{33}}. \tag{3.408}$$

To find x_3 and y_3 we can use the equations

$$x'^2 + y'^2 + z'^2 = 1, \quad \omega_x^2 + \omega_y^2 + \omega_z^2 = a_x + a_y + a_z. \tag{3.409}$$

The first of these is obvious, while the second is obtained from the first equalities (3.400) and (3.401) or directly from the first three equations (3.398) by adding them.

According to equalities (3.402)

$$\left.\begin{aligned} &\omega_x = \frac{x_1 + y_1}{2}, \quad \omega_y = \frac{x_2 + y_2}{2}, \quad \omega_z = \frac{x_3 + y_3}{2}; \\ &x' = \frac{y_1 - x_1}{\sqrt{3k}}, \quad y' = \frac{y_2 - x_2}{\sqrt{3k}}, \quad z' = \frac{y_3 - x_3}{\sqrt{3k}}. \end{aligned}\right\} \tag{3.410}$$

Substituting here x_1, x_2 and y_1, y_2 from relations (3.408) and introducing the resulting expressions for ω_x, ω_y, ω_z, x', y', and z' into equalities (3.409), we obtain the following equations for x_3 and y_3:

$$\left.\begin{aligned} &y_3^2(b_{13}^2 + b_{13}^2 b_{21}^2 + 1) + x_3^2(a_{13}^2 + a_{13}^2 a_{21}^2 + 1) - \\ &\qquad - 2y_3x_3(b_{13}a_{13} + b_{13}b_{21}a_{13}a_{21} + 1) = 12k_0, \\ &y_3^2(b_{13}^2 + b_{13}^2 b_{21}^2 + 1) + x_3^2(a_{13}^2 + a_{13}^2 a_{21}^2 + 1) + \\ &+ 2y_3x_3(b_{13}a_{13} + b_{13}b_{21}a_{13}a_{21} + 1) = 4(a_x + a_y + a_z). \end{aligned}\right\} \tag{3.411}$$

Dropping y_3 (or x_3) from these equations, we obtain a biquadratic equation for x_3 (or y_3).

Up to this point we have analyzed a system without gyroscopic sensing elements, and which determined the Cartesian coordinates ξ_*, η_*, ζ_* or ξ, η, ζ. The conversion to curvilinear coordinates may be effected in exactly the same way as with systems using gyroscopic elements. We will not consider this case in detail, confining ourselves to the following considerations which may prove useful in understanding the problems arising in converting to an arbitrary reference grid.

In the general case complete information is given, as was shown, by twelve newtonometers located at four points not lying in the same plane. The directions of the axes of sensitivity of the sets of three newtonometers at each of the four points should be non-coplaner within each set. If the tetrahedron whose vertices are the four points at which the newtonometers are located, is rigid, the task of the inertial system reduces, essentially, to determination of the coordinates of the vertices of the tetrahedron. To determine the coordinates of any of the vertices, the three non-coplaner newtonometers whose sensing masses are located at this point are sufficient (if the earth's gravitational field is known). It is also necessary to know at each moment of time the orientation of the directions of the newtonometer axes as a function of the coordinates being determined and of time. The orientation parameters of the tetrahedron relative to the coordinate system $O_1\xi_*\eta_*\zeta_*$ become known, clearly, as soon as the coordinates of its vertices are known. The orientation of the newtonometers in relation to the tetrahedron, on the other hand, should be given as a function of the coordinates determined by the system, and of time.

In conclusion, let us consider one more problem. In compiling the ideal equations we assumed that the earth's gravitational field was known. Under this assumption, it was possible, on the basis of relations (3.368), (3.370) and (3.376) to determine the first and second partial derivatives of the force function of the gravitational field with respect to coordinates x, y and z which entered into equations (3.387), (3.388) and (3.389). At the same time under this assumption we obtained a superfluous system of equations for the navigation parameters of interest to us, since we had available the nine equations (3.388), (3.389) and (3.390) for ω_x, ω_y and ω_z. The question arises as to whether it is possible in compiling the operational algorithm of an inertial system to have preliminary knowledge of only some of the characteristics of the gravitational field, and to determine the missing ones from equations (3.388), (3.389) and (3.390).

It is evident that it is impossible to fully determine the characteristics of the gravitational field from equations (3.387), (3.388), (3.389) and (3.390). In fact, if we assume that in (3.387)

$$v_x = v_y = v_z = 0, \tag{3.412}$$

then the fifteen equations listed above are sufficient to find ω_x, ω_y, ω_z, x, y, and z and the nine first and second derivatives of V at point O with respect to coordinates x, y and z. There are, however, no superfluous equations, and if equality (3.412) does not obtain, the number of unknowns increases by 3 (v_x, v_y, and v_z), and the number of equations becomes insufficient. We note that the introduction of gyroscopic sensing elements in addition to the newtonometers leaves the situation unchanged. This is due to the fact that the system of equations (3.390) does not contain the parameters of the gravitational field. Thus, if ω_x, ω_y and ω_z are determined using the gyroscopic gauges of absolute angular velocity, equations (3.390) simply become superfluous.

On the other hand, equations (3.388), (3.389) and (3.390) enable us to determine ω_x, ω_y and ω_z and the second derivatives of V as functions of time, such that if the second derivatives are known as functions of coordinates (for example, of ξ, η and ζ), equations (3.388), (3.389) and (3.390), together with equations (3.61), (3.62) and (3.64), also enable us to find the coordinates x, y, z as functions of time. In this case, obviously, equations (3.387) may not be used. If equations (3.387) are used, the first derivatives of the force function, which enter into these equations, should be given as functions of the coordinates, i.e., the projections of the strength vector g of the earth's gravitational field on the axes of the coordinate system attached to the earth should be known (as functions of the coordinates ξ, η and ζ).

In conclusion we note that, for the practical realization of systems which do not contain gyroscopic sensing elements, extremely accurate newtonometers, with a range of measurement from g to $(10^{-9}$ to $10^{-11})g$ are required.

NOTES

1. Andreyev, V. D., On the general equations of inertial navigation, Prikladnaya matematika i mekhanika, Vol. XXVIII, Issue 2, 1964.

2. A similar system was first considered as far as is known by the author, in 1944 by L. I. Tkachev.

3. Here and below the coordinates of some point O, near which the sensitivity masses of the newtonometers are located, will be understood as the coordinates of the object.

4. Lur'ye, A. I., Analiticheskaya mekhanika (Analytic Mechanics), Fizmatgiz, 1961.

5. Compare, for example, Lur'ye, A. I., ibid.

6. Fridlender, G. O.,Inertsial'nyye sistemy navigatsii (Inertial Systems of Navigation), Fizmatgiz, 1961.

7. Andreyev, V. D., On equations of nonperturbed operation of an inertial system determining curvilinear coordinates. Prikladnaya matematika i mekhanika, vol. XXIX, Issue 5, 1965.

8. The necessary information from tensor analysis can be found, for example, in: Lur'ye, A. I., op. cit.; Kil'chevskiy, N. A., Elementy tenzornogo ischisleniya i ego prilozheniya k mekhanika (Elements of Tensor Calculus and its Applications to Mechanics), Gostekhizdat, 1954; and Kochin, N. Ye. Vektornoye ischisleniye i nachala tenzornogo ischisleniya (Vector Calculus and Introductory Tensor Calculus), Press of the Academy of Sciences of the USSR, 1951.

9. Rashevskiy, P. K., Kurs differentsial'noy geometrii (Course on Differential Geometry), Gostekhizdat, 1956.

10. Andreyev, V. D., Devyanin, Ye. A., Dem'yanovskiy, A. P., On the theory of inertial systems not containing gyroscopic sensing elements, Academy of Sciences of the USSR, Inzhenernyy Zhurnal, Mekhanika tverdogo tela, No. 1, 1966.

11. Taylor, H. L., Satellite orientation by inertial techniques, J. Aerospace Science, vol. 28, No. 6, June 1961.

12. More precisely, the function V satisfies the Laplace equation both on and outside the surface of the earth.

13. Of course, if the sign of at least one of the projections ω_x, ω_y, or ω_z is not known from external information sources.

14. Compare, for example, Kochin, N. Ye., op. cit.

15. The preceding results can also be obtained by other methods. Compare the article of Ye. A. Devyanin and A. P. Dem'yanovskiy, Determination of absolute angular velocity and distance to a center of attraction and construction of the vertical by inertial methods, and also the supplement by Ye. A. Devyanin to this article (Inzhenernyy Zhurnal, Mekhanika tverdogo tela, Nos. 2, 5, 1966).

Chapter 4

THE DERIVATION AND TRANSFORMATION OF THE ERROR EQUATIONS OF INERTIAL NAVIGATION SYSTEMS

§4.1. The Perturbation Mode of Inertial Systems. Basic Instrument Errors.

The equations describing the ideal functioning of inertial navigation systems examined in the preceding chapter constitute algorithms on the basis of which various systems may be constructed. In order to realize the algorithms it is necessary, clearly, to have available the required instruments and devices. These are, primarily, inertial sensing elements: newtonometers and gyroscopes. Further, computational, including integrating, devices will always form a part of such a system. In order to effect time integration and the synthesis of time functions, an inertial system should include a timer from which time signals are fed to the computer; these signals should mark absolute (newtonian) time, which may be assumed as corresponding to the astronomical siderial time. Finally, the system should include devices which effect the interconnections between the various elements and instruments, including servo devices based on one or another principle of operation.

The equations describing the ideal functioning of inertial systems include the initial values of the coordinates and their rates of change, i.e., the initial conditions of the motion of the object in which the inertial system is placed. These initial conditions should be known. Moreover, the sensing elements of the system should be oriented in a particular way at the moment at which the system begins to function. Their initial orientation should correspond to the selected algorithm describing the functioning of the system.

The ideal equations are sufficient for describing the functioning of an inertial navigation system only when all of its elements and devices are error-free (ideal) and when the initial conditions of the system correspond precisely to the initial conditions of the motion of the object.

In real systems these conditions are fulfilled only to a certain level of approximation. Therefore, the mode of functioning of an inertial system differs from that described by the ideal equations, and the navigation parameters are imperfectly determined by the system. This mode of functioning, or, in other words, the motion of the inertial system, determined taking into account errors in initial conditions and instrument measurements, may be termed the perturbed motion of a navigational system.

Since the algorithm characterizing the unperturbed motion of the system is known, in dealing with perturbed systems we are primarily interested in their deviations from unperturbed motion.

Equations defining the deviations of variables describing the state of an inertial navigation system from their ideal values will henceforth be termed error equations. These equations determine the stability of the inertial system as a whole. They also establish the connection between errors associated with system elements and errors in the initial conditions, on the one hand, and errors in the systems' determination of the navigation parameters, on the other. Thus, the properties of the error equations determine, in the final analysis, the functional accuracy of the inertial system. Analysis of the properties of the error equations constitutes, therefore, one of the fundamental goals of the analysis of an inertial system.

Analysis of the error equations permits determination of the requirements on the system elements which must be met if the system is to achieve a previously specified level of accuracy. Study of the error equations further permits systematic selection of the algorithm describing the ideal functioning of the system (including the reference grid in which the position of the object is determined), and the orientation of the sensing elements. Finally, the error equations permit, as we will see below, rigorous determination of the acceptability of various simplifications of the algorithm determining the functioning of an interial system. Moreover, it is only on the basis of the properties of the error equations that it is possible to judge the need for corrections in an inertial system, as well as the effectiveness of various correction procedures.

Before proceeding to derive the error equations, it is necessary to examine in somewhat greater detail the basic sources of error which perturb the functioning of an inertial system.

The essence of the functioning of an inertial navigation system consists in the processing according to a specific algorithm of the information contained in the readings of inertial sensing elements: newtonometers and gyroscopes. It is to be expected that the instrument errors associated with newtonometers and gyroscopes are the primary sources of error in the functioning of an inertial system.

The primary content of the algorithm determining the functioning of an inertial system is the integration of the fundamental equation of inertial navigation. This integration presupposes knowledge of the initial conditions of motion of the object. Error in these initial conditions also leads to perturbations in the functioning of an inertial system. The algorithm selected to integrate the fundamental equation (different algorithms may be used in different systems) presupposes a specific orientation of the sensing elements of the system, beginning at the moment at which the system begins to function. This applies equally to errors in the specification (or the pre-start computation) of the numerical values of the initial conditions, as well as to errors in the realizations of these values in the system.

Further, the solution of the fundamental equation depends on a priori knowledge of the gravitational field of the earth, i.e., the magnitude of the gravitational pull as a function of position in a earth body-axis coordinate system. Solution of the fundamental equation also presupposes a given motion of the earth around its center of gravity. Errors in specification of the gravitational field and the earth rate give rise, clearly, to error.

Finally, the instrumentational realization of the algorithm for integrating the fundamental equation and the appropriate orientation of the sensing elements gives rise in a real system to error. This is due to the instrument error of the timer, the computing and integrating devices, and the servo and transform systems. Engineering inaccuracies in the mechanical (kinematic) elements of the system are also of relevance here: inaccuracies in dimensions, angles between datum planes and alignment directions, coaxial misalignments between elements, slack, elastic deformations, etc.

An inertial navigation system includes, as a rule, a large number of elements and devices. All of these elements and devices contribute their error to the functioning of the system. However, it would be incorrect to attempt to reflect as large a number of elements as possible in the error equations. A more effective analysis of the error equations would result from the reduction, if possible, of the error contributions of all of the elements to a few characteristic ones covering all possible sources of error. In other words, it is always expedient to use the smallest possible number of independent parameters defining the state of the system.

In an inertial navigation system, error in the specification of initial conditions and the instrument error of the sensing elements, the newtonometers and gyroscopes, may be taken as characteristic error sources of this type. Instrument error in the inertial sensing elements will henceforth be termed basic instrument error. The instrument error of all other elements and devices in the system can in the overwhelming majority of instances be reduced to a few equivalent basic error types, i.e., error in the sources of primary information. Error in the specification of the gravitational field and the earth rate likewise reduce to equivalent basic error.

The possibility of reducing the instrument error of any element or device in an inertial system to an equivalent error in the sensing elements is not, generally speaking, obvious. It will be evident from the following that this possibility occurs only when all of the elements and devices in the system fulfill, even though with a certain degree of error, their functions, i.e., all elements and devices transform the information fed to them in accordance with those portions of the ideal equations which they realize. This means that at the output of any device there is always, along with an error signal, a basic, useful signal. Error in this case may always be represented as some additive error introduced into the output of the device.

It is evident that such types of error in the functioning of the elements and devices of the system are also possible when the algorithm defining the ideal functioning of the system (or of some part of it) breaks down. This occurs, for example, in zones of instrument dead time, in air gaps and stagnation friction zones. In these zones the elements in question may not fulfill their function in the system: the useful signal may be absent at their output in spite of the presence of an input signal.

We will return to this problem in Section 4.6. In this section we will use concrete examples to show how the error of system elements and devices may be reduced to an equivalent basic error. In the meantime we will consider that, as a rule, the only instrument error in an inertial system is the instrument error of the sensing elements: the newtonometers, the geometrical sum of whose errors we will designate by the vector $\Delta\vec{n}$, and the gyroscope for measuring absolute angular velocity, the vector sum of whose errors we will designate as $\Delta\vec{m}$.

The physical sources of instrument error in the sensing elements were discussed in Chapter 1, which included a derivation of the equations describing their operation, and we will not consider this question further here. We will consider $\Delta\vec{n}$ and $\Delta\vec{m}$ as given functions of time. They may be either determined or random. The form of these functions may of course be a function of the parameters of motion of the object on which the inertial system is mounted, in particular velocities and accelerations (g-loads).

It has already been noted that the error equations link error in the determination of the navigation parameters with the instrument error of the elements of the system and error in the specification of the initial conditions. In developing the equations describing the ideal operation of inertial navigation systems it is assumed that they should solve two fundamental problems: first, to determine the coordinates of a moving object and their rate of change, and, second, to guarantee the required orientation of the inertial sensing elements and to define the orientation parameters of the object. It is accordingly necessary to obtain equations defining both error in the determination of the coordinates of the object and defining error in the parameters characterizing the orientation of the inertial elements and the object in space. In the general case these two groups of equations are related. However, a number of considerations make it expedient to begin by deriving the equations defining error in the specification of the coordinates, and this will be the subject of the next three sections of this chapter.

§4.2. Equations Describing Error in the Specification of Cartesian Coordinates

4.2.1. The vector form of the error equations. We will derive the equations describing error in coordinate specification for the system examined in §3.1.

Equations (3.53) -- (3.58) or (3.59) -- (3.65) constitute, in essence, the functional algorithm of this system, i.e., the equation defining its ideal operation. Equations (3.53) -- (3.58) and (3.59) -- (3.65) are fully equivalent and differ only in their form: equations (3.53) -- (3.58) are a vector description of the operational algorithm of the system, while equations (3.59) -- (3.65) are in scalar form.

Equations (3.59) -- (3.65) based on the newtonometer readings n_x, n_y, n_z and the absolute angular velocity readings m_x, m_y, m_z permit us to obtain the Cartesian coordinates of the object x, y, z in the coordinate system O_1xyz, and also the coordinates ξ_*, η_*, ζ_* in the basic Cartesian system and the coordinates ξ, η, ζ in the coordinate system $O_1\xi\eta\zeta$ rigidly bound to the earth.

We recall that the force function $\varepsilon(\xi, \eta, \zeta)$ of the non-spherical component of the gravitational field which enters into equations (3.56) and (3.65) is considered as known. Also considered as known are the projections u_ξ, u_η, u_ζ of the absolute angular velocity of the earth around its axis, which enter into relations (3.67).

The initial conditions of the ideal equations (3.59) -- (3.65) are:

$v_x(0)$, $v_y(0)$, $v_z(0)$ -- the values of the projections of the absolute linear velocity of the object (more precisely, the velocity of the apex of the trihedron O_{xyz} connected to the platform) around axes x, y, z at the initial moment of operation of the inertial system;

$x(0)$, $y(0)$, $z(0)$ -- the coordinates of point O in the coordinate system O_1xyz at the initial moment;

$\alpha_{ij}(0)$, $\beta_{ij}(0)$ -- the initial values of the direction cosines between axes x, y, z and axes ξ_*, η_*, ζ_* and ξ, η, ζ.

Now let Δm_x, Δm_y, Δm_z, Δn_x, Δn_y, Δn_z be defined as the instrument error of the device measuring absolute angular velocity and the newtonometers, respectively. We will consider these quantities to be functions of time, either determined or random. At the initial moment of operation of the inertial system they may be different from zero. We will denote their initial values by $\Delta m_x(0)$, $\Delta m_y(0)$, $\Delta m_z(0)$, $\Delta n_x(0)$, $\Delta n_y(0)$, $\Delta n_z(0)$.

Also let $\delta v_x(0)$, $\delta v_y(0)$, $\delta v_z(0)$, $\delta x(0)$, $\delta y(0)$, $\delta z(0)$, $\delta \alpha_{ij}(0)$, $\delta \beta_{ij}(0)$, $\delta \xi_*$, $\delta \eta_*$, $\delta \zeta_*$, $\delta \xi$, $\delta \eta$, $\delta \zeta$ be the error in the specification of the corresponding initial conditions.

We will denote deviations of variables and functions from their values corresponding to the unperturbed, ideal functioning of the system by δv_x, δv_y, δv_z, δx, δy, δz, $\delta \alpha_{ij}$, $\delta \beta_{ij}$, $\delta \xi_*$, $\delta \eta_*$, $\delta \zeta_*$, $\delta \xi$, $\delta \eta$, $\delta \zeta$.

In order to obtain the error equations, i.e., the equations corresponding to perturbations δx, δy, δz etc., it is necessary to substitute $x + \delta x$, $y + \delta y$, $z + \delta z$, ... for x, y, z ... in equations (3.59) -- (3.65), and $m_x + \delta m_x$, $n_x + \delta n_x$, ... for m_x, n_x, ... and to subtract equations (3.59) -- (3.65) from the resulting equations. If we ignore the squares and products of the perturbation, the procedure for deriving the error equations reduces to the derivation of the usual equations in the variants corresponding to equations (3.59) -- (3.65).

The modification of equations (3.59) -- (3.65) may proceed in a completely formal manner since we are considering the general case. The significance of this remark consists in the following. Let us assume that we are considering not the general, but a specific case, in which the orientation of trihedron O_{xyz} has been selected in some special fashion. In this case a number of terms may drop out. For example, if we assume that the z axis of trihedron O_{xyz} is directed along vector $\vec{g}$, then in the first and second of equations (3.59) projections g_x and g_y will be absent. In this case the formal modification of these equations will not enable us to obtain δg_x and δg_y which, clearly, should enter into the error equation, since in the perturbed mode the conditions under which the z axis and the vector $\vec{g}$ would coincide will not be fulfilled. As is well known, in the compilation of equations describing the variations in dynamic systems for the purpose of investigating their stability or the transient processes occurring in them, it is always necessary, before attempting to derive the perturbation equations by means of formal variation of the initial unperturbed equations, to make certain that there are no forces acting on the system other than those occurring in the equation describing the unperturbed motion of the system.

In our case, as has already been noted, equations expressing to a first approximation the influence of perturbations δx, δy, δz, etc. may be obtained equations of variations of (3.59) -- (3.65).

We now turn to the derivation of these variations equations for (3.59) -- (3.65). These equations may be obtained either by varying the scalar equations (3.59) -- (3.65) or by varying the equivalent equations (3.53) -- (3.58). We will begin by deriving the variations equations from the vector equations (3.53) -- (3.58).

Varying equations (3.53) -- (3.58), we obtain:

$$\left.\begin{aligned} \delta\boldsymbol{v} &= \int_0^t (\Delta\boldsymbol{n} - \Delta\boldsymbol{m}\times\boldsymbol{v} - \boldsymbol{m}\times\delta\boldsymbol{v} + \delta\boldsymbol{g} + \Delta\boldsymbol{g})\,dt + \delta\boldsymbol{v}(0), \\ \delta\boldsymbol{r} &= \int_0^t (\delta\boldsymbol{v} - \boldsymbol{m}\times\delta\boldsymbol{r} - \Delta\boldsymbol{m}\times\boldsymbol{r})\,dt + \delta\boldsymbol{r}(0); \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \delta\boldsymbol{\xi}_0 &= \int_0^t (\delta\boldsymbol{\xi}_0\times\boldsymbol{m} + \boldsymbol{\xi}_0\times\Delta\boldsymbol{m})\,dt + \delta\boldsymbol{\xi}(0), \\ \delta\boldsymbol{\eta}_0 &= \int_0^t (\delta\boldsymbol{\eta}_0\times\boldsymbol{m} + \boldsymbol{\eta}_0\times\Delta\boldsymbol{m})\,dt + \delta\boldsymbol{\eta}(0), \\ \delta\boldsymbol{\zeta}_0 &= \int_0^t (\delta\boldsymbol{\zeta}_0\times\boldsymbol{m} + \boldsymbol{\zeta}_0\times\Delta\boldsymbol{m})\,dt + \delta\boldsymbol{\zeta}(0); \end{aligned}\right\} \tag{4.2}$$

$$\left.\begin{aligned} \delta\boldsymbol{\xi} &= \int_0^t [\delta\boldsymbol{\xi}\times(\boldsymbol{m}-\boldsymbol{u}) + \boldsymbol{\xi}\times(\Delta\boldsymbol{m} - \delta\boldsymbol{u} - \Delta\boldsymbol{u})]\,dt + \delta\boldsymbol{\xi}(0), \\ \delta\boldsymbol{\eta} &= \int_0^t [\delta\boldsymbol{\eta}\times(\boldsymbol{m}-\boldsymbol{u}) + \boldsymbol{\eta}\times(\Delta\boldsymbol{m} - \delta\boldsymbol{u} - \Delta\boldsymbol{u})]\,dt + \delta\boldsymbol{\eta}(0), \\ \delta\boldsymbol{\zeta} &= \int_0^t [\delta\boldsymbol{\zeta}\times(\boldsymbol{m}-\boldsymbol{u}) + \boldsymbol{\zeta}\times(\Delta\boldsymbol{m} - \delta\boldsymbol{u} - \Delta\boldsymbol{u})]\,dt + \delta\boldsymbol{\zeta}(0); \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} \delta\boldsymbol{g} &= -\delta\left(\frac{\mu\boldsymbol{r}}{r^3}\right) + \delta\,\mathrm{grad}\,\varepsilon(\xi,\ \eta,\ \zeta), \\ \delta\boldsymbol{u} &= \boldsymbol{u}_\xi\delta\xi + \boldsymbol{u}_\eta\delta\eta + \boldsymbol{u}_\zeta\delta\zeta; \end{aligned}\right\} \tag{4.4}$$

$$\left.\begin{gathered} \delta\xi_0 = \delta\boldsymbol{r}\cdot\boldsymbol{\xi}_0 + \boldsymbol{r}\cdot\delta\boldsymbol{\xi}_0, \quad \delta\eta_0 = \delta\boldsymbol{r}\cdot\boldsymbol{\eta}_0 + \boldsymbol{r}\cdot\delta\boldsymbol{\eta}_0, \\ \delta\zeta_0 = \delta\boldsymbol{r}\cdot\boldsymbol{\zeta}_0 + \boldsymbol{r}\cdot\delta\boldsymbol{\zeta}_0, \\ \delta\xi = \delta\boldsymbol{r}\cdot\boldsymbol{\xi} + \boldsymbol{r}\cdot\delta\boldsymbol{\xi}, \quad \delta\eta = \delta\boldsymbol{r}\cdot\boldsymbol{\eta} + \boldsymbol{r}\cdot\delta\boldsymbol{\eta}, \\ \delta\zeta = \delta\boldsymbol{r}\cdot\boldsymbol{\zeta} + \boldsymbol{r}\cdot\delta\boldsymbol{\zeta}. \end{gathered}\right\} \tag{4.5}$$

The variations equations (4.1) -- (4.5) include, in addition to the vectors

$$\left.\begin{aligned} \Delta\boldsymbol{m} &= \Delta m_x\boldsymbol{x} + \Delta m_y\boldsymbol{y} + \Delta m_z\boldsymbol{z}, \\ \Delta\boldsymbol{n} &= \Delta n_x\boldsymbol{x} + \Delta n_y\boldsymbol{y} + \Delta n_z\boldsymbol{z} \end{aligned}\right\} \tag{4.6}$$

of the instrument error of the device measuring absolute angular velocity and the newtonometers, the error vectors $\Delta\vec{g}$ and $\Delta\vec{u}$. These errors may be expressed by means of the equivalent errors $\Delta\vec{n}$ and $\Delta\vec{m}$. For now, however, it is expedient to preserve them in the equations. There are two reasons for this. First, the errors $\Delta\vec{g}$ and $\Delta\vec{u}$ are in themselves characteristic of inertial systems. They reflect incomplete knowledge or specification of the force function of the gravitational field and the earth rate. Second, using the errors $\Delta\vec{g}$ and $\Delta\vec{u}$ it is possible to demonstrate the procedure of reducing specific types of errors to equivalent basic errors.

The variations of the variables entering into equations (4.1) -- (4.6) are isochronous. We continue as yet to regard the timer in the inertial system as ideal. We will return below to this question and will show, in particular, that error in the specification of time may also be reduced to certain equivalent basic errors.

We now move from the integral equations (4.1) -- (4.3) to their differential forms, which we obtain by differentiating them with respect to time in the coordinate system O_1xyz, i.e., in the same coordinate system in which the integration of relations (3.53) -- (3.55), and consequently relations (4.1) -- (4.3), was performed. Considering equality (3.1) and shifting terms not containing variations of variables to the right side, we obtain:

$$\begin{aligned} \delta\ddot{r} + 2\omega\times\delta\dot{r} + \omega\times(\omega\times\delta r) + \dot{\omega}\times\delta r - \delta g = {} \\ = \Delta g + \Delta n - 2\Delta m\times\dot{r} + \Delta\dot{m}\times r - {} \\ - \Delta m\times(\omega\times r) - \omega\times(\Delta m\times r); \end{aligned} \tag{4.7}$$

$$\left.\begin{aligned} \delta\dot{\xi}_0 + \omega\times\delta\xi_0 = \xi_0\times\Delta m, \quad \delta\dot{\eta}_0 + \omega\times\delta\eta_0 = \eta_0\times\Delta m, \\ \delta\dot{\zeta}_0 + \omega\times\delta\zeta_0 = \zeta_0\times\Delta m; \end{aligned}\right\} \tag{4.8}$$

$$\left.\begin{aligned} \delta\dot{\xi} + (\omega - u)\times\delta\xi = \xi\times(\Delta m - \Delta u - \delta u), \\ \delta\dot{\eta} + (\omega - u)\times\delta\eta = \eta\times(\Delta m - \Delta u - \delta u), \\ \delta\dot{\zeta} + (\omega - u)\times\delta\zeta = \zeta\times(\Delta m - \Delta u - \delta u). \end{aligned}\right\} \tag{4.9}$$

In equations (4.7) -- (4.9) the dots denote, as before, local differentiation in the O_1xyz coordinate system. The initial conditions of the differential equations (4.7) -- (4.9) derive from the integral equations (4.1) -- (4.3). The initial conditions are:

$$\left.\begin{aligned} &\delta r(0),\ \delta\dot{r}(0) = \frac{d\,\delta r(0)}{dt} - \omega(0)\times\delta r(0) - \Delta m(0)\times r(0),\\ &\delta\xi_0(0),\ \delta\eta_0(0),\ \delta\zeta_0(0),\ \delta\xi(0),\ \delta\eta(0),\ \delta\zeta(0). \end{aligned}\right\} \tag{4.10}$$

Since the error in the computation of the initial value of the velocity appearing in the second equality (4.10) is

$$\frac{d\,\delta r(0)}{dt} = \delta\dot{r}_0 + \omega(0)\times\delta r(0) + \delta\omega(0)\times r(0). \tag{4.11}$$

$\delta\dot{\vec{r}}(0)$ may be represented as follows:

$$\delta\dot{r}(0) = \delta\dot{r}_0 + [\delta\omega(0) - \Delta m(0)]\times r(0). \tag{4.12}$$

where $\delta\dot{\vec{r}}_0$ is the error resulting from the introduction of the initial value $\dot{\vec{r}}(0)$ into the system computer.

If the initial value $\vec{\omega}(0)$ is measured by a gyroscope measuring absolute angular velocity, and is not a calculated value,

$$\delta\omega(0) = \Delta m(0),\quad \delta\dot{r}(0) = \delta\dot{r}_0. \tag{4.13}$$

Let us now transform equations (4.4), (4.5) and (4.7) -- (4.9). Let us first determine the variation of $\delta\vec{g}$. From the first equality (4.4) we have:

$$\left.\begin{aligned} \delta g &= \delta\,\mathrm{grad}\,\frac{\mu}{r} + \delta\,\mathrm{grad}\,\varepsilon(\xi,\eta,\zeta),\\ \delta\,\mathrm{grad}\,\varepsilon &= \frac{\partial\varepsilon}{\partial\xi}\delta\xi + \frac{\partial\varepsilon}{\partial\eta}\delta\eta + \frac{\partial\varepsilon}{\partial\zeta}\delta\zeta + \mathrm{grad}\,\delta\varepsilon. \end{aligned}\right\} \tag{4.14}$$

We introduce the vectors

$$\left.\begin{aligned} \delta r_1 &= (r\cdot\delta\xi_0)\xi_0 + (r\cdot\delta\eta_0)\eta_0 + (r\cdot\delta\zeta_0)\zeta_0\\ \delta r_2 &= (r\cdot\delta\xi)\xi + (r\cdot\delta\eta)\eta + (r\cdot\delta\zeta)\zeta \end{aligned}\right\} \tag{4.15}$$

Then relations (4.5) may be written in the form

$$\delta r_3 = \delta r + \delta r_1, \quad \delta r_4 = \delta r + \delta r_2. \tag{4.16}$$

where vector $\delta\vec{r}$ is defined by equation (4.7) and the vectors $\delta\vec{r}_3$ and $\delta\vec{r}_4$ are introduced through the equalities

$$\left.\begin{aligned} \delta r_3 &= \xi_* \delta\xi_* + \eta_* \delta\eta_* + \zeta_* \delta\zeta_*, \\ \delta r_4 &= \xi \delta\xi + \eta \delta\eta + \zeta \delta\zeta \end{aligned}\right\} \tag{4.17}$$

From the equalities (4.17) it is easy to see the physical significance of the vectors $\delta\vec{r}_3$ and $\delta\vec{r}_4$: these designate the total error in the determination of the coordinates of the object in the $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$ coordinate systems, respectively, consisting of the error $\delta\vec{r}$ in the determination of the x, y, z coordinates in the O_1xyz coordinate system and the errors $\delta\vec{r}_1$ and $\delta\vec{r}_2$ in the conversion of the x, y, z coordinates to the ξ_*, η_*, ζ_* and ξ, η, ζ coordinates.

Taking account of equalities (4.17) causes the first formula (4.4) to take the form

$$\begin{aligned} \delta \operatorname{grad} \varepsilon &= \frac{\partial\varepsilon}{\partial\xi}\delta\xi + \frac{\partial\varepsilon}{\partial\eta}\delta\eta + \frac{\partial\varepsilon}{\partial\zeta}\delta\zeta + \\ &+ \left(\delta r_4 \cdot \operatorname{grad}\frac{\partial\varepsilon}{\partial\xi}\right)\xi + \left(\delta r_4 \cdot \operatorname{grad}\frac{\partial\varepsilon}{\partial\eta}\right)\eta + \left(\delta r_4 \cdot \operatorname{grad}\frac{\partial\varepsilon}{\partial\zeta}\right)\zeta \end{aligned} \tag{4.18}$$

Let us turn to (4.8) and (4.9).

The trihedra $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$ are rigid, so the vector triple $\vec{\xi}_* + \delta\vec{\xi}_*$, $\vec{\eta}_* + \delta\vec{\eta}_*$, $\vec{\zeta}_* + \delta\vec{\zeta}_*$ and $\vec{\xi} + \delta\vec{\xi}$, $\vec{\eta} + \delta\vec{\eta}$, $\vec{\zeta} + \delta\vec{\zeta}$ likewise form orthogonal trihedra. Since the vectors $\delta\vec{\xi}_*$, $\delta\vec{\eta}_*$, $\delta\vec{\zeta}_*$, and $\delta\vec{\xi}$, $\delta\vec{\eta}$, and $\delta\vec{\zeta}$ are small, it is possible to introduce small rotation vectors $\vec{\theta}_1$ and $\vec{\theta}_2$, such that the equalities

$$\left.\begin{aligned} \delta\xi_* &= -\theta_1 \times \xi_*, \quad \delta\eta_* = -\theta_1 \times \eta_*, \quad \delta\zeta_* = -\theta_1 \times \zeta_*; \\ \delta\xi &= -\theta_2 \times \xi, \quad \delta\eta = -\theta_2 \times \eta, \quad \delta\zeta = -\theta_2 \times \zeta \end{aligned}\right\} \tag{4.19}$$

will be valid to within the second order of smallness relative to the vectors $\delta\vec{\xi}_*$, $\delta\vec{\eta}_*$, $\delta\vec{\zeta}_*$, $\delta\vec{\xi}$, $\delta\vec{\eta}$, and $\delta\vec{\zeta}$.

Substituting relations (4.19) into formulas (4.15), we find:

$$\Delta r_1 = \theta_1 \times r, \quad \Delta r_2 = \theta_2 \times r. \tag{4.20}$$

In order to determine $\vec{\theta}_1$ and $\vec{\theta}_2$ we proceed as follows. We substitute the first equality (4.19) into the first equation (4.18). This substitution and a few simple transformations yield:

$$\dot{\theta}_1 \times \xi_* + \theta_1 \times \dot{\xi}_* + \omega \times (\theta_1 \times \xi_*) = \Delta m \times \xi_*. \tag{4.21}$$

We now substitute into relation (4.21) the expression for $\dot{\vec{\xi}}_*$ from the first equation (3.54). This gives relation (4.21) the form:

$$\dot{\theta}_1 \times \xi_* + \theta_1 \times (\xi_* \times \omega) + \omega \times (\theta_1 \times \xi_*) = \xi_* \times \Delta m. \tag{4.22}$$

It is easy to verify that the following identity holds:

$$\theta_1 \times (\xi_* \times \omega) + \omega \times (\theta_1 \times \xi_*) = (\omega \times \theta_1) \times \xi_*. \tag{4.23}$$

Using this identity, we derive the following equation from equality (4.22):

$$(\dot{\theta}_1 + \omega \times \theta_1 - \Delta m) \times \xi_* = 0. \tag{4.24}$$

But since the vector quantity in parentheses is essentially arbitrary, it follows that the vector $\vec{\theta}_1$ should satisfy the equality

$$\dot{\theta}_1 + \omega \times \theta_1 = \Delta m. \tag{4.25}$$

For $\vec{\theta}_2$, analogously, from the third equality (4.19), the first formula (3.55) and the relation (4.9) we obtain

$$\dot{\theta}_2 \times \xi + \theta_2 \times [\xi \times (\omega - u)] + (\omega - u) \times (\theta_2 \times \xi) = \\ = (\Delta u - \Delta m + \delta u) \times \xi. \tag{4.26}$$

which, using identity (4.23), may be reduced to a form analogous to (4.25):

$$\dot{\theta}_2 + (\omega - u) \times \theta_2 = \Delta m - \Delta u - \delta u. \tag{4.27}$$

But in accordance with the last equation (4.9) and the last three equalities (4.19),

$$\delta u = -\theta_2 \times u. \tag{4.28}$$

We therefore finally obtain in place of (4.27):

$$\dot{\theta}_2 + \omega \times \theta_2 = \Delta m - \Delta u. \tag{4.29}$$

In deducing equations (4.25) and (4.29) we made use of the first equalities (4.8), (3.54), (4.9) and (3.55). The result, of course, is the same if we use the second or third of these equalities. In fact, if, for example, in equation (4.24) ξ_* were replaced with η_* or ζ_*, the final result, i.e., equation (4.25), would be unchanged.

We now return to equalities (4.14) and (4.18) If in relations (4.14) we insert $\delta\xi$, $\delta\eta$, $\delta\zeta$ from the last three equalities (4.19), we obtain the expression

$$\delta\,\mathrm{grad}\, r = -\theta_2 \times \mathrm{grad}\, r + \mathrm{grad}\,\delta\varepsilon, \tag{4.30}$$

in which the term grad $\delta\Sigma$ may be represented, according to relation (4.18), in the following form:

$$\mathrm{grad}\,\delta\varepsilon = \left(\delta r_1 \cdot \mathrm{grad}\,\frac{\partial\varepsilon}{\partial\xi}\right)\xi + \\ + \left(\delta r_1 \cdot \mathrm{grad}\,\frac{\partial\varepsilon}{\partial\eta}\right)\eta + \left(\delta r_1 \cdot \mathrm{grad}\,\frac{\partial\varepsilon}{\partial\zeta}\right)\zeta \tag{4.31}$$

Let us now collect the transformed equations in a system equivalent to equations (4.1) -- (4.5). For this we will use relations (4.7), (4.25), (4.29), (4.20), (4.16) and (4.30). The error equations of the system under consideration will then reduce to the following system of vector equations on variations:

$$\begin{aligned} \delta\ddot{r} + 2\omega \times \delta\dot{r} + \omega \times (\omega \times \delta r) + \dot{\omega} \times \delta r - \\ - \delta \operatorname{grad} \frac{\mu}{r} + \theta_2 \times \operatorname{grad} \varepsilon - \operatorname{grad} \delta\varepsilon = \\ = \Delta g + \Delta n - 2\Delta m \times \dot{r} - \Delta\dot{m} \times r - \\ - \Delta m \times (\omega \times r) - \omega \times (\Delta m \times r); \end{aligned} \quad (4.32)$$

$$\left.\begin{aligned} \dot{\theta}_1 + \omega \times \theta_1 = \Delta m, \\ \dot{\theta}_2 + \omega \times \theta_2 = \Delta m - \Delta u; \end{aligned}\right\} \quad (4.33)$$

$$\left.\begin{aligned} \delta r_1 = \theta_1 \times r, \quad \delta r_2 = \theta_2 \times r, \\ \delta r_3 = \delta r + \delta r_1, \quad \delta r_4 = \delta r + \delta r_2. \end{aligned}\right\} \quad (4.34)$$

Tne initial conditions of the differential equations entering into the system (4.32) -- (4.34) are given by relations (4.10) -- (4.13).

The solution of the error equations (4.32) -- (4.34) enables us to find the vectors $\delta\vec{r}$, $\delta\vec{r}_3$, $\delta\vec{r}_4$, i.e., the error in the determination of the Cartesian coordinates in the O_1xyz, $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$ coordinate systems as a function of the error in the specification of the initial conditions, the basic instrument errors $\Delta\vec{m}$ and $\Delta\vec{n}$, and the errors $\Delta\vec{g}$ and $\Delta\vec{u}$ in the specification of the gravitational field and the earth rate.

4.2.2. Equations defining error in projections onto the platform axes. Let us now turn from the vector error equations (4.32) -- (4.34) to the scaler error equations. The most convenient way of effecting this transition is by projecting these equations on the the x, y, z axes of the platform of the inertial system, since the local derivatives in equations (4.32) -- (4.34) were taken in this coordinate system.

Expanding the vector products in equations (4.32), we find:

$$
\left.\begin{aligned}
&\delta\ddot{x}-(\omega_y^2+\omega_z^2)\delta x+(\omega_x\omega_y-\dot{\omega}_z)\delta y-2\omega_z\delta\dot{y}+\\
&\quad+(\omega_x\omega_z+\dot{\omega}_y)\delta z+2\omega_y\delta\dot{z}-\\
&\quad-\left(\delta\,\mathrm{grad}\,\frac{\mu}{r}-\theta_2\times\mathrm{grad}\,r+\mathrm{grad}\,\delta r\right)_x=\\
&\quad=\Delta g_x+\Delta n_x-2(\Delta m_y\dot{z}-\Delta m_z\dot{y})-\\
&\quad-\Delta\dot{m}_y z+\Delta\dot{m}_z y-\omega_x(\Delta m_y y+\Delta m_z z)-\\
&\qquad-\Delta m_x(\omega_y y+\omega_z z)+2x(\omega_y\Delta m+\omega_z\Delta m_z).\\
&\delta\ddot{y}-(\omega_x^2+\omega_z^2)\delta y+(\omega_y\omega_z-\dot{\omega}_x)\delta z-2\omega_x\delta\dot{z}+\\
&+(\omega_y\omega_x+\dot{\omega}_z)\delta x+2\omega_z\delta\dot{x}-\\
&-\left(\delta\,\mathrm{grad}\,\frac{\mu}{r}-\theta_2\times\mathrm{grad}\,r+\mathrm{grad}\,\delta r\right)_y=\Delta g_y+\Delta n_y-\\
&-2(\Delta m_z\dot{x}-\Delta m_x\dot{z})-\Delta\dot{m}_z x+\Delta\dot{m}_x z-\\
&-\omega_y(\Delta m_z z+\Delta m_x x)-\Delta m_y(\omega_z z+\omega_x x)+\\
&\qquad\qquad\qquad\qquad+2y(\omega_z\Delta m_z+\omega_x\Delta m_x).\\
&\delta\ddot{z}-(\omega_x^2+\omega_y^2)+(\omega_z\omega_x-\dot{\omega}_y)\delta x-2\omega_y\delta\dot{x}+\\
&+(\omega_z\omega_y+\dot{\omega}_x)\delta y+2\omega_x\delta\dot{y}-\\
&-\left(\delta\,\mathrm{grad}\,\frac{\mu}{r}-\theta_2\times\mathrm{grad}\,r+\mathrm{grad}\,\delta r\right)_z=\\
&=\Delta g_z+\Delta n_z-2(\Delta m_x\dot{y}-\Delta m_y\dot{x})-\Delta\dot{m}_x y+\Delta\dot{m}_y x-\\
&-\omega_z(\Delta m_x x+\Delta m_y y)-\Delta m_z(\omega_x x+\omega_y y)+\\
&\qquad\qquad\qquad\qquad+2z(\omega_x\Delta m_x+\omega_y\Delta m_y).
\end{aligned}\right\}\tag{4.35}
$$

From equations (4.33) and (3.64) we obtain:

$$
\left.\begin{gathered}
\dot{\theta}_{1x}+\omega_y\theta_{1z}-\omega_z\theta_{1y}=\Delta m_x,\quad \dot{\theta}_{1y}+\omega_z\theta_{1x}-\omega_x\theta_{1z}=\Delta m_y,\\
\dot{\theta}_{1z}+\omega_x\theta_{1y}-\omega_y\theta_{1x}=\Delta m_z;
\end{gathered}\right\}\tag{4.36}
$$

$$
\left.\begin{aligned}
&\dot{\theta}_{2x}+\omega_y\theta_{2z}-\omega_z\theta_{2y}=\Delta m_x-\Delta u_x,\\
&\dot{\theta}_{2y}+\omega_z\theta_{2x}-\omega_x\theta_{2z}=\Delta m_y-\Delta u_y,\\
&\dot{\theta}_{2z}+\omega_x\theta_{2y}-\omega_y\theta_{2x}=\Delta m_z-\Delta u_z,\\
&\Delta u_x=\Delta u_\xi\beta_{11}+\Delta u_\eta\beta_{21}+\Delta u_\zeta\beta_{31},\\
&\Delta u_y=\Delta u_\xi\beta_{12}+\Delta u_\eta\beta_{22}+\Delta u_\zeta\beta_{32},\\
&\Delta u_z=\Delta u_\xi\beta_{13}+\Delta u_\eta\beta_{23}+\Delta u_\zeta\beta_{33}.
\end{aligned}\right\}\tag{4.37}
$$

Finally, from relations (4.34) we find:

$$
\left.\begin{aligned}
\delta x_1&=\theta_{1y}z-\theta_{1z}y, & \delta x_3&=\delta x+\delta x_1,\\
\delta y_1&=\theta_{1z}x-\theta_{1x}z, & \delta y_3&=\delta y+\delta y_1,\\
\delta z_1&=\theta_{1x}y-\theta_{1y}x, & \delta z_3&=\delta z+\delta z_1;
\end{aligned}\right\}\tag{4.38}
$$

$$
\left.\begin{aligned}
\delta x_2&=\theta_{2y}z-\theta_{2z}y, & \delta x_4&=\delta x+\delta x_2,\\
\delta y_2&=\theta_{2z}x-\theta_{2x}z, & \delta y_4&=\delta y+\delta y_2,\\
\delta z_2&=\theta_{2x}y-\theta_{2y}x, & \delta z_4&=\delta z+\delta z_2.
\end{aligned}\right\}\tag{4.39}
$$

The initial conditions of equations (4.35) -- (4.39) are the quantities: $\delta x(0)$, $\delta y(0)$, $\delta z(0)$, $\delta\dot{x}(0)$, $\delta\dot{y}(0)$, $\delta\dot{z}(0)$; $\theta_{1x}(0)$, $\theta_{1y}(0)$, $\theta_{1z}(0)$; $\theta_{2x}(0)$, $\theta_{2y}(0)$, $\theta_{2z}(0)$.

In accordance with (4.11)

$$\left.\begin{aligned}\delta\dot{x}(0) &= \delta\dot{x}_0 + [\delta\omega_y(0) - \Delta m_y(0)]\, z(0) - \\ &\quad - [\delta\omega_z(0) - \Delta m_z(0)]\, y(0), \\ \delta\dot{y}(0) &= \delta\dot{y}_0 + [\delta\omega_z(0) - \Delta m_z(0)]\, x(0) - \\ &\quad - [\delta\omega_x(0) - \Delta m_x(0)]\, z(0), \\ \delta\dot{z}(0) &= \delta\dot{z}_0 + [\delta\omega_x(0) - \Delta m_x(0)]\, y(0) - \\ &\quad - [\delta\omega_y(0) - \Delta m_y(0)]\, x(0).\end{aligned}\right| \tag{4.40}$$

In equation (4.35) the projections

$$\left.\begin{aligned}&\left(\delta\,\mathrm{grad}\,\frac{\mu}{r} - \theta_2 \times \mathrm{grad}\,\varepsilon + \mathrm{grad}\,\delta\varepsilon\right)_x, \\ &\left(\delta\,\mathrm{grad}\,\frac{\mu}{r} - \theta_2 \times \mathrm{grad}\,\varepsilon + \mathrm{grad}\,\delta\varepsilon\right)_y, \\ &\left(\delta\,\mathrm{grad}\,\frac{\mu}{r} - \theta_2 \times \mathrm{grad}\,\varepsilon + \mathrm{grad}\,\delta\varepsilon\right)\end{aligned}\right| \tag{4.41}$$

have intentionally been left unexpanded. Before expanding expressions (4.41), it is necessary to consider the following. For a gravitational field as close to spherical as that of the earth,

$$|\mathrm{grad}\,\varepsilon| \ll \left|\mathrm{grad}\,\frac{\mu}{r}\right|. \tag{4.42}$$

and so

$$|\delta\,\mathrm{grad}\,\varepsilon| \ll \left|\delta\,\mathrm{grad}\,\frac{\mu}{r}\right|. \tag{4.43}$$

Equations (4.35) -- (4.39) are the essence of the equation in variations. They were obtained by formal variations of the equations describing the ideal operation of the equations (3.53) -- (3.58) describing the ideal operation of the system. Only terms which are linear relative to variations of the variables were retained. Terms containing squares of variations, products of variations with each other and products of variations and instrument error quantities were considered to be sufficiently small to be ignored.

Thus, in the perturbation $\delta \operatorname{grad} \frac{\mu}{r}$ only the linear terms are retained, i.e.,

$$\delta \operatorname{grad} \frac{\mu}{r} = -\mu\delta \frac{r}{r^3} = \frac{\mu}{r^3}\left(-\delta r + 3r \frac{\delta r}{r}\right). \tag{4.44}$$

At the same time $\delta \operatorname{grad} \varepsilon$ has the same order of smallness (which may be verified by performing the corresponding calculation) as the quadratic terms of the series expansion of the spherical component of the gravitational field, which were ignored in equality (4.44), the more complete form of which is:

$$\operatorname{grad} \frac{\mu}{r+\delta r} - \operatorname{grad} \frac{\mu}{r} = \delta \operatorname{grad} \frac{\mu}{r} = \frac{\mu}{r^3}\left(-\delta r + 3r \frac{\delta r}{r}\right). \tag{4.45}$$

Consequently, if only the linear terms are left in the expansion of the spherical component of the gravitational field, there is no need to retain the variation of the non-spherical component. The variation of the correction for non-sphericity of the gravitational field is retained in the error equations, so that it is necessary at the same time to retain the quadratic terms in the expansion of the spherical component, i.e., in place of $\delta \operatorname{grad} \frac{\mu}{r}$, determined by equality (4.44), the quantity

$$\operatorname{grad} \frac{\mu}{r+\delta r} - \operatorname{grad} \frac{\mu}{r} = -\frac{\mu}{r^3}\delta r + \frac{\mu r}{r^3}\frac{3\delta r}{r} + \frac{3\mu\delta r}{2r^4}\delta r - \frac{12\mu\delta r^2}{r^5} r. \tag{4.46}$$

must be used.

We note that in relations (4.44) -- (4.46) the quantity δr denotes not the modulus of vector $\delta\vec{r}$, but rather is defined by the equality

$$\delta r = |r + \delta r| - r \tag{4.47}$$

It seems that it is senseless to speak of retaining the quadratic terms of the expansion of (4.46), for the reason that the entire set of equations (4.35) -- (4.39) are linearized equations. If, however, we return to the ideal equations (3.53) -- (3.55), we notice that these equations are linear in $\vec{r}$, $v = d\vec{r}/dt$, $\vec{\xi}_*$, $\vec{\eta}_*$, $\vec{\zeta}_*$, $\vec{\xi}$, $\vec{\eta}$, $\vec{\zeta}$, excepting vector $\vec{g}$, which is a non-linear function of these variables. Therefore, when the instrument errors are absent, equations (4.1) -- (4.3) and consequently also equations (4.7) -- (4.9), if the perturbation $\delta\vec{g}$ is abstracted out, are exact equations for the perturbations $\delta\vec{r}$, $\delta\vec{v}$, $\delta\vec{\xi}_*$, $\delta\vec{\eta}_*$, $\delta\vec{\zeta}_*$, $\delta\vec{\xi}$, $\delta\vec{\eta}$, $\delta\vec{\zeta}$. In the investigation of the stability of an inertial system discussion will center around solution of the homogeneous error equations, based on the assumption that instrument error is zero, and only errors in the initial conditions are the source of perturbation. Thus, with regard to the homogeneous equation (4.7) -- (4.9) their accuracy is determined only by the accuracy of the expansion of (4.46).

At the same time the homogeneous equations (4.36) and (4.37) are first order equations, i.e., linearized equations, since in introducing the small rotation of vectors $\vec{\theta}_1$ and $\vec{\theta}_2$ and in making the transition from equations (4.8) and (4.9) to equations (4.36) and (4.37) we ignored terms containing the squares and products of small angles. But the homogeneous equations (4.35) remained exact in the sense that the degree of their approximation is determined only by the error in the approximate representation of (4.46).

The stability of the system of homogeneous equations (4.35) -- (4.39) is determined, obviously, primarily by the differential equations (4.35) -- (4.37). It is easy to see that equations (4.36) and (4.37) are solved separately from the others. The solution of equations (4.37) takes part only in the formation of δgrade in equation (4.35). Equations (4.37) are not associated in any other way with equations (4.35). But δgrade is a quantity of the second order of smallness. The use of linear equations (4.35) -- (4.39) in its formation can give only a third approximation of the error, but the second approximation remains exact.

It follows from these considerations that in the first approximation the quantity $\delta \operatorname{grad} \frac{\mu}{r}$ in equations (4.35) may be substituted for using equality (4.44), and the quantity $\delta \operatorname{grad} \varepsilon$ may be considered to be sufficiently small to be ignored. In this case expressions (4.41) in equations (4.35) are replaced by:

$$\left.\begin{aligned} \delta \operatorname{grad}_x \frac{\mu}{r} &= \frac{\mu}{r^3}\left(-\delta x + 3x\frac{\delta r}{r}\right), \\ \delta \operatorname{grad}_y \frac{\mu}{r} &= \frac{\mu}{r^3}\left(-\delta y + 3y\frac{\delta r}{r}\right), \\ \delta \operatorname{grad}_z \frac{\mu}{r} &= \frac{\mu}{r^3}\left(-\delta z + 3z\frac{\delta r}{r}\right), \end{aligned}\right\} \tag{4.48}$$

where

$$r^2 = x^2 + y^2 + z^2, \quad \delta r = |r + \delta r| - r = \frac{r \cdot \delta r}{r}. \tag{4.49}$$

The equalities (4.48) and (4.49) are equivalent to the vector equality

$$\delta \operatorname{grad} \frac{\mu}{r} = \frac{\mu}{r^3}\left(-\delta r + 3r\frac{r \cdot \delta r}{r^2}\right). \tag{4.50}$$

The following conclusions may be drawn from the above.

If we ignore the variations of the non-spherical component of the gravitational field, the differential equaiton (4.35) may be solved independently of equations (4.36) and (4.37). The corresponding vector equations (4.32), as well as the first and second equations (4.33) may be dealt with in the same manner. Thus, $\delta\vec{r}_1$ and $\delta\vec{r}_2$ become independent.

If there are insufficient first order equations, their subsequent specification should consist in the following: equations (4.8) and (4.9) should be taken instead of equations (4.36) and (4.37), and in equations (4.35) the projections of the vector $\delta \operatorname{grad} \varepsilon$ and the quadratic terms of the series expansion of the spherical component of the gravitational field (4.46) should be retained.

We obtained the error equations (4.32) -- (4.34) or, equivalently, equations (4.35) -- (4.39), on th basis of the ideal vector equations (3.53) -- (3.58). It is possible, of course, to obtain these equations directly from the ideal scalar equations (3.59) -- (3.65). In view of the importance of the error equations in the solution of the fundamental problems of inertial navigation, we will repeat their derivation in analytic (scalar) form. This will permit a clearer understanding of the transformations and assumptions made in the derivation process. Moreover, we will henceforth need certain scalar relations, the derivation of which will require the re-derivation of a large portion of the error equations in scalar form.

In order to derive equations (4.35) it is necessary to differentiate the scalar equations (3.59) with respect to time, and then to vary them except for δv_x, δv_y, δv_z, and to substitute expressions (4.41) for δg_x, δg_y, δg_z.

In order to obtain equations (4.36), we must obtain the differentiated equations (3.60), replacing m_x, m_y, m_z, with ω_x, ω_y, ω_z. Performing the differentiation and varying, we obtain

$$\left.\begin{aligned}
&\delta\dot{a}_{11}+\delta a_{13}\omega_y-\delta a_{12}\omega_z=a_{12}\Delta m_z-a_{13}\Delta m_y,\\
&\delta\dot{a}_{12}+\delta a_{11}\omega_z-\delta a_{13}\omega_x=a_{13}\Delta m_x-a_{11}\Delta m_z,\\
&\delta\dot{a}_{13}+\delta a_{12}\omega_x-\delta a_{11}\omega_y=a_{11}\Delta m_y-a_{12}\Delta m_x;\\
&\delta\dot{a}_{21}+\delta a_{23}\omega_y-\delta a_{22}\omega_z=a_{22}\Delta m_z-a_{23}\Delta m_y,\\
&\delta\dot{a}_{22}+\delta a_{21}\omega_z-\delta a_{23}\omega_x=a_{23}\Delta m_x-a_{21}\Delta m_z,\\
&\delta\dot{a}_{23}+\delta a_{22}\omega_x-\delta a_{21}\omega_y=a_{21}\Delta m_y-a_{22}\Delta m_x;\\
&\delta\dot{a}_{31}+\delta a_{33}\omega_y-\delta a_{32}\omega_z=a_{32}\Delta m_z-a_{33}\Delta m_y,\\
&\delta\dot{a}_{32}+\delta a_{31}\omega_z-\delta a_{33}\omega_x=a_{33}\Delta m_x-a_{31}\Delta m_z,\\
&\delta\dot{a}_{33}+\delta a_{32}\omega_x-\delta a_{31}\omega_y=a_{31}\Delta m_y-a_{32}\Delta m_x.
\end{aligned}\right\}\tag{4.51}$$

In order to further transform equations (4.51), we introduce the notation

$$\left.\begin{aligned} \theta_{1x} &= -\alpha_{12}\delta\alpha_{13} - \alpha_{22}\delta\alpha_{23} - \alpha_{32}\delta\alpha_{33}, \\ \theta_{1y} &= \alpha_{11}\delta\alpha_{13} + \alpha_{21}\delta\alpha_{23} + \alpha_{31}\delta\alpha_{33}, \\ \theta_{1z} &= -\alpha_{11}\delta\alpha_{12} - \alpha_{21}\delta\alpha_{22} - \alpha_{31}\delta\alpha_{32}. \end{aligned}\right\} \tag{4.52}$$

By varying the obvious equalities

$$\left.\begin{aligned} \alpha_{12}\alpha_{13} + \alpha_{22}\alpha_{23} + \alpha_{32}\alpha_{33} = 0, \\ \alpha_{11}\alpha_{13} + \alpha_{21}\alpha_{23} + \alpha_{31}\alpha_{33} = 0, \\ \alpha_{11}\alpha_{12} + \alpha_{21}\alpha_{22} + \alpha_{31}\alpha_{32} = 0, \end{aligned}\right\} \tag{4.53}$$

we see that, with an accuracy to within terms of the second order of smallness relative to $\delta\alpha_{ij}$, the projections θ_{1x}, θ_{1y}, θ_{1z} may also be represented in the following form:

$$\left.\begin{aligned} \theta_{1x} &= \alpha_{13}\delta\alpha_{12} + \alpha_{23}\delta\alpha_{22} + \alpha_{33}\delta\alpha_{32}, \\ \theta_{1y} &= -\alpha_{13}\delta\alpha_{11} - \alpha_{23}\delta\alpha_{21} - \alpha_{33}\delta\alpha_{31}, \\ \theta_{1z} &= \alpha_{12}\delta\alpha_{11} + \alpha_{22}\delta\alpha_{21} + \alpha_{32}\delta\alpha_{31}. \end{aligned}\right\} \tag{4.54}$$

It is easy to see that θ_{1x}, θ_{1y}, θ_{1z}, defined by equalities (4.52) or (4.54), are the x, y, z components of some small rotation vector $\vec{\theta}_1$ of the trihedron O_1xyz relative to the trihedron $O_1\xi_*\eta_*\zeta_*$. This small rotation is characterized in a change by a magnitude $\delta\alpha_{ij}$ of the direction cosine α_{ij} between the x, y, z and ξ_*, η_*, ζ_* axes.

We now multiply the second equation (4.51) by α_{13}, the fifth by α_{23} and the eighth by α_{33} and add. Grouping, we arrive at the equality

$$\begin{aligned} &\delta\dot{\alpha}_{13}\alpha_{13} + \delta\dot{\alpha}_{23}\alpha_{23} + \delta\dot{\alpha}_{33}\alpha_{33} - \omega_x(\alpha_{13}\delta\alpha_{12} + \delta\alpha_{22}\alpha_{23} + \delta\alpha_{32}\alpha_{33}) + \\ &\quad + \omega_y(\alpha_{13}\delta\alpha_{11} + \alpha_{23}\delta\alpha_{21} + \alpha_{33}\delta\alpha_{31}) = \Delta m_x(\alpha_{13}^2 + \alpha_{23}^2 + \alpha_{33}^2) - \\ &\qquad - \Delta m_z(\alpha_{11}\alpha_{13} + \alpha_{21}\alpha_{23} + \alpha_{31}\alpha_{33}). \end{aligned} \tag{4.55}$$

Taking into account relations (4.53) and (4.54), as well as

$$\alpha_{13}^2 + \alpha_{23}^2 + \alpha_{33}^2 = 1 \tag{4.56}$$

and, consequently, to within terms of the second order of smallness

$$\alpha_{13}\delta\alpha_{13} + \alpha_{23}\delta\alpha_{23} + \alpha_{33}\delta\alpha_{33} = 0, \tag{4.57}$$

we obtain from equation (4.55):

$$\alpha_{13}\delta\dot{\alpha}_{12} + \alpha_{23}\delta\dot{\alpha}_{22} + \alpha_{33}\delta\dot{\alpha}_{32} - \omega_z\theta_{1y} = \Delta m_z. \tag{4.58}$$

But according to the first relation (4.54),

$$\begin{aligned}\alpha_{13}\delta\dot{\alpha}_{12} + \alpha_{23}\delta\dot{\alpha}_{22} + \alpha_{33}\delta\dot{\alpha}_{32} &= \\ = \dot{\theta}_{1x} - \dot{\alpha}_{13}\delta\alpha_{12} - \dot{\alpha}_{23}\delta\alpha_{22} - \dot{\alpha}_{33}\delta\alpha_{32}.\end{aligned} \tag{4.59}$$

Replacing now $\dot{\alpha}_{13}$, $\dot{\alpha}_{23}$, $\dot{\alpha}_{33}$ in the right side of equation (4.59) with their variations from the third, sixth and ninth equations (3.60), we find that

$$\begin{aligned}\dot{\alpha}_{13}\delta\alpha_{12} + \dot{\alpha}_{23}\delta\alpha_{22} + \dot{\alpha}_{33}\delta\alpha_{32} &= \\ = \omega_y(\alpha_{11}\delta\alpha_{12} + \alpha_{21}\delta\alpha_{22} + \alpha_{31}\delta\alpha_{32}) &- \\ - \omega_x(\alpha_{12}\delta\alpha_{12} + \alpha_{22}\delta\alpha_{22} + \alpha_{32}\delta\alpha_{32}).\end{aligned} \tag{4.60}$$

Since the second term of the right side is equal to zero,

$$\begin{aligned}\dot{\alpha}_{13}\delta\alpha_{12} + \dot{\alpha}_{23}\delta\alpha_{22} + \dot{\alpha}_{33}\delta\alpha_{32} &= \\ = \omega_y(\alpha_{11}\delta\alpha_{12} + \alpha_{21}\delta\alpha_{22} + \alpha_{31}\delta\alpha_{32})\end{aligned} \tag{4.61}$$

We now insert relation (4.61) into equality (4.59), and the latter relation into equality (4.58). Taking equality (4.52) into account, we obtain

$$\dot{\theta}_{1x} + \omega_y \theta_{1z} - \omega_z \theta_{1y} = \Delta m_x. \tag{4.62}$$

If we multiply the first equation (4.51) by α_{13}, the fourth by α_{23}, and the seventh by α_{33} and add, then, using equalities (3.60), (4.52), (4.54) and relations (4.53), (4.56), (4.57), we arrive at the equation

$$\dot{\theta}_{1y} + \omega_z \theta_{1x} - \omega_x \theta_{1z} = \Delta m_y. \tag{4.63}$$

Finally, multiplying the third equation (4.51) by α_{11}, the sixth by α_{21} and the ninth by α_{31}, we find that

$$\dot{\theta}_{1z} + \omega_x \theta_{1y} - \omega_y \theta_{1x} = \Delta m_z. \tag{4.64}$$

Comparing equations (4.62) -- (4.64) with equations (4.36), we see that they coincide.

Equations (4.37) may be obtained from the scaler equations (3.61) in a manner completely analogous to the derivation of equations (4.62) -- (4.64). This requires only the introduction of the small rotation vector $\vec{\theta}_2$, the projections of which on the x, y, z axes are

$$\left.\begin{aligned}
\theta_{2x} &= \beta_{13}\delta\beta_{12} + \beta_{23}\delta\beta_{22} + \beta_{33}\delta\beta_{32} = \\
&= -\beta_{12}\delta\beta_{13} - \beta_{22}\delta\beta_{23} - \beta_{32}\delta\beta_{33}, \\
\theta_{2y} &= -\beta_{13}\delta\beta_{11} - \beta_{23}\delta\beta_{21} - \beta_{33}\delta\beta_{31} = \\
&= \beta_{11}\delta\beta_{13} + \beta_{21}\delta\beta_{23} + \beta_{31}\delta\beta_{33}, \\
\theta_{2z} &= \beta_{12}\delta\beta_{11} + \beta_{22}\delta\beta_{21} + \beta_{32}\delta\beta_{31} = \\
&= -\beta_{11}\delta\beta_{12} - \beta_{21}\delta\beta_{22} - \beta_{31}\delta\beta_{32}.
\end{aligned}\right| \tag{4.65}$$

and, in addition, the use of equality (3.64).

We note that the homogeneous equations (4.51) are exact . In going from equations (4.51) to equations (4.62) -- (4.64), we modified equalities of the type (4.53) -- (4.56) and ignored squares of variations of the direction cosines $\delta\alpha_{ij}$ and their products. The homogeneous equations (4.62) -- (4.64) are therefore first approximations of the effect of the perturbations. Analogously, equations of the form (4.51), which may be obtained from the equalities (3.61), will be precise, while equations (4.37) will be first approximations. This confirms the considerations expressed above * regarding the accuracy of equations (4.8), (4.9), and (4.36), (4.37), since equations (4.51), like the analogous equations for $\delta\beta_{ij}$, are projections of the vector equations (4.8) and (4.9) on the x, y, z axes.

In the process of deducing equations (4.36) and (4.37) directly from the scalar ideal equations (3.60) and (3.61), we obtained relations (4.52), (5.54) and (5.65), which link the variations $\delta\alpha_{ij}$ and $\delta\beta_{ij}$ of the direction cosines to θ_{1x}, θ_{1y}, θ_{1z} and θ_{2x}, θ_{2y}, θ_{2z}, respectively. These relations permit, in particular, expression of the initial values $\theta_{1x}(0)$, $\theta_{1y}(0)$, $\theta_{1z}(0)$, $\theta_{2x}(0)$, $\theta_{2y}(0)$, $\theta_{2z}(0)$, in terms of $\delta\alpha_{ij}(0)$ and $\delta\beta_{ij}(0)$.

In addition, we will show how equations (4.37) are obtained from relations (3.31), (3.32), (3.33), (3.41) and (3.21), (3.22), (3.23).

From formula (3.41) it follows that

$$
\begin{aligned}
\delta\dot{\beta}_{11} = {} & \alpha_{11}\delta\dot{\alpha}'_{11} + \alpha_{21}\delta\dot{\alpha}'_{21} + \alpha_{31}\delta\dot{\alpha}'_{31} + \\
& + \dot{\alpha}_{21}\delta\alpha'_{21} + \dot{\alpha}_{31}\delta\alpha'_{31} + \dot{\alpha}_{11}\delta\alpha'_{11} + \\
& + \dot{\alpha}'_{11}\delta\alpha_{11} + \dot{\alpha}'_{21}\delta\alpha_{21} + \dot{\alpha}'_{31}\delta\alpha_{11} + \\
& + \alpha'_{11}\delta\dot{\alpha}_{11} + \alpha'_{21}\delta\dot{\alpha}_{21} + \alpha'_{31}\delta\dot{\alpha}_{31}.
\end{aligned}
$$

---

* See page 300 [of translation].

Substituting here the values $\dot{\alpha}_{ij}$, $\dot{\alpha}'_{ij}$, $\delta\dot{\alpha}_{ij}$, $\delta\dot{\alpha}'_{ij}$ from expressions (3.31) -- (3.33) and (3.21) -- (3.23), and noting that

$$\beta_{12} = \alpha'_{11}\alpha_{12} + \alpha'_{21}\alpha_{22} + \alpha'_{31}\alpha_{32},$$
$$\beta_{13} = \alpha'_{11}\alpha_{13} + \alpha'_{21}\alpha_{23} + \alpha'_{31}\alpha_{33},$$

and making use of the orthogonality property of the tables of direction cosines (3.16) and (3.27), we arrive at the equality

$$\begin{aligned}\delta\dot{\beta}_{11} = {} & \delta\beta_{12}(\omega_z - u_z) - \delta\beta_{13}(\omega_y - u_y) + \\ & + \beta_{12}[\Delta m_z - (\Delta u_\xi\beta_{11} + \Delta u_\eta\beta_{21} + \Delta u_\zeta\beta_{31}) - \\ & - (u_\xi\delta\beta_{11} + u_\eta\delta\beta_{21} + u_\zeta\delta\beta_{31})] - \\ & - \beta_{13}[\Delta m_y - (\Delta u_\xi\beta_{12} + \Delta u_\eta\beta_{22} + \Delta u_\zeta\beta_{32}) - \\ & \qquad - (u_\xi\delta\beta_{12} + u_\eta\delta\beta_{22} + u_\zeta\delta\beta_{32})],\end{aligned}$$

which also follows from relations (3.61), (3.64). The remaining analogous equalities following from these same relations are obtained in a similar manner. The further derviation of equations (4.37) is obvious.

We now derive relations (4.52), (4.54), (4.65) directly from relations (3.17). According to (3.17)

$$\left.\begin{aligned}\delta\xi_0 &= \delta\alpha_{11}x + \delta\alpha_{12}y + \delta\alpha_{13}z,\\ \delta\eta_0 &= \delta\alpha_{21}x + \delta\alpha_{22}y + \delta\alpha_{23}z,\\ \delta\zeta_0 &= \delta\alpha_{31}x + \delta\alpha_{32}y + \delta\alpha_{33}z.\end{aligned}\right\} \tag{4.66}$$

Further from relations (3.17) and the first three equalities (4.19) we find:

$$\left.\begin{aligned}\delta\alpha_{11} &= -\theta_{1y}\alpha_{13} + \theta_{1z}\alpha_{12},\\ \delta\alpha_{12} &= -\theta_{1z}\alpha_{11} + \theta_{1x}\alpha_{13},\\ \delta\alpha_{13} &= -\theta_{1x}\alpha_{12} + \theta_{1y}\alpha_{11};\\ \delta\alpha_{21} &= -\theta_{1y}\alpha_{23} + \theta_{1z}\alpha_{22},\\ \delta\alpha_{22} &= -\theta_{1z}\alpha_{21} + \theta_{1x}\alpha_{23},\\ \delta\alpha_{23} &= -\theta_{1x}\alpha_{22} + \theta_{1y}\alpha_{21};\\ \delta\alpha_{31} &= -\theta_{1y}\alpha_{33} + \theta_{1z}\alpha_{32},\\ \delta\alpha_{32} &= -\theta_{1z}\alpha_{31} + \theta_{1x}\alpha_{33},\\ \delta\alpha_{33} &= -\theta_{1x}\alpha_{32} + \theta_{1y}\alpha_{31}.\end{aligned}\right\} \tag{4.67}$$

If we now multiply the third equality (4.67) by α_{12}, the sixth by α_{22}, and the ninth by α_{32}, and add the resulting equalities, taking into account the relations

$$\left.\begin{aligned} \alpha_{12}^2 + \alpha_{22}^2 + \alpha_{32}^2 = 1, \\ \alpha_{12}\alpha_{11} + \alpha_{22}\alpha_{21} + \alpha_{32}\alpha_{31} = 0, \end{aligned}\right\} \tag{4.68}$$

we obtain the expression for θ_{1x}

$$\theta_{1x} = -\alpha_{12}\delta\alpha_{13} - \alpha_{22}\delta\alpha_{23} - \alpha_{32}\delta\alpha_{33}, \tag{4.69}$$

which coincides with the first formula (4.52), as required.

The remaining five formulas (4.52) and (4.54) may be obtained in an analogous way from equalities (4.67).

In order to obtain formulas (4.65) from the latter three equalities (4.19), the relations

$$\left.\begin{aligned} \delta\xi_0 = \delta\beta_{11}x + \delta\beta_{12}y + \delta\beta_{13}z, \\ \delta\eta_0 = \delta\beta_{21}x + \delta\beta_{22}y + \delta\beta_{23}z, \\ \delta\zeta_0 = \delta\beta_{31}x + \delta\beta_{32}y + \delta\beta_{33}z, \end{aligned}\right\} \tag{4.70}$$

must be used to write the equations linking the variations $\delta\beta_{ij}$ of the direction cosines with θ_{2x}, θ_{2y}, θ_{2z}, analogously to equations (4.67), and the same operations must be carried out on them as in the derivation of formulas (4.52) and (4.54).

Thus, we have obtained equations (4.35) -- (4.37) from the ideal scaler equations (3.59) -- (3.65). It remains to obtain equalities (4.38) and (4.39) and, in addition, to obtain from relations (3.65) the projections (4.41). From equalities (3.62) we have:

$$
\left.\begin{aligned}
\delta x &= \delta\alpha_{11}\xi_* + \delta\alpha_{21}\eta_* + \delta\alpha_{31}\zeta_* + \\
&\qquad + \alpha_{11}\delta\xi_* + \alpha_{21}\delta\eta_* + \alpha_{31}\delta\zeta_*, \\
\delta y &= \delta\alpha_{12}\xi_* + \delta\alpha_{22}\eta_* + \delta\alpha_{32}\zeta_* + \\
&\qquad + \alpha_{12}\delta\xi_* + \alpha_{22}\delta\eta_* + \alpha_{32}\delta\zeta_*, \\
\delta z &= \delta\alpha_{13}\xi_* + \delta\alpha_{23}\eta_* + \delta\alpha_{33}\zeta_* + \\
&\qquad + \alpha_{13}\delta\xi_* + \alpha_{23}\delta\eta_* + \alpha_{33}\delta\zeta_*.
\end{aligned}\right\} \tag{4.71}
$$

But according to expressions (4.17)

$$
\left.\begin{aligned}
\alpha_{11}\delta\xi_* + \alpha_{21}\delta\eta_* + \alpha_{31}\delta\zeta_* &= \delta x_3, \\
\alpha_{12}\delta\xi_* + \alpha_{22}\delta\eta_* + \alpha_{32}\delta\zeta_* &= \delta y_3, \\
\alpha_{13}\delta\xi_* + \alpha_{23}\delta\eta_* + \alpha_{33}\delta\zeta_* &= \delta z_3.
\end{aligned}\right\} \tag{4.72}
$$

On the other hand, using ξ_*, η_*, ζ_* from relations (3.62), we find:

$$
\begin{aligned}
&\delta\alpha_{11}\xi_* + \delta\alpha_{21}\eta_* + \delta\alpha_{31}\zeta_* = \\
&\quad = (\alpha_{11}\delta\alpha_{11} + \alpha_{21}\delta\alpha_{21} + \alpha_{31}\delta\alpha_{31})x + (\alpha_{12}\delta\alpha_{11} + \alpha_{22}\delta\alpha_{21} + \\
&\qquad + \alpha_{32}\delta\alpha_{31})y + (\alpha_{13}\delta\alpha_{11} + \alpha_{23}\delta\alpha_{21} + \alpha_{33}\delta\alpha_{31})z.
\end{aligned} \tag{4.73}
$$

As a result of the orthogonality of the table of direction cosines α_{ij} it is possible to write with an accuracy to within the second order of smallness:

$$
\alpha_{11}\delta\alpha_{11} + \alpha_{21}\delta\alpha_{21} + \alpha_{31}\delta\alpha_{31} = 0,
$$

but according to equalities (4.52) and (4.54),

$$
\begin{aligned}
\alpha_{12}\delta\alpha_{11} + \alpha_{22}\delta\alpha_{21} + \alpha_{32}\delta\alpha_{31} &= \theta_{1z}, \\
\alpha_{13}\delta\alpha_{11} + \alpha_{23}\delta\alpha_{21} + \alpha_{33}\delta\alpha_{31} &= -\theta_{1y}.
\end{aligned}
$$

Thus, taking into account the first equality (4.72), the first equality (4.71) assumes the form:

$$
\delta x = \theta_{1z} y - \theta_{1y} z + \delta x_3.
$$

But from formula (4.38):

$$\theta_{1z} y - \theta_{1y} z = -\delta x_1.$$

We therefore finally obtain:

$$\delta x_1 = \theta_{1y} z - \theta_{1z} y, \quad \delta x_2 = \delta x + \delta x_1. \tag{4.74}$$

which coincides with the corresponding equalities (4.38). The remaining equalities of this group may be obtained in an analogous manner.

In order to obtain the last three relations (4.39), it is necessary to use the equalities

$$\left.\begin{aligned}
\delta x &= \delta\beta_{11}\xi + \delta\beta_{21}\eta + \delta\beta_{31}\zeta + \beta_{11}\delta\xi + \beta_{21}\delta\eta + \beta_{31}\delta\zeta, \\
\delta y &= \delta\beta_{12}\xi + \delta\beta_{22}\eta + \delta\beta_{32}\zeta + \beta_{12}\delta\xi + \beta_{22}\delta\eta + \beta_{32}\delta\zeta, \\
\delta z &= \delta\beta_{13}\xi + \delta\beta_{21}\eta + \delta\beta_{33}\zeta + \beta_{13}\delta\xi + \beta_{23}\delta\eta + \beta_{33}\delta\zeta,
\end{aligned}\right\} \tag{4.75}$$

that follow from formulas (3.63).

Completing the derivation of equations (4.35) -- (4.39) from (3.59) -- (3.65), we obtain from the latter equations the projections (4.41) of the variations in the intensity of the gravitational field of the earth on the x, y, z axes, i.e., we show that the following expressions for δg_x, δg_y, δg_z obtain:

$$\left.\begin{aligned}
\delta g_x &= \left(\delta \operatorname{grad} \frac{\mu}{r} - \theta_2 \times \operatorname{grad} \varepsilon + \operatorname{grad} \delta\varepsilon\right)_x, \\
\delta g_y &= \left(\delta \operatorname{grad} \frac{\mu}{r} - \theta_2 \times \operatorname{grad} \varepsilon + \operatorname{grad} \delta\varepsilon\right)_y, \\
\delta g_z &= \left(\delta \operatorname{grad} \frac{\mu}{r} - \theta_2 \times \operatorname{grad} \varepsilon + \operatorname{grad} \delta\varepsilon\right)_z.
\end{aligned}\right\} \tag{4.76}$$

The three equalities (4.76) are clearly equivalent to the single vector equality

$$\delta g = \delta \operatorname{grad} \frac{\mu}{r} - \theta_2 \times \operatorname{grad} \varepsilon + \operatorname{grad} \delta\varepsilon. \tag{4.77}$$

the correctness of which we are now certain on the basis of equations (3.65).

From equations (3.65) it follows that

$$\begin{aligned}\delta g = &-\delta\frac{\mu r}{r^3} + \delta\frac{\partial\varepsilon}{\partial\xi_*}(\beta_{11}x + \beta_{12}y + \beta_{13}z) + \\ &+ \delta\frac{\partial\varepsilon}{\partial\eta_*}(\beta_{21}x + \beta_{22}y + \beta_{23}z) + \delta\frac{\partial\varepsilon}{\partial\zeta_*}(\beta_{31}x + \beta_{32}y + \beta_{33}z) + \\ &+ \frac{\partial\varepsilon}{\partial\xi_*}(x\,\delta\beta_{11} + y\,\delta\beta_{12} + z\,\delta\beta_{13}) + \\ &+ \frac{\partial\varepsilon}{\partial\eta_*}(x\,\delta\beta_{21} + y\,\delta\beta_{22} + z\,\delta\beta_{23}) + \\ &\qquad\qquad + \frac{\partial\varepsilon}{\partial\zeta_*}(x\,\delta\beta_{31} + y\,\delta\beta_{32} + z\,\delta\beta_{33}),\end{aligned} \tag{4.78}$$

whence

$$\delta g = \delta\,\mathrm{grad}\,\frac{\mu}{r} + \mathrm{grad}\,\delta\varepsilon + \frac{\partial\varepsilon}{\partial\xi_*}\delta\xi_* + \frac{\partial\varepsilon}{\partial\eta_*}\delta\eta_* + \frac{\partial\varepsilon}{\partial\zeta_*}\delta\zeta_*. \tag{4.79}$$

But according to the final three equalities of (4.19)

$$\frac{\partial\varepsilon}{\partial\xi_*}\delta\xi_* + \frac{\partial\varepsilon}{\partial\eta_*}\delta\eta_* + \frac{\partial\varepsilon}{\partial\zeta_*}\delta\zeta_* = -\theta_1 \times \mathrm{grad}\,\varepsilon. \tag{4.80}$$

The validity of equality (4.77) follows from formulas (4.80) and (4.79).

In conclusion we will write the error equations in vector and scalar form.

If the coordinates being determined are the ξ_*, η_*, ζ_* coordinates in the fundamental Cartesian coordinate system, then the error vector equations form the system

$$\left.\begin{aligned}&\delta\ddot{r} + 2\omega\times\delta\dot{r} + \dot{\omega}\times\delta r + \omega\times(\omega\times\delta r) + \\ &+ \frac{\mu}{r^3}\delta r - \frac{\mu r}{r^3}\,\frac{3(r\cdot\delta r)}{r^2} = \Delta n + \Delta g - 2\,\Delta m\times\dot{r} - \\ &\qquad - \Delta\dot{m}\times r - \Delta m\times(\omega\times r) - \omega\times(\Delta m\times r), \\ &\qquad\qquad \dot{\theta}_1 + \omega\times\theta_1 = \Delta m, \\ &\qquad \delta r_1 = \theta_1\times r, \quad \delta r_* = \delta r + \delta r_1.\end{aligned}\right| \tag{4.81}$$

The initial conditions of system (4.81) will be the quantities

$$\delta r(0) = \delta r^0, \quad \delta \dot{r}(0) = \dot{r}^0, \quad \theta_1(0) = \theta_1^0. \tag{4.82}$$

The following scalar equations correspond to the vector equation (4.81):

$$\left.\begin{aligned}
&\delta\ddot{x} + \left[\frac{\mu}{r^5}(y^2 + z^2 - 2x^2) - \omega_y^2 - \omega_z^2\right]\delta x + \\
&\quad + \left(\omega_x\omega_y - \dot{\omega}_z - \frac{3\mu xy}{r^5}\right)\delta y - 2\omega_z\delta\dot{y} + \\
&\quad + \left(\omega_x\omega_z + \dot{\omega}_y - \frac{3\mu xz}{r^5}\right)\delta z + 2\omega_y\delta\dot{z} = \\
&= \Delta n_x + \Delta g_x - 2(\Delta m_y\dot{z} - \Delta m_z\dot{y}) - \Delta\dot{m}_y z + \Delta\dot{m}_z y - \\
&\quad - \omega_x(\Delta m_y y + \Delta m_z z) - \Delta m_x(\omega_y y + \omega_z z) + \\
&\qquad + 2x(\omega_y\Delta m_y + \omega_z\Delta m_z), \\
&\delta\ddot{y} + \left[\frac{\mu}{r^5}(z^2 + x^2 - 2y^2) - \omega_z^2 - \omega_x^2\right]\delta y + \\
&\quad + \left(\omega_y\omega_z - \dot{\omega}_x - \frac{3\mu yz}{r^5}\right)\delta z - 2\omega_x\delta\dot{z} + \\
&\quad + \left(\omega_y\omega_x + \dot{\omega}_z - \frac{3\mu yx}{r^5}\right)\delta x + 2\omega_z\delta\dot{x} = \\
&\quad = \Delta n_y + \Delta g_y - 2(\Delta m_z\dot{x} - \Delta m_x\dot{z}) - \\
&\quad - \Delta\dot{m}_z x + \Delta\dot{m}_x z - \omega_y(\Delta m_z z + \Delta m_x x) - \\
&\qquad - \Delta m_y(\omega_z z + \omega_x x) + 2y(\omega_z\Delta m_z + \omega_x\Delta m_x), \\
&\delta\ddot{z} + \left[\frac{\mu}{r^5}(x^2 + y^2 - 2z^2) - \omega_x^2 - \omega_y^2\right]\delta z + \\
&\quad + \left(\omega_z\omega_x - \dot{\omega}_y - \frac{3\mu zx}{r^5}\right)\delta x - 2\omega_y\delta\dot{x} + \\
&\quad + \left(\omega_z\omega_y + \dot{\omega}_x - \frac{3\mu zy}{r^5}\right)\delta y + 2\omega_x\delta\dot{y} = \\
&\quad = \Delta n_z + \Delta g_z - 2(\Delta m_x\dot{y} - \Delta m_y\dot{x}) - \\
&\quad - \Delta\dot{m}_x y + \Delta\dot{m}_y x - \omega_z(\Delta m_x x + \Delta m_y y) - \\
&\qquad - \Delta m_z(\omega_x x + \omega_y y) + 2z(\omega_x\Delta m_x + \omega_y\Delta m_y);
\end{aligned}\right| \tag{4.83}$$

$$\left.\begin{aligned}
&\dot{\theta}_{1x} + \omega_y\theta_{1z} - \omega_z\theta_{1y} = \Delta m_x, \\
&\dot{\theta}_{1y} + \omega_z\theta_{1x} - \omega_x\theta_{1z} = \Delta m_y, \\
&\dot{\theta}_{1z} + \omega_x\theta_{1y} - \omega_y\theta_{1x} = \Delta m_z.
\end{aligned}\right\} \tag{4.84}$$

$$\left.\begin{aligned}
&\delta x_1 = \theta_{1y} z - \theta_{1z} y, \quad \delta y_1 = \theta_{1z} x - \theta_{1x} z, \\
&\qquad \delta z_1 = \theta_{1x} y - \theta_{1y} x; \\
&\delta x_3 = \delta x + \delta x_1, \quad \delta y_3 = \delta y + \delta y_1, \\
&\qquad \delta z_3 = \delta z + \delta z_1
\end{aligned}\right| \tag{4.85}$$

The initial conditions of these equations will be the quantities:

$$\left.\begin{aligned}&\delta x(0)=\delta x^0, \quad \delta y(0)=\delta y^0, \quad \delta z(0)=\delta z^0,\\&\delta \dot{x}(0)=\delta \dot{x}^0, \quad \delta \dot{y}(0)=\delta \dot{y}^0, \quad \delta \dot{z}(0)=\delta \dot{z}^0,\\&\theta_{1x}(0)=\theta^0_{1x}=a^0_{13}\delta a^0_{12}+a^0_{23}\delta a^0_{22}+a^0_{33}\delta a^0_{32},\\&\theta_{1y}(0)=\theta^0_{1y}=a^0_{11}\delta a^0_{13}+a^0_{21}\delta a^0_{23}+a^0_{31}\delta a^0_{33},\\&\theta_{1z}(0)=\theta^0_{1z}=a^0_{12}\delta a^0_{11}+a^0_{22}\delta a^0_{21}+a^0_{32}\delta a^0_{31}.\end{aligned}\right\} \tag{4.86}$$

where, in accordance with relations (4.40)

$$\left.\begin{aligned}&\delta \dot{x}^0=\delta \dot{x}_0+(\delta\omega^0_y-\Delta m^0_y)z^0-(\delta\omega^0_z-\Delta m^0_z)y^0,\\&\delta \dot{y}^0=\delta \dot{y}_0+(\delta\omega^0_z-\Delta m^0_z)x^0-(\delta\omega^0_x-\Delta m^0_x)z^0,\\&\delta \dot{z}^0=\delta \dot{z}_0+(\delta\omega^0_x-\Delta m^0_x)y^0-(\delta\omega^0_y-\Delta m^0_y)x^0.\end{aligned}\right\} \tag{4.87}$$

4.2.3. <u>Additional remarks</u>. If the coordinates being determined are the Cartesian coordinates ξ, η, ζ in the coordinate system attached to the earth, then the first equation (4.81) is retained, but the final three equations are replaced by

$$\left.\begin{aligned}&\dot{\theta}_2+\omega\times\theta_2=\Delta m-\Delta u,\\&\delta r_2=\theta_2\times r, \quad \delta r_4=\delta r+\delta r_2\end{aligned}\right\} \tag{4.88}$$

with the initial condition $\vec{\theta}_2(0) = \vec{\theta}_2^0$.

Corresponding to equation (4.88) are the scaler equations

$$\left.\begin{aligned}&\dot{\theta}_{2x}+\omega_y\theta_{2z}-\omega_z\theta_{2y}=\Delta m_x-\Delta u_x,\\&\dot{\theta}_{2y}+\omega_z\theta_{2x}-\omega_x\theta_{2z}=\Delta m_y-\Delta u_y,\\&\dot{\theta}_{2z}+\omega_x\theta_{2y}-\omega_y\theta_{2x}=\Delta m_z-\Delta u_z;\end{aligned}\right\} \tag{4.89}$$

$$\left.\begin{aligned}&\delta x_2=\theta_{2y}z-\theta_{2z}y, \quad \delta y_2=\theta_{2z}x-\theta_{2x}z,\\&\delta z_2=\theta_{2x}y-\theta_{2y}x;\\&\delta x_4=\delta x+\delta x_2, \quad \delta y_4=\delta y+\delta y_2, \quad \delta z_4=\delta z+\delta z_2.\end{aligned}\right\} \tag{4.90}$$

The initial conditions of equation (4.89) will be the quantities defined by the following equalities, analogous to the final three equalities (4.86):

$$\left.\begin{aligned}\theta_{2x}(0)&=\theta^0_{2x}=\beta^0_{13}\delta\beta^0_{12}+\beta^0_{23}\delta\beta^0_{22}+\beta^0_{33}\delta\beta^0_{32},\\ \theta_{2y}(0)&=\theta^0_{2y}=\beta^0_{11}\delta\beta^0_{13}+\beta^0_{21}\delta\beta^0_{23}+\beta^0_{31}\delta\beta^0_{33},\\ \theta_{2z}(0)&=\theta^0_{2z}=\beta^0_{12}\delta\beta^0_{11}+\beta^0_{22}\delta\beta^0_{21}+\beta^0_{32}\delta\beta^0_{31}.\end{aligned}\right| \qquad (4.91)$$

It may be seen from equations (4.83) and (4.89), that the errors $\Delta\vec{g}$ and $\Delta\vec{u}$ may in fact be replaced by the equivalent basic instrument errors $\Delta\vec{n}$ and $\Delta\vec{m}$. One feature of this substitution should be noted, however. If equations (4.83) and (4.89) are compared, the following circumstance is revealed . The right side of equations (4.89) includes the projections of the vector $\Delta\vec{m} - \Delta\vec{u}$, while the right side of equations (4.83) contains only the projections of vector $\Delta\vec{m}$. Therefore the substitution of the equivalent value $\Delta\vec{m}$ for $\Delta\vec{u}$ is effected in the following manner: first, in equations (4.89) the following substitution is performed

$$\Delta m' = \Delta m - \Delta u. \qquad (4.92)$$

then $\Delta\vec{m}'$ is substituted for $\Delta\vec{m}$ in the right side of equations (4.83), while other terms containing $\Delta\vec{u}$ are dealt with by the corresponding equivalent variation of the error vector $\Delta\vec{n}$.

Equations (4.81) for the corresponding scaler equations (4.83) -- (4.85) do not differ in essence from the system represented by the first equation (4.81) and equation (4.88). They differ only in the right sides of the first equation (4.88) and the second equation (4.81). By virtue of the reducibility of the error $\Delta\vec{u}$ to the equivalent errors $\Delta\vec{m}$ and $\Delta\vec{n}$, this difference is insignificant. Therefore only the system of equations (4.83) -- (4.85) will be considered in the following discussion.

Up to this point in this section, the discussion has concerned the derivation and transformation of the error equations of inertial navigation systems containing gyroscopic sensing elements. In §3.4, however, it was shown that it is possible to construct an inertial system in which gyroscopic sensing elements are absent. In these so-called gravimetric inertial navigation systems, the only sensing elements are newtonometers. All initial information on the basis of which the operational algorithm of the system is constructed derives from these elements.

The operational algorithms of inertial systems lacking gyroscopic sensing elements may be of different sorts. These systems may contain, as was shown above, from 6 to 12 newtonometers depending on the algorithm. The minimum number of newtonometers -- six -- corresponds, naturally, to the number of degrees of freedom of an object freely moving in space. As was shown in §3.4, the equations describing the ideal operation of an inertial system containing no gyroscopic sensing elements determining the Cartesian coordinates, may differ from equations (3.59) -- (3.65) in that relations for the calculation of ω_x, ω_y, ω_z in accordance with the newtonometer readings, are added to the latter.

The error equations of an inertial system without gyroscopic elements determining the Cartesian coordinates will therefore differ from equations (4.83) -- (4.85) as well. The difference will consist only in the addition to equations (4.83) -- (4.85) of expressions derived by variation of the relations used to determine ω_x, ω_y, ω_z.

As was stated in §3.4, the basic means for determining ω_x, ω_y, ω_z is the use of equations (3.392). In this case the error equations (4.83) -- (4.85) remain valid. It is necessary only to substitute for Δm_x, Δm_y, Δm_z the following expressions following from relations (3.392):

$$\left.\begin{aligned}
\Delta m_x &= \delta\omega_x = \frac{1}{2l}\int_0^t \left(\Delta n_z^{(2)} - \Delta n_y^{(1)}\right)dt + \delta\omega_x(0),\\
\Delta m_y &= \delta\omega_y = \frac{1}{2l}\int_0^t \left(\Delta n_x^{(3)} - \Delta n_z^{(1)}\right)dt + \delta\omega_y(0),\\
\Delta m_z &= \delta\omega_z = \frac{1}{2l}\int_0^t \left(\Delta n_y^{(3)} - \Delta n_x^{(2)}\right)dt + \delta\omega_z(0)
\end{aligned}\right\} \qquad (4.93)$$

As is evident, this changes only the right sides of equations (4.83) and (4.84).

§4.3 Error Equations in the Determination of Curvilinear Coordinates*

4.3.1. The general case of non-stationary oblique curvilinear coordinates. We will derive the error equations for an inertial system determining arbitrary curvilinear and, in the, general case, non-orthogonal and non-stationary, coordinates $\varkappa^1$, $\varkappa^2$, $\varkappa^3$ of an object in the basic Cartesian coordinate system $O_1\xi_*\eta_*\zeta_*$. As in §4.2, we will confine ourselves for the moment to that portion of the error equations which relates to the determination of the coordinates of an object.

In order to solve this problem, it is sufficient to examine the inertial navigation system described in §3.2, in which a free gyro-stablized platform was taken as the basis of the kinematic system, and the newtonometers were oriented along the vectors $\vec{r}^1$, $\vec{r}^2$, $\vec{r}^3$ of the mutually based trihedron. The operational algorithm of this inertial navigation system is given by equations (3.172) and (3.163) or (3.164) and the table of direction cosines (3.173) characterizing the orientation of the axes of sensitivity of the newtonometers $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ relative to the stablized platform, i.e., relative to the axes of the basic Cartesian coordinate system $O_1\xi^1\xi^2\xi^3$.

As in the derivation of the error equations of an inertial system defining Cartesian coordinates which was performed in §4.2, we will reduce the instrument error of the elements and devices of the system to a few basic instrument errors. As before, we will take as the basic errors the errors of the sensing elements. In this case, these will be the errors Δn_{e_s} of the newtonometers and the orientation errors

---

* V. D. Andreyev, Error equations of an inertial system determining arbitrary curvilinear coordinates of a moving object. Izv. AN SSSR, Mekhanika, No. 4, 1965.

of the gyrostablized platform. The latter may be given as three angles characterizing the deviation of the trihedron associated with the platform from the required position. It is more convenient, however, to define, as before, the orientation errors of the gyrostablized platform by means of the projections $\Delta m_{\xi}s$ of the absolute angular velocity of its rotation in inertial space around the ξ^{s} axis of the gyrostablized platform.

Varying the basic inertial navigation equation (1.88) and taking into account the basic instrument errors discussed above, we obtain

$$\left.\begin{aligned} \delta\left(\frac{d^2 r}{dt^2} - g\right) &= \Delta n - 2\Delta m \times \frac{dr}{dt} - \frac{d\,\Delta m}{dt} \times r, \\ \Delta m &= \Delta m_{\xi}\xi_1 + \Delta m_{\eta}\eta_1 + \Delta m_{\zeta}\zeta_1. \end{aligned}\right\} \tag{4.94}$$

On the other hand,

$$\delta\left(\frac{d^2 r}{dt^2} - g\right) = \delta n = \delta(n^s r_s). \tag{4.95}$$

From expressions (3.132) for the contravariant components of the vector $\vec{n}$ in the basic coordinate system, and from which the formulas (3.172) were derived, we find:

$$\begin{aligned} \delta n = r_i \delta(\ddot{x}^i + \Gamma^i_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^i_{0n}\dot{x}^n + \Gamma^i_{00} - g^i) + \\ + (\ddot{x}^i + \Gamma^i_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^i_{0n}\dot{x}^n + \Gamma^i_{00} - g^i)\delta r_i. \end{aligned} \tag{4.96}$$

From relations (4.94), (4.95) and (4.96) we now obtain the following three equations (k = 1, 2, 3):

$$\begin{aligned} \delta(\ddot{x}^k + \Gamma^k_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^k_{0n}\dot{x}^n + \Gamma^k_{00} - g^k) + \\ + (\ddot{x}^i + \Gamma^i_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^i_{0n}\dot{x}^n + \Gamma^i_{00} - g^i) r^k \cdot \delta r_i = \\ = \Delta n^k - \left(2\Delta m \times \frac{dr}{dt} - \frac{d\,\Delta m}{dt} \times r\right) \cdot r^k. \end{aligned} \tag{4.97}$$

Equations (4.97) will be the first group of error equations for an inertial system determining arbitrary curvilinear coordinates. They clearly correspond to equations (4.1).

Let us now expand the expressions appearing in equations (4.97). Since Γ_{mn}^{k}, Γ_{0n}^{k}, Γ_{00}^{k} are functions of the coordinates x^1, x^2, x^3 and time t, and the variations are, as before, isochronic,

$$\begin{aligned}\delta(\ddot{x}^k + \Gamma_{mn}^{k}\dot{x}^m\dot{x}^n + 2\Gamma_{0n}^{k}\dot{x}^n + \Gamma_{00}^{k} - g^k) = \\ = \delta\ddot{x}^k + 2\Gamma_{mn}^{k}\dot{x}^m\delta\dot{x}^n + \frac{\partial}{\partial x^s}(\Gamma_{mn}^{k})\dot{x}^m\dot{x}^n\delta x^s + \\ + \frac{\partial}{\partial x^s}(\Gamma_{00}^{k})\delta x^s + 2\frac{\partial}{\partial x^s}(\Gamma_{0n}^{k})\dot{x}^n\delta x^s + 2\Gamma_{0n}^{k}\delta\dot{x}^n - \delta g^k.\end{aligned} \quad (4.98)$$

According to the definition of $\vec{r}_s$:

$$\delta r_s = \frac{\partial^2 r}{\partial x^s\,\partial x^n}\delta x^n. \quad (4.99)$$

From this, recalling the definition of the Christoffel symbols, we obtain:

$$r^k \cdot \delta r_s = \delta x^n \Gamma_{sn}^{k}. \quad (4.100)$$

Let us introduce the vector $\vec{q}$ and its contravariant components $(\vec{q})^k$ according to the equalities:

$$\left(2\Delta m \times \frac{dr}{dt} + \frac{d\Delta m}{dt} \times r\right) \cdot r^k = q \cdot r^k = (q)^k \quad (4.101)$$

and compute the values of $(\vec{q})^k$.

For the vectors $\vec{\Delta m}$ and $d\vec{r}/dt$ appearing in expression (4.101) we have:

$$\left.\begin{aligned}\Delta m = \Delta m^s r_s = \Delta m_s r^s, \\ \frac{dr}{dt} = \dot{x}^s r_s + \frac{\partial r}{\partial t}.\end{aligned}\right\} \quad (4.102)$$

From the second equality (4.102), it follows:

$$
\left.\begin{aligned}
\left(\frac{dr}{dt}\right)^k &= \dot{x}^k + \frac{\partial r}{\partial t}\cdot r^k = \dot{x}^k + a_0^k, \\
\left(\frac{dr}{dt}\right)_k &= a_{kl}(\dot{x}^l + a_0^l).
\end{aligned}\right\} \qquad (4.103)
$$

where $\left(\frac{d\vec{r}}{dt}\right)^k$ and $\left(\frac{d\vec{r}}{dt}\right)_k$ denote the contravariant and covariant components of the vector $d\vec{r}/dt$. Further,

$$
\left.\begin{aligned}
\left(\frac{d\Delta m}{dt}\right)^k &= \Delta\dot{m}^k + \Delta m^l(\Gamma_{ls}^k \dot{x}^s + \Gamma_{0l}^k). \\
\left(\frac{d\Delta m}{dt}\right)_k &= a_{nk}\left(\frac{d\Delta m}{dt}\right)^n
\end{aligned}\right\} \qquad (4.104)
$$

and, finally,

$$
\left.\begin{aligned}
(r)_k &= \frac{1}{2}\frac{\partial}{\partial x^k} r^2, \\
(r)^k &= a^{sk}(r)_s.
\end{aligned}\right\} \qquad (4.105)
$$

Substituting (4.102), (4.103), (4.104) and (4.105) into the left side of equality (4.`01) and introducing the Levi-Civita symbols (3.150), (3.151), (3.152), we obtain the following representations of expressions (4.101):

$$
\begin{aligned}
(q)^k = \epsilon^{stk}\left[2(\Delta m)_s\left(\frac{dr}{dt}\right)_t + \left(\frac{d\Delta m}{dt}\right)_s (r)_t\right] = \\
= \epsilon^{stk}\Big[2\Delta m_s a_{tl}(\dot{x}^l + a_0^l) + \\
+ a_{ns}(\Delta\dot{m}^n + \Delta m^l(\Gamma_{lm}^n \dot{x}^m + \Gamma_{0l}^n))\frac{1}{2}\frac{\partial}{\partial x^t} r^2\Big].
\end{aligned} \qquad (4.106)
$$

Let us find δg^k. According the formulas (3.11) and (3.15)

$$
g = \mathrm{grad}\,\frac{\mu}{r} + \mathrm{grad}\,\varepsilon(\eta^1, \eta^2, \eta^3) \qquad (4.107)
$$

Therefore,

$$
g^k = -\frac{\mu}{r^3} r \cdot r^k + \mathrm{grad}^l \varepsilon\, \eta_l^k = -\frac{\mu}{2r^3} a^{lk}\frac{\partial r^2}{\partial x^l} + \mathrm{grad}^l \varepsilon\, \eta_l^k. \qquad (4.108)
$$

where η_l^k is defined by equations (3.163) and (3.164)

We now obtain

$$
\delta g^k = -\delta\left(\frac{\mu}{r^3} r\cdot r^k\right) + \mathrm{grad}^l \delta\varepsilon\, \eta_l^k + \mathrm{grad}^l \varepsilon\, \delta\eta_l^k. \qquad (4.109)
$$

In order to find $\delta\eta_{\ell}^{k}$ is it necessary to vary equations (3.163) or their equivalents; the variations of the first components in the right side of equalities (4.109) may be expanded immediately:

$$\delta\left(\frac{\mu}{r^3} r \cdot r^k\right) = \mu r^k \cdot \delta\left(\frac{r}{r^3}\right) + \frac{\mu}{r^3} r \cdot \delta r^k. \tag{4.110}$$

Using formula (4.44) we expand the first terms of the right sides of equalities (4.110):

$$\begin{aligned} \mu r^k \cdot \delta\left(\frac{r}{r^3}\right) &= \frac{\mu}{r^3}\left(\delta r - 3r \frac{r \cdot \delta r}{r^2}\right) \cdot r^k = \\ &= \frac{\mu}{r^3} r^k \cdot r_q \delta x^q - \frac{3\mu}{r^5} r \cdot r_n a^{kn} r \cdot r_q \delta x^q = \\ &= \frac{\mu}{r^3} \delta x^k - \frac{3\mu}{4r^5} a^{kn} \frac{\partial r^2}{\partial x^n} \frac{\partial r^2}{\partial x^q} \delta x^q. \end{aligned} \tag{4.111}$$

In order to expand the second terms of (4.110), we consider the equalities

$$r = (r)^m r_m, \quad r_m \cdot \delta r^k = -r^k \cdot \delta r_m. \tag{4.112}$$

Then

$$\frac{\mu}{r^3} r \cdot \delta r^k = -\frac{\mu}{2r^3} \frac{\partial r^2}{\partial x^n} a^{mn} \Gamma_{mq}^{k} \delta x^q. \tag{4.113}$$

Substituting (4.113) and (4.111) in (4.110) and also in (4.109) we arrive at the following expressions for δg^k.

$$\begin{aligned} \delta g^k = &-\frac{\mu}{r^3} \delta x^k + \frac{3\mu}{4r^5} a^{kn} \frac{\partial r^2}{\partial x^n} \frac{\partial r^2}{\partial x^q} \delta x^q + \\ &+ \frac{\mu}{2r^3} \frac{\partial r^2}{\partial x^n} a^{mn} \Gamma_{mq}^{k} \delta x^q + \operatorname{grad}^l \delta r \eta_l^k + \operatorname{grad}^l r \delta \eta_l^k. \end{aligned} \tag{4.114}$$

In considering in §4.2 the error equations of inertial systems determining Cartesian coordinates, we concluded that in the first approximation of the error equations the variations of the nonspherical components of the earth's gravitational field may be ignored. It is evident that this conclusion is valid here as well.

In place of relations (4.114) we will then have:

$$\delta g^k = -\frac{\mu}{r^3}\delta x^k + \frac{3\mu}{4r^5}a^{kn}\frac{\partial r^2}{\partial x^n}\frac{\partial r^2}{\partial x^n}\delta x^n + \frac{\mu}{2r^3}\frac{\partial r^2}{\partial x^n}a^{mn}\Gamma^k_{m0}\delta x^n. \tag{4.115}$$

Taking these equalities and equalities (4.106), (4.100), (4.98) into account, we obtain from relations (4.97) the equations:

$$\begin{gathered}
\delta\ddot{x}^k + 2\Gamma^k_{mn}\dot{x}^m\delta x^n + \frac{\partial}{\partial x^a}(\Gamma^k_{mn})\dot{x}^m\dot{x}^n\delta x^n + \frac{\partial}{\partial x^n}(\Gamma^k_{00})\delta x^n + \\
+ 2\frac{\partial}{\partial x^n}(\Gamma^k_{0n})\dot{x}^n\delta x^n + 2\Gamma^k_{0n}\delta x^n + \frac{\mu}{r^3}\delta x^k - \\
- \frac{3\mu}{4r^5}a^{kn}\frac{\partial r^2}{\partial x^n}\frac{\partial r^2}{\partial x^n}\delta x^n - \frac{\mu}{2r^3}\frac{\partial r^2}{\partial x^n}a^{mn}\Gamma^k_{mn}\delta x^n + \\
+ \Big(\ddot{x}^s + \Gamma^s_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^s_{0n}\dot{x}^n + \Gamma^s_{00} + \\
+ \frac{\mu}{2r^3}a^{sn}\frac{\partial r^2}{\partial x^n}\Big)\Gamma^k_{s0}\delta x^n = \Lambda n^k - \epsilon^{slk}\Big\{2\Lambda m_s a_{ll}(\dot{x}^l + a^l_0) + \\
+ a_{ns}\Big[\Lambda\dot{m}^n + (\Gamma^n_{lm}\dot{x}^m + \Gamma^n_{0l})\Lambda m^l\Big]\frac{1}{2}\frac{\partial r^2}{\partial x^l}\Big\}.
\end{gathered} \tag{4.116}$$

Equations (4.116) constitute the first group of error equations of the inertial navigation system under consideration. To them it is necessary to add the equations

$$\left.\begin{aligned}
&\dot{\theta}^k_1 + \theta^l_1(\Gamma^k_{li}\dot{x}^i + \Gamma^k_{0l}) = \Lambda m^k, \\
&\delta x^k_1 = \epsilon^{slk}\theta_{1s}\frac{1}{2}\frac{\partial r^2}{\partial x^l}, \\
&\delta x^k_3 = \delta x^k + \delta x^k_1.
\end{aligned}\right\} \tag{4.117}$$

The first group of equations (4.117) is analogous to equations (4.84) of the preceding section. The equations of this group define the small rotation vector θ_1 of the hydrostabilized platform in inertial space, resulting from its free drift.

The second and third groups of equations (4.117) define, analogous to (4.85), the errors δx^s_1, in addition to δx^s, in the determination of the coordinates x^s caused by the fact that the angle θ_1 is different from 0, and total errors δx^s_3 in the determination of the coordinates analogous to the errors δx_3, δy_3, δz_3 in equation (4.85).

Before analyzing equations (4.116) and (4.117) and examining special cases, let us transform them somewhat. As a result of the transformations we obtain the following system of equations:

$$
\left.
\begin{aligned}
&\delta\ddot{x}^k + \frac{\mu}{r^3}\delta x^k + 2\left(\Gamma^k_{mn}\dot{x}^m + \Gamma^k_{0n}\right)\delta\dot{x}^n + \\
&+\Bigg[\left(\frac{\partial}{\partial x^n}\Gamma^k_{mn} + \Gamma^s_{mn}\Gamma^k_{sn}\right)\dot{x}^m\dot{x}^n + \\
&+ 2\left(\frac{\partial}{\partial x^n}\Gamma^k_{0n} + \Gamma^s_{0n}\Gamma^k_{s0}\right)\dot{x}^n + \Gamma^k_{sn}\left(\ddot{x}^s + \Gamma^s_{00}\right) + \\
&+ \frac{\partial}{\partial x^n}\Gamma^k_{00} - \frac{3\mu}{4r^5}a^{kn}\frac{\partial r^2}{\partial x^n}\frac{\partial r^2}{\partial x^n}\Bigg]\delta x^n = \\
&= \Delta n^k - \in^{srk}\left\{a_{sp}\Delta m^p a_{rl}\left(\dot{x}^l + a^l_0\right) + \right. \\
&\left. + a_{ns}\left[\Delta\dot{m}^n + \left(\Gamma^n_{lm}\dot{x}^m + \Gamma^m_{0l}\right)\Delta m^l\right]\frac{1}{2}\frac{\partial r^2}{\partial x^l}\right\}, \\
&\qquad \dot{\theta}^k_1 + \theta^l_1\left(\Gamma^k_{ls}\dot{x}^s + \Gamma^k_{0l}\right) = \Delta m^k, \\
&\qquad \delta x^k_1 = \in^{srk}a_{ns}\theta^n_1\frac{1}{2}\frac{\partial r^2}{\partial x^l}, \\
&\qquad \delta x^k_3 = \delta x^k + \delta x^k_1.
\end{aligned}
\right\}
\qquad (4.118)
$$

The initial conditions of equations (4.118) are obvious.

Equations (4.118) contain the contravariant components Δn^s and Δm^s of vectors $\Delta\vec{n}$ and $\Delta\vec{m}$ of the instrument errors of the sensing elements. The projections Δn_{e_s} of vector $\Delta\vec{n}$ on the axes of the sensing newtonometers and the projections $\Delta m_{\xi s}$ of vector m on the axes of the gyrostabilized platform, i.e., on the axes of the basic Cartesian coordinate system, are known.

Since in the system under consideration the unit vectors $\vec{e}_s$ of the newtonometer axes are disposed along the vectors $\vec{r}^s$ of the common basis,

$$
\Delta n_{e_s} = \Delta n\, e_s = \frac{\Delta n^s}{\sqrt{a^{ss}}}. \qquad (4.119)
$$

Hence (not summing over s!)

$$\Delta n^s = \Delta n_s \sqrt{a^{ss}}. \tag{4.120}$$

From equalities (4.94) and (3.88), (3.89)

$$\Delta m^s = \Delta m \cdot r^s = \Delta m_{1'} \frac{\partial \xi^s}{\partial \varkappa^m} a^{ms}. \tag{4.121}$$

We note that the orientation errors of the newtonometers in the class of systems under consideration may be obtained by varying the elements of table (3.173). We will not present here the relations obtained from this variation; this will be done below.

4.3.2. <u>Orthogonal coordinates. Cartesian and geocentric coordinates</u>. Let us examine several special cases of the derived ideal equations of an inertial system using curvilinear coordinates. For stationary coordinates, when

$$\Gamma_{0n}^{s} = 0, \quad \Gamma_{00}^{s} = 0, \quad a_0^l = 0, \tag{4.122}$$

the first two groups of equations (4.118) assume the form:

$$\left.\begin{aligned}
&\delta\ddot{\varkappa}^s + \frac{\mu}{r^3}\delta\varkappa^s + 2\Gamma_{m0}^{s}\dot{\varkappa}^m\delta\dot{\varkappa}^0 + \\
&\quad + \Big[\dot{\varkappa}^m\dot{\varkappa}^n\Big(\frac{\partial}{\partial\varkappa^n}\Gamma_{mn}^{s} + \Gamma_{mn}^{\sigma}\Gamma_{\sigma n}^{s}\Big) + \ddot{\varkappa}^\sigma\Gamma_{\sigma n}^{s} - \\
&\quad - \frac{3\mu}{4r^5}a^{sn}\frac{\partial r^2}{\partial\varkappa^n}\frac{\partial r^2}{\partial\varkappa^n}\Big]\delta\varkappa^n = \\
&\quad = \Delta n^s - \epsilon^{sls}\Big[2\Delta m^p a_{sp}a_{ll}\dot{\varkappa}^l + a_{ns}(\Delta\dot{m}^n + \\
&\qquad\qquad + \Delta m^l\Gamma_{lm}^{n}\dot{\varkappa}^m)\frac{1}{2}\frac{\partial r^l}{\partial\varkappa^l}\Big], \\
&\qquad \dot{\theta}_l^s + \theta_l^l\Gamma_{sl}^{s}\dot{\varkappa}^l = \Delta m^s.
\end{aligned}\right| \tag{4.123}$$

The third and fourth groups of equations (4.118) do not change, nor do relations (4.120) and (4.121).

If the coordinates $\varkappa^s$ are non-stationary, but are orthogonal, then only the diagonal components of the metric tensor are different from zero. As a result, the first and third groups of equations (4.118) may be written in the following form:

$$\left.\begin{aligned}
&\delta\ddot{x}^k + \frac{\mu}{r^3}\delta x^k + 2(\Gamma^k_{m0}\dot{x}^m + \Gamma^k_{00})\delta\dot{x}^n + \\
&\quad + \Big[\dot{x}^m\dot{x}^s\left(\frac{\partial}{\partial x^n}\Gamma^k_{ms} + \Gamma^l_{ms}\Gamma^k_{ln}\right) + \\
&\quad + 2\dot{x}^m\left(\frac{\partial}{\partial x^n}\Gamma^k_{0m} + \Gamma^l_{0m}\Gamma^k_{ln}\right) + (\ddot{x}^l + \Gamma^l_{00})\Gamma^k_{ln} + \\
&\quad + \frac{\partial}{\partial x^n}\Gamma^k_{00} - \frac{3\mu}{4r^5}a^{kk}\frac{\partial r^2}{\partial x^k}\frac{\partial r^2}{\partial x^n}\Big]\delta x^n = \\
&\quad = \Lambda n^k - \epsilon^{slk}\Big\{2a_{ss}\Lambda m^s a_{ll}(\dot{x}^l + a^l_0) + \\
&\quad + a_{ss}\left[\Lambda\dot{m}^s + \Lambda m^l(\Gamma^s_{lm}\dot{x}^m + \Gamma^s_{0l})\right]\frac{1}{2}\frac{\partial r^2}{\partial x^l}\Big\}, \\
&\qquad\qquad \delta x^k_1 = \epsilon^{slk}a_{ss}\theta^s_1\frac{1}{2}\frac{\partial r^2}{\partial x^l}.
\end{aligned}\right\} \tag{4.124}$$

The second and fourth groups of equations (4.118) and relations (4.120) do not change, but equalities (4.121) take the form:

$$\Lambda m^s = \Lambda m_{1^s}\frac{\partial \xi^s}{\partial x^s}a^{ss}. \tag{4.125}$$

Finally, if the coordinates are stationary and orthogonal,

$$\Gamma^k_{0k} = \Gamma^k_{00} = a^l_0 = 0, \tag{4.126}$$

must be substituted into equation (4.124), as a result of which these equations take the following form:

$$\begin{aligned}
&\delta\ddot{x}^k + \frac{\mu}{r^3}\delta x^k + 2\Gamma^k_{mn}\dot{x}^m\delta\dot{x}^n + \\
&\qquad + \Big[\dot{x}^m\dot{x}^s\left(\frac{\partial}{\partial x^n}\Gamma^k_{ms} + \Gamma^k_{ln}\Gamma^l_{ms}\right) + \Gamma^k_{ln}\ddot{x}^l - \\
&\quad - \frac{3\mu}{4r^5}a^{kk}\frac{\partial r^2}{\partial x^k}\frac{\partial r^2}{\partial x^n}\Big]\delta x^n = \Lambda n^k - \epsilon^{slk}\Big[2a_{ss}a_{ll}\Lambda m^s\dot{x}^l + \\
&\qquad\qquad + a_{ss}(\Lambda\dot{m}^s + \Lambda m^l\dot{x}^m\Gamma^s_{lm})\frac{1}{2}\frac{\partial r^2}{\partial x^l}\Big].
\end{aligned} \tag{4.127}$$

Equations (4.124) and (4.127) correspond to the ideal equations (3.210) -- (3.213) in orthogonal curvilinear coordinates.

If x^1, x^2, x^3 are non-stationary Cartesian coordinates, equations (4.124) transform, as expected, into equations (4.83) -- (4.85). In fact, in this case the vectors $\vec{r}_3$ are unitary, and the orthogonal trihedron formed by them rotates as a unit. All of the Christoffel symbols are equal to zero, while the non-zero diagonal elements of the metric tensor are equal to one. Taking this into account, we obtain from equation (4.124):

$$\left.\begin{aligned}
&\delta\ddot{x}^k + \frac{\mu}{r^3}\,\delta x^k + 2\Gamma^k_{0\sigma}\,\delta\dot{x}^\sigma + \\
&\quad + \left[\frac{\partial}{\partial x^\sigma}\left(2\Gamma^k_{0s}\dot{x}^s + \Gamma^k_{00}\right) - \frac{3\mu}{4r^5}\,\frac{\partial r^2}{\partial x^k}\,\frac{\partial r^2}{\partial x^\sigma}\right]\delta x^\sigma = \\
&\quad = \Delta n^k - \epsilon_{slk}\Big[2\,\Delta m^s\left(\dot{x}^l + a^l_0\right) + \\
&\qquad\qquad + \left(\Delta\dot{m}^s + \Delta m^t\Gamma^s_{0t}\right)\frac{1}{2}\,\frac{\partial r^2}{\partial x^l}\Big], \\
&\quad \dot{\theta}^k_i + \theta^t_i\Gamma^k_{0t} = \Delta m^k, \\
&\quad \delta x^k_i = \epsilon_{slk}\theta^s_i\,\frac{1}{2}\,\frac{\partial r^2}{\partial x^l}, \quad \delta x^k_\Sigma = \delta x^k + \delta x^k_i.
\end{aligned}\right| \qquad (4.128)$$

Since the coordinates are Cartesian,

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \qquad (4.129)$$

and

$$\frac{\partial r^2}{\partial x^k} = 2x^k. \qquad (4.130)$$

The symbols Γ^k_{0s} and Γ^k_{00} for the case in question have already been calculated in §3.3. Turning to formulas (3.233) -- (3.237) we find:

$$\left.\begin{aligned}
&\frac{\partial}{\partial x^\sigma}\left(2\Gamma^k_{0s}\dot{x}^s + \Gamma^k_{00}\right) = \\
&\qquad = (\dot{\omega} \times r_\sigma)\cdot r_k + \omega_k\omega_\sigma - r_k\cdot r_\sigma\omega^2, \\
&\quad a^l_0 = \frac{\partial r}{\partial t}\cdot r^l = (\omega \times r)\cdot r^l.
\end{aligned}\right| \qquad (4.131)$$

Substituting (4.131) and (4.130) into (4.128) we arrive at the equations

$$\left.\begin{aligned}
&\delta\ddot{x}^k + \frac{\mu}{r^5}\left(r^2\delta x^k - 3x^k x_\sigma\,\delta x^\sigma\right) + \\
&\quad + 2(\omega \times r_\sigma)\cdot r^k\,\delta\dot{x}^\sigma + \\
&\quad + [(\dot{\omega} \times r_\sigma)\cdot r^k + \omega^k\omega_\sigma - r^k r_\sigma\omega^2]\,\delta x^\sigma = \\
&\quad = \Delta n^k - \epsilon^{slk}\,[2\,\Delta m_s\,[\dot{x}_l + (\omega \times r)\cdot r_l] + \\
&\qquad\qquad + [\Delta\dot{m}_s + \Delta m^t(\omega \times r_t)\cdot r_s]\,x_l], \\
&\dot{\theta}^k_i + \theta^t(\omega \times r_t)\cdot r^k = \Delta m^k, \quad \delta x^k_i = \epsilon_{slk}\theta^s_i x^l, \\
&\qquad\qquad \delta x^k_\Sigma = \delta x^k + \delta x^k_i.
\end{aligned}\right| \qquad (4.132)$$

Since the coordinates are Cartesian, $\vec{r}^{s} = \vec{r}_{s}$ and, consequently, the contravariant and covariant components are equal; also equal are the Levi-Civita symbols ϵ_{stk} and ϵ^{stk}. The indices in equations (4.132) are therefore disposed in such a way as to guarantee the summing rule. Expanding the mixed products in these equations, then summing and noting that quantities $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$; $\varkappa^1$, $\varkappa^2$, $\varkappa^3$; ω_1, ω_2, ω_3; Δm_1, Δm_2, Δm_3; θ_1^1, θ_1^2, θ_1^3 in equations (4.132) correspond to the quantities $\vec{x}$, $\vec{y}$, $\vec{z}$; x, y, z; ω_x, ω_y, ω_z; Δm_x, Δm_y, Δm_z; θ_{1x}, θ_{1y}, θ_{1z} in equations (4.83) -- (4.85), we easily convince ourselves of the identity of equations (4.132) and (4.83) -- (4.85).

The error equations (4.118) were obtained by considering an inertial system the kinematic basis of which was taken to be a free gyrostabilized platform. In §3.2 it was also shown that a maneuverable gyroplatform may also serve as the kinematic basis in the determination of orthogonal curvilinear coordinates. Equation (4,118) remain valid, of course, in this case as well, and only the instrument error $\Delta\vec{m}$ changes, being specified not as projections Δm_{ξ}s, but as projections $\Delta m_{(s)}$ on the axes of the maneuverable platform, i.e., on the $\vec{r}_s$ directions. As a result, in calculating the contravariant components of the vector $\Delta\vec{m}$ in the basic coordinate system, instead of formulas (4.125) we will have formulas

$$\Delta m^{s} = \Delta m_{(s)} \sqrt{a^{ss}}. \tag{4.133}$$

where $\Delta m_{(s)}$ denotes the errors in the specifications of the projections of the angular presession velocity of the maneuverable gyroplatform on its axes (the instrument errors in the formation of the controlling moments and the free drifts taken with inverse sign). According to formulas (3.209) and (1.78a),

$$\left.\begin{aligned}
\Delta m_{(1)} &= -\Delta\left(\frac{Mf_{1y}^{5}}{H}\right) + \frac{Mf_{2y}^{5}}{H}, \\
\Delta m_{(2)} &= \Delta\left(\frac{Mf_{1x}^{4}}{H}\right) - \frac{Mf_{2x}^{4}}{H}, \\
\Delta m_{(3)} &= \Delta\left(\frac{Mf_{1x}^{6}}{H}\right) - \frac{Mf_{2x}^{6}}{H}.
\end{aligned}\right\} \tag{4.134}$$

For purposes of illustration, let us derive from equations (4.118) the error equations for the geocentric coordinates r, λ, φ in a trihedral bound to the earth.

Since the geocentric reference grid is orthogonal, instead of relation (4.118) we may begin from equation (4.124) and the second and fourth groups of equation (4.118).

We will use the values (3.252) for the Christoffel symbols calculated above for a geocentric reference grid, the values (3.267) of the symbols Γ_{0s}^{k}, Γ_{00}^{k}, and the values (3.250) of the diagonal components of the metric tensor. Substituting these into the first group of equations (4.124), we obtain the following values for the individual terms in the left side of these equations.

For k = 1:

$$\left.\begin{aligned}
&2\left(\Gamma_{mn}^{1}\dot{x}^{m}+\Gamma_{0n}^{1}\right)\delta\dot{x}^{n}=\\
&\qquad=2\left[\left(\Gamma_{22}^{1}\dot{x}^{2}+\Gamma_{01}^{2}\right)\delta\dot{x}^{2}+\Gamma_{33}^{1}\dot{x}^{3}\delta\dot{x}^{3}\right]=\\
&\qquad\qquad=-2r\left[\dot{\varphi}\delta\dot{\varphi}+(\dot{\lambda}+u)\cos^{2}\varphi\,\delta\dot{\lambda}\right].\\
&\left[\dot{x}^{m}\dot{x}^{n}\left(\frac{\partial}{\partial x^{\eta}}\Gamma_{mn}^{1}+\Gamma_{mn}^{s}\Gamma_{s\eta}^{1}\right)+\right.\\
&\left.+2\dot{x}^{n}\left(\frac{\partial}{\partial x^{\eta}}\Gamma_{0n}^{1}+\Gamma_{0n}^{s}\Gamma_{s\eta}^{1}\right)\right]\delta x^{\eta}=\\
&=\frac{\partial}{\partial x^{\eta}}\left[\Gamma_{22}^{1}(\dot{x}^{2})^{2}+\Gamma_{33}^{1}(\dot{x}^{3})^{2}+2\Gamma_{02}^{1}\dot{x}^{2}\right]\delta x^{\eta}+\\
&+\Gamma_{22}^{1}\delta x^{2}\left[2\dot{x}^{1}\left(\Gamma_{01}^{2}+\Gamma_{21}^{2}\dot{x}^{2}\right)+2\dot{x}^{3}\left(\Gamma_{03}^{2}+\Gamma_{23}^{2}\dot{x}^{2}\right)\right]+\\
&+\Gamma_{33}^{1}\delta x^{3}\left[2\Gamma_{31}^{3}\dot{x}^{1}\dot{x}^{3}+\dot{x}^{2}\left(2\Gamma_{02}^{3}+\Gamma_{22}^{3}\dot{x}^{2}\right)\right]=\\
&=-\delta r\left[\dot{\varphi}^{2}+(\dot{\lambda}^{2}+2\dot{\lambda}u)\cos^{2}\varphi\right]-\\
&-2\delta\lambda(u+\dot{\lambda})(\dot{r}\cos^{2}\varphi-r\dot{\varphi}\sin\varphi\cos\varphi)-\\
&\qquad\qquad-\delta\varphi\left[2\dot{r}\dot{\varphi}-r(\dot{\lambda}^{2}+2\dot{\lambda}u)\sin\varphi\cos\varphi\right].\\
&\left[\frac{\partial}{\partial x^{\eta}}\Gamma_{00}^{1}+\Gamma_{s\eta}^{1}\left(\ddot{x}^{s}+\Gamma_{00}^{s}\right)\right]\delta x^{\eta}=\\
&=\frac{\partial}{\partial x^{\eta}}\Gamma_{00}^{1}\delta x^{\eta}+\Gamma_{22}^{1}\ddot{x}^{2}\delta x^{2}+\Gamma_{33}^{1}\left(\ddot{x}^{3}+\Gamma_{00}^{3}\right)\delta x^{3}=\\
&=-\delta r u^{2}\cos^{2}\varphi+\delta\varphi\, r\left(u^{2}\sin\varphi\cos\varphi-\ddot{\varphi}\right)-\delta\lambda\, r\ddot{\lambda}\cos^{2}\varphi.\\
&\qquad-\frac{3\mu}{r^{5}}a^{11}\frac{\partial r^{2}}{\partial x^{1}}\frac{\partial r^{2}}{\partial x^{\eta}}\delta x^{\eta}=-\frac{3\mu}{r^{3}}\delta r.
\end{aligned}\right\} \tag{4.135}$$

For k = 2:

$$\left.\begin{aligned}
&2\left(\Gamma_{m0}^{2}\dot{x}^{m}+\Gamma_{0n}^{2}\right)\delta\dot{x}^{n}=\\
&\quad=2\left[\left(\Gamma_{21}^{2}\dot{x}^{2}+\Gamma_{01}^{2}\right)\delta\dot{x}^{1}+\left(\Gamma_{23}^{2}\dot{x}^{2}+\Gamma_{03}^{2}\right)\delta\dot{x}^{3}+\right.\\
&\quad\left.+\left(\Gamma_{12}^{2}\dot{x}^{1}+\Gamma_{32}^{2}\dot{x}^{3}\right)\delta\dot{x}^{2}\right]=\\
&=2\left[\delta\dot{r}\,\frac{\lambda+u}{r}-\delta\dot{\varphi}(\lambda+u)\tan\varphi+\delta\dot{\lambda}\left(\frac{\dot{r}}{r}-\dot{\varphi}\tan\varphi\right)\right].\\
&\left[\left(\frac{\partial}{\partial x^{n}}\Gamma_{mn}^{2}+\Gamma_{mn}^{s}\Gamma_{s0}^{2}\right)\dot{x}^{m}\dot{x}^{n}+\right.\\
&\left.+2\left(\frac{\partial}{\partial x^{0}}\Gamma_{0n}^{2}+\Gamma_{0n}^{s}\Gamma_{s0}^{2}\right)\dot{x}^{n}\right]\delta x^{n}=\\
&=\frac{\partial}{\partial x^{0}}\left[2\dot{x}^{1}\left(\Gamma_{21}^{2}\dot{x}^{2}+\Gamma_{01}^{2}\right)+2\dot{x}^{3}\left(\Gamma_{23}^{2}\dot{x}^{2}+\Gamma_{03}^{2}\right)\right]\delta x^{n}+\\
&+\Gamma_{12}^{2}\delta x^{2}\left|\dot{x}^{2}\left(\Gamma_{22}^{1}\dot{x}^{2}+2\Gamma_{02}^{1}\right)+(\dot{x}^{3})^{2}\Gamma_{33}^{1}\right|+\\
&+\Gamma_{32}^{2}\delta x^{2}\left|\dot{x}^{2}\left(\Gamma_{22}^{3}\dot{x}^{2}+2\Gamma_{02}^{3}\right)+2\Gamma_{31}^{3}\dot{x}^{1}\dot{x}^{3}\right|+\\
&+\Gamma_{21}^{2}\delta x^{1}\left|\dot{x}^{1}\left(\Gamma_{21}^{2}\dot{x}^{2}+2\Gamma_{01}^{2}\right)+\dot{x}^{3}\left(\Gamma_{23}^{2}\dot{x}^{2}+2\Gamma_{03}^{2}\right)+\right.\\
&\left.+\Gamma_{12}^{2}\dot{x}^{1}\dot{x}^{2}+\Gamma_{32}^{2}\dot{x}^{3}\dot{x}^{2}\right|+\Gamma_{23}^{2}\delta x^{3}\left|\dot{x}^{1}\left(\Gamma_{21}^{2}\dot{x}^{2}+2\Gamma_{01}^{2}\right)+\right.\\
&\left.+\dot{x}^{3}\left(\Gamma_{23}^{2}\dot{x}^{2}+2\Gamma_{03}^{2}\right)+\Gamma_{12}^{2}\dot{x}^{1}\dot{x}^{2}+\Gamma_{32}^{2}\dot{x}^{3}\dot{x}^{2}\right|=\\
&=-2\delta r\,\frac{\lambda+u}{r}\,\dot{\varphi}\tan\varphi-2\,\delta\varphi\,(\dot{\lambda}+u)\left(\dot{\varphi}+\frac{\dot{r}}{r}\tan\varphi\right)-\\
&\qquad-\delta\lambda\left[\dot{\lambda}(\lambda+2u)+\dot{\varphi}^{2}+2\,\frac{\dot{r}}{r}\,\dot{\varphi}\tan\varphi\right].\\
&\qquad-\frac{3\mu}{4r^{5}}\,a^{22}\,\frac{\partial r^{2}}{\partial x^{2}}\,\frac{\partial r^{2}}{\partial x^{n}}\,\delta x^{n}=0.\\
&\left[\frac{\partial}{\partial x^{0}}\Gamma_{00}^{s}+\Gamma_{s0}^{2}\left(\ddot{x}^{s}+\Gamma_{00}^{s}\right)\right]\delta x^{s}=\\
&=\Gamma_{31}^{2}\ddot{x}^{2}\,\delta x^{1}+\Gamma_{12}^{2}\left(\ddot{x}^{1}+\Gamma_{00}^{1}\right)\delta x^{2}+\Gamma_{32}^{2}\left(\ddot{x}^{3}+\Gamma_{00}^{3}\right)\delta x^{2}+\\
&+\Gamma_{23}^{2}\ddot{x}^{2}\,\delta x^{3}=\delta r\,\frac{\ddot{\lambda}}{r}+\delta\lambda\left(\frac{\ddot{r}}{r}-\ddot{\varphi}\tan\varphi-u^{2}\right)-\delta\varphi\ddot{\lambda}\tan\varphi.
\end{aligned}\right|\qquad(4.136)$$

For k = 3:

$$\left.\begin{aligned}
&2\left(\Gamma_{m0}^{3}\dot{x}^{m}+\Gamma_{0n}^{3}\right)\delta\dot{x}^{n}=\\
&\quad=2\left[\left(\Gamma_{22}^{3}\dot{x}^{2}+\Gamma_{02}^{3}\right)\delta\dot{x}^{2}+\Gamma_{31}^{3}\dot{x}^{3}\delta\dot{x}^{1}+\Gamma_{13}^{3}\dot{x}^{1}\delta\dot{x}^{3}\right]=\\
&\qquad\qquad=2\left[\delta\dot{r}\,\frac{\dot{\varphi}}{r}+\delta\dot{\lambda}(\dot{\lambda}+u)\sin\varphi\cos\varphi+\delta\dot{\varphi}\frac{\dot{r}}{r}\right].\\
&\left[\left(\frac{\partial}{\partial x^{n}}\Gamma_{mn}^{3}+\Gamma_{mn}^{s}\Gamma_{s0}^{3}\right)\dot{x}^{m}\dot{x}^{n}+\right.\\
&\left.+2\left(\frac{\partial}{\partial x^{n}}\Gamma_{0n}^{3}+\Gamma_{0n}^{s}\Gamma_{s0}^{3}\right)\dot{x}^{n}\right]\delta x^{n}=\\
&=\frac{\partial}{\partial x^{0}}\left|\dot{x}^{2}\left(\Gamma_{22}^{3}\dot{x}^{2}+2\Gamma_{02}^{3}\right)+2\Gamma_{13}^{3}\dot{x}^{1}\dot{x}^{3}\right|\delta x^{n}+\\
&+\Gamma_{31}^{3}\left|2\Gamma_{31}^{3}\dot{x}^{1}\dot{x}^{3}+\dot{x}^{2}\left(\Gamma_{22}^{3}\dot{x}^{2}+2\Gamma_{02}^{3}\right)\right|\delta x^{1}+\\
&+\Gamma_{13}^{3}\left|\dot{x}^{2}\left(\Gamma_{22}^{1}\dot{x}^{2}+2\Gamma_{02}^{1}\right)+\Gamma_{33}^{1}(\dot{x}^{3})^{2}\right|\delta x^{3}+\\
&+\Gamma_{22}^{3}\left|2\dot{x}^{1}\left(\Gamma_{21}^{2}\dot{x}^{2}+\Gamma_{01}^{2}\right)+2\dot{x}^{3}\left(\Gamma_{23}^{2}\dot{x}^{2}+\Gamma_{03}^{2}\right)\right|\delta x^{2}=\\
&=\delta r\,\frac{1}{r}\left(\dot{\lambda}+2u\dot{\lambda}\right)\sin\varphi\cos\varphi+\\
&+2\,\delta\lambda(\dot{\lambda}+u)\left(\frac{\dot{r}}{r}\cos\varphi-\dot{\varphi}\sin\varphi\right)\sin\varphi-\\
&\qquad\qquad-\delta\varphi\left[\left(\dot{\lambda}^{2}+2u\dot{\lambda}\right)\sin^{2}\varphi+\dot{\varphi}^{2}\right],\\
&\qquad-\frac{3\mu}{4r^{5}}\,a^{33}\,\frac{\partial r^{2}}{\partial x^{3}}\,\frac{\partial r^{2}}{\partial x^{n}}\,\delta x^{n}=0.\\
&\left[\frac{\partial}{\partial x^{0}}\Gamma_{00}^{3}+\Gamma_{sn}^{3}\left(\ddot{x}^{s}+\Gamma_{00}^{s}\right)\right]\delta x^{n}=\\
&\quad=\frac{\partial}{\partial x^{0}}\Gamma_{00}^{3}\,\delta x^{n}+\Gamma_{31}^{3}\left(\ddot{x}^{3}+\Gamma_{00}^{3}\right)\delta x^{1}+\\
&\quad+\Gamma_{13}^{3}\left(\ddot{x}^{1}+\Gamma_{00}^{1}\right)\delta x^{3}+\Gamma_{22}^{3}\ddot{x}^{2}\,\delta x^{2}=\\
&\quad=\delta r\,\frac{1}{r}\left(\ddot{\varphi}+u^{2}\sin\varphi\cos\varphi\right)+\ddot{\lambda}\sin\varphi\cos\varphi\,\delta\lambda+\\
&\qquad\qquad+\delta\varphi\left(\frac{\ddot{r}}{r}-u^{2}\sin^{2}\varphi\right).
\end{aligned}\right|\qquad(4.137)$$

Noting that for the right sides of the first group of equations (4.124) the following relations hold

$$\frac{\partial r^2}{\partial x^1} = 2x^1, \quad \frac{\partial r^2}{\partial x^2} = \frac{\partial r^2}{\partial x^3} = 0 \tag{4.138}$$

and taking into account expressions (3.183) for the Levi-Civita symbols ϵ^{stk}, we have:

$$\left.\begin{aligned}
&\Delta n^1 - \epsilon^{231}[2a_{22}\Delta m^2 a_{33}(\dot{x}^3 + a_0^3)] - \\
&\qquad - \epsilon^{321}[2a_{33}\Delta m^3 a_{22}(\dot{x}^2 + a_0^2)] = \\
&= \Delta n^1 + 2\sqrt{\frac{a_{22}a_{33}}{a_{11}}}[\Delta m^3(\dot{x}^2 + a_0^2) - \Delta m^2(\dot{x}^3 + a_0^3)], \\
&\Delta n^2 - \epsilon^{312}(2a_{33}\Delta m^3 a_{11}(\dot{x}^1 + a_0^1) + \\
&+ a_{33}[\Delta\dot{m}^3 + \Delta m^l(\Gamma_{ml}^3\dot{x}^m + \Gamma_{0l}^3)]x^1] - \\
&- \epsilon^{132}[2a_{11}\Delta m^1 a_{33}(\dot{x}^3 + a_0^3)] = \\
&= \Delta n^2 + 2\sqrt{\frac{a_{11}a_{33}}{a_{22}}}[\Delta m^1(\dot{x}^3 + a_0^3) - \Delta m^3(\dot{x}^1 + a_0^1)] - \\
&- \sqrt{\frac{a_{33}}{a_{11}a_{22}}}[\Delta\dot{m}^3 + \Delta m^2(\dot{x}^2\Gamma_{22}^3 + \Gamma_{02}^3) + \\
&\qquad + \Delta m^3\dot{x}^1\Gamma_{31}^3 + \Delta m^1\dot{x}^3\Gamma_{13}^3]x^1, \\
&\Delta n^3 - \epsilon^{123}[2a_{11}\Delta m^1 a_{22}(\dot{x}^2 + a_0^2)] - \\
&- \epsilon^{213}[2a_{22}\Delta m^2 a_{11}(\dot{x}^1 + a_0^1) + \\
&+ a_{22}(\Delta\dot{m}^2 + \Delta m^l(\Gamma_{ml}^2\dot{x}^m + \Gamma_{0l}^2))x^1] = \\
&= \Delta n^3 + 2\sqrt{\frac{a_{11}a_{22}}{a_{33}}}[\Delta m^2(\dot{x}^1 + a_0^1) - \Delta m^1(\dot{x}^2 + a_0^2)] + \\
&+ \sqrt{\frac{a_{22}}{a_{11}a_{33}}}[\Delta\dot{m}^2 + \Delta m^3(\Gamma_{23}^2\dot{x}^2 + \Gamma_{03}^2) + \\
&\qquad + \Delta m^1(\Gamma_{21}^2\dot{x}^2 + \Gamma_{01}^2) + \Delta m^2(\Gamma_{32}^2\dot{x}^3 + \Gamma_{12}^2\dot{x}^1)],
\end{aligned}\right| \tag{4.139}$$

where, according to the first group of equalities (4.103),

$$a_0^1 = \frac{\partial r}{\partial t} \cdot r^1. \tag{4.140}$$

Comparing (4.140) with (3.232) and (3.130), we have:

$$a_0^1 = ua^{11}\left(\eta^1\frac{\partial\eta^2}{\partial x^1} - \eta^2\frac{\partial\eta^1}{\partial x^1}\right). \tag{4.141}$$

Whence, using relations (3.246), (3.247) and (3.250), we find

$$a_0^1 = 0, \quad a_0^2 = u, \quad a_0^3 = 0. \tag{4.142}$$

We may now write explicit expressions for the right sides of the first group of equations (4.124) for the case of geocentric coordinates. Substituting into formulas (4.139) the values a_0^s from equalities (4.142), the values of the symbols Γ_{sk}^m, Γ_{0k}^m, Γ_{00}^m from relations (3.252) and (3.267) and the values a_{ss}, a^{ss} from (3.250), we arrive at the following expression:

$$\left.\begin{aligned}
&\Delta n^1 + 2r^2 \cos\varphi\, [\Delta m^3 (\dot{\lambda} + u) - \Delta m^2 \dot{\varphi}],\\
&\Delta n^2 + \frac{2}{\cos\varphi} (\Delta m^1 \dot{\varphi} - \Delta m^3 \dot{r}) -\\
&\quad - \frac{r}{\cos\varphi} [\Delta \dot{m}^3 + \Delta m^2 (\dot{\lambda} + u) \sin\varphi \cos\varphi +\\
&\qquad\qquad + \Delta m^3 \frac{\dot{r}}{r} + \Delta m^1 \frac{\dot{\varphi}}{r}],\\
&\Delta n^3 + 2\cos\varphi\, [\Delta m^2 \dot{r} - \Delta m^1 (\dot{\lambda} + u)] +\\
&\quad + r\cos\varphi\, [\Delta \dot{m}^2 + \Delta m^3 (\dot{\lambda} + u) \tan\varphi + \Delta m^1 \frac{\dot{\lambda} + u}{r} +\\
&\qquad\qquad + \Delta m^2 \left(\frac{\dot{r}}{r} - \dot{\varphi} \tan\varphi\right)].
\end{aligned}\right| \tag{4.143}$$

Here Δn^s and Δm^s are the contravariant components of the vectors $\Delta\vec{n}$ and $\Delta\vec{m}$ of the instrument errors in the basic coordinate system. Using formulas (4.120) and (4.133) they may be expressed in terms of the projections $\Delta n_{(s)}$ and $\Delta m_{(s)}$ of the vectors $\Delta\vec{n}$ and $\Delta\vec{m}$ on the directions $\vec{r}_s$.

Substituting $\Delta n_{(s)}$ and $\Delta m_{(s)}$ for Δn^s and Δm^s we obtain the expressions

$$\left.\begin{aligned}
&\Delta n_{(1)} + 2r\, [\Delta m_{(3)} (\dot{\lambda} + u) \cos\varphi - \Delta m_{(2)} \dot{\varphi}],\\
&\frac{\Delta n_{(2)}}{r\cos\varphi} + \frac{1}{\cos\varphi} \Big[\Delta m_{(1)} \dot{\varphi} - 2\Delta m_{(3)} \frac{\dot{r}}{r} -\\
&\qquad - \Delta \dot{m}_{(3)} - \Delta m_{(2)} (\dot{\lambda} + u) \sin\varphi\Big],\\
&\frac{\Delta n_{(3)}}{r} + 2\frac{\dot{r}}{r} \Delta m_{(2)} - \Delta m_{(1)} (\dot{\lambda} + u) \cos\varphi +\\
&\qquad + \Delta \dot{m}_{(2)} - \Delta m_{(3)} (\dot{\lambda} + u) \sin\varphi.
\end{aligned}\right\} \tag{4.144}$$

Combining equalities (4.135), (4.136), (4.137) and (4.144), we obtain the desired first group of error equations for a geocentric reference grid.

$$
\left.
\begin{aligned}
&\delta\ddot{r} - \left[\frac{2\mu}{r^3} + \dot{\varphi}^2 + (\dot{\lambda} + u)^2 \cos^2\varphi\right]\delta r - \\
&\quad - [2(\dot{\lambda} + u)(\dot{r}\cos^2\varphi - r\dot{\varphi}\sin\varphi\cos\varphi) + r\ddot{\lambda}\cos^2\varphi]\,\delta\lambda + \\
&\quad + [r(\dot{\lambda} + u)^2 \sin\varphi\cos\varphi - 2\dot{r}\dot{\varphi} - r\ddot{\varphi}]\,\delta\varphi - \\
&\quad - 2r(\dot{\lambda} + u)\,\delta\dot{\lambda}\cos^2\varphi - 2r\dot{\varphi}\,\delta\dot{\varphi} = \\
&\quad = \Delta n_{(1)} + 2r\,[\Delta m_{(3)}(\dot{\lambda} + u)\cos\varphi - \Delta m_{(2)}\dot{\varphi}]. \\
&\delta\ddot{\lambda} + \left[\frac{\mu}{r^3} - (\dot{\lambda} + u)^2 - \dot{\varphi}^2 + \frac{\ddot{r}}{r} - 2\frac{\dot{r}}{r}\dot{\varphi}\tan\varphi - \ddot{\varphi}\tan\varphi\right]\delta\lambda + \\
&\quad + \frac{1}{r}[\ddot{\lambda} - 2\dot{\varphi}(\dot{\lambda} + u)\tan\varphi]\,\delta r - \\
&\quad - \left[2(\dot{\lambda} + u)\dot{\varphi} + 2\frac{\dot{r}}{r}(\dot{\lambda} + u)\tan\varphi - \ddot{\lambda}\tan\varphi\right]\delta\varphi + \\
&\quad + 2\left(\frac{\dot{r}}{r} - \dot{\varphi}\tan\varphi\right)\delta\dot{\lambda} - 2(\dot{\lambda} + u)\,\delta\dot{\varphi}\tan\varphi + 2\frac{1}{r}(\dot{\lambda} + u)\,\delta\dot{r} = \\
&\quad = \frac{\Delta n_{(3)}}{r\cos\varphi} + \frac{1}{\cos\varphi}\left[\Delta m_{(1)}\dot{\varphi} - 2\Delta m_{(3)}\frac{\dot{r}}{r} - \right. \\
&\quad \left. - \Delta\dot{m}_{(3)} - \Delta m_{(2)}(\dot{\lambda} + u)\sin\varphi\right]. \\
&\delta\ddot{\varphi} + \left[\frac{\mu}{r^3} - (\dot{\lambda} + u)^2 \sin^2\varphi - \dot{\varphi}^2 + \frac{\ddot{r}}{r}\right]\delta\varphi + \\
&\quad + \left[2\frac{\dot{r}}{r}(\dot{\lambda} + u)\sin\varphi\cos\varphi - 2\dot{\varphi}(\dot{\lambda} + u)\sin^2\varphi + \right. \\
&\quad \left. + \ddot{\lambda}\sin\varphi\cos\varphi\right]\delta\lambda + \frac{1}{r}[(\dot{\lambda} + u)^2 \sin\varphi\cos\varphi + \ddot{\varphi}]\,\delta r + \\
&\quad + 2(\dot{\lambda} + u)\sin\varphi\cos\varphi\,\delta\dot{\lambda} + 2\frac{1}{r}\dot{\varphi}\,\delta\dot{r} + 2\frac{\dot{r}}{r}\delta\dot{\varphi} = \\
&\quad = \frac{\Delta n_{(2)}}{r} + 2\Delta m_{(2)}\frac{\dot{r}}{r} - \Delta m_{(1)}(\dot{\lambda} + u)\cos\varphi + \\
&\quad + \Delta\dot{m}_{(2)} - \Delta m_{(3)}(\dot{\lambda} + u)\sin\varphi.
\end{aligned}
\right| \tag{4.145}
$$

Equations (4.145) correspond to the first group of equations (4.124) or, equivalently, to the first group of equations (4.118).

The second group of equations (4.118), if only the non-zero Christoffel symbols and $\Gamma^{k}_{0\ell}$ symbols are retained, take the form:

$$
\left.
\begin{aligned}
&\dot{\theta}^1_1 + \theta^2_1(\Gamma^1_{22}\dot{x}^2 + \Gamma^1_{02}) + \theta^3_1\Gamma^1_{33}\dot{x}^3 = \Delta m^1, \\
&\dot{\theta}^2_1 + \theta^1_1(\Gamma^2_{21}\dot{x}^2 + \Gamma^2_{01}) + \theta^2_1(\Gamma^2_{12}\dot{x}^1 + \Gamma^2_{32}\dot{x}^3) + \\
&\qquad + \theta^3_1(\Gamma^2_{23}\dot{x}^2 + \Gamma^2_{03}) = \Delta m^2, \\
&\dot{\theta}^3_1 + \theta^1_1\Gamma^3_{31}\dot{x}^3 + \theta^2_1(\Gamma^3_{23}\dot{x}^2 + \Gamma^3_{02}) + \theta^3_1\Gamma^3_{13}\dot{x}^1 = \Delta m^3.
\end{aligned}
\right| \tag{4.146}
$$

Introducing the values (3.252) and (3.267) of the symbols Γ_{mn}^{k} and Γ_{0n}^{k} and expressing Δm^{s} in terms of $\Delta m_{(s)}$ in accordance with formulas (4.133), we obtain:

$$\left.\begin{aligned}
&\dot{\theta}_1^1 - \theta_1^2 r(\dot{\lambda}+u)\cos^2\varphi - \theta_1^3 r\dot{\varphi} = \Delta m_{(1)},\\
&\dot{\theta}_1^2 + \theta_1^1 \frac{\dot{\lambda}+u}{r} + \theta_1^2\left(\frac{\dot{r}}{r} - \dot{\varphi}\tan\varphi\right) -\\
&\qquad\qquad - \theta_1^3(\dot{\lambda}+u)\tan\varphi = \frac{\Delta m_{(2)}}{r\cos\varphi},\\
&\dot{\theta}_1^3 + \theta_1^1\frac{\dot{\varphi}}{r} + \theta_1^2(\dot{\lambda}+u)\sin\varphi\cos\varphi + \theta_1^3\frac{\dot{r}}{r} = \frac{\Delta m_{(3)}}{r}.
\end{aligned}\right\} \tag{4.147}$$

Equations (4.147) correspond to the second group of the error equations (4.118).

Finally, the third and fourth groups of equations (4.118) may be written, using formulas (4.138), (3.250), (3.251) and (3.183), in the following manner:

$$\delta r_1 = 0, \quad \delta\lambda_1 = \theta_1^3\frac{r}{\cos\varphi}, \quad \delta\varphi_1 = -\theta_1^2 r\cos\varphi. \tag{4.148}$$

$$\delta r_3 = \delta r + \delta r_1, \quad \delta\lambda_3 = \delta\lambda + \delta\lambda_1, \quad \delta\varphi_3 = \delta\varphi + \delta\varphi_1. \tag{4.149}$$

Equations (4.145), (4.147) -- (4.149) constitute a complete system of error equations for coordinate determination in a geocentric reference grid.

We note that equations (4.145) and (4.147) -- (4.149) may be obtained from formulas (4.83) -- (4.85), since the geocentric reference grid is orthogonal. We also note that this grid is spherical, i.e., vector $\vec{r}_1$ is directed along the radius of the earth from its center, and vectors $\vec{r}_2$, $\vec{r}_3$ are normal to this direction. Moreover, we will consider that the x, y, z axes in projections on which equations (4.83) -- (4.85) are written, are superposed with vectors $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_1$, respectively of the main basis of the geocentric reference grid. In this case

$$y = x = 0, \quad z = r. \tag{4.150}$$

should be substituted in equations (4.83).

According to (3.269) and the correspondence of the x, y, z axes and the vectors $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_1$,

$$\left.\begin{array}{c}\omega_{(1)}=\omega_z=(\dot{\lambda}+u)\sin\varphi, \quad \omega_{(2)}=\omega_x=-\dot{\varphi}, \\ \omega_{(3)}=\omega_y=(\dot{\lambda}+u)\cos\varphi;\end{array}\right\} \tag{4.151}$$

$$\left.\begin{array}{lll}\Delta m_{(1)}=\Delta m_z, & \Delta m_{(2)}=\Delta m_x, & \Delta m_{(3)}=\Delta m_y, \\ \Delta n_{(1)}=\Delta n_z, & \Delta n_{(2)}=\Delta n_x, & \Delta n_{(3)}=\Delta n_y\end{array}\right\} \tag{4.152}$$

In addition, it is evident that the relations

$$\delta y=r\,\delta\varphi, \quad \delta z=\delta r, \quad \delta x=\delta\lambda\, r\cos\varphi. \tag{4.153}$$

are valid to within the second order of smallness.

Substituting expressions (4.153), (4.152), (4.151) and (4.150) into equations (4.83) and dividing the second of the resulting equations by $r\cos\varphi$ and the third by r, we obtain equations (4.145).

In order to obtain equations (4.147) from equations (4.84), we must take into account the further fact that in equations (4.84) the quantities θ_{1x}, θ_{1y}, θ_{1z} are projections of the vector $\vec{\theta}_1$ on the x, y, z axes, and that in (4.147) the contravariant components of this vector in the main basis are expressed in terms of θ_1^1, θ_1^2, θ_1^3. Therefore, in analogy with formulas (4.120) and (4.133):

$$\theta_{1z}=\theta_{1(1)}=\theta_1^1, \quad \theta_{1x}=\theta_{1(2)}=\theta_1^2 r\cos\varphi, \quad \theta_{1y}=\theta_{1(3)}=r\theta_1^3. \tag{4.154}$$

The substitution of (4.154) in (4.84) gives equation (4.147).

Finally, from relations (4.85), with the aid of relations (4.154) and (4.150), relations (4.148) and (4.149) are obtained.

In the same way that (4.145), (4.147) -- (4.149) were obtained, the general equation (4.118) may be used to derive the error equations for any reference grid, in particular, of course, for those reference grids discussed in §3.3.

§4.4 Reduction of Error Equations for Curvilinear Coordinates to Error Equations for Cartesian Coordinates.

4.4.1. The possibility of reduction. The error equation (4.118) and the resulting error equations for specific reference grids are extraordinarily unwieldy and complex. One may easily verify this by referring to equations(4.145) and (4.147) -- (4.149) for geocentric coordinates. Especially complex are the first group of equations (4.118). This is not surprising, since these equations are essentially none other than variations of the general case of the motion of an object (mass point) in curvilinear coordinates in a spherical gravitational field and under the influence of several arbitrary forces, i.e., variations of Newton's general equations in curvilinear, non-orthogonal, and non-stationary coordinates.

Although equations (4.118) are linear, their coefficients, determined by the trajectory of the object with which the system is associated, are complex functions of time. Equations (4.118) and the equations derived from them for curvilinear coordinates do not possess, as a rule, a symmetry which is clearly enough expressed to facilitate their analysis. Simplification of these equations by ignoring various terms is in general impossible, since various of these terms may be of decisive significance depending on the character of the motion of the object.

Another possibility is to simplify equations (4.118) by dividing the possible trajectories of motion of an object into several classes on the basis of their practical interest, i.e., to simplify the equations by reference to the various categories of objects for which the inertial system is designed.

However, even for a given class of trajectories the concrete form of equations (4.118) will vary as a function of the structure of the inertial navigation system, i.e., as a function of the coordinates in which it operates. Thus, the problem in any case reduces to the consideration of a large number of equations of the form (4.118).

This may avoided if analysis of the error equations for any reference grid is reduced to their analysis in any one reference grid selected in an appropriate fashion. This approach, in conjunction with the reduction of the real instrument errors of the system elements to the equivalent instrument errors of the sensing elements, makes possible a general analysis of the error equations of inertial systems, as we will see.

The expediency of this approach derives from the fact that there is no direct need to consider the error equations in the form of variations of the coordinates determined by the system. In fact, with regard to the synthesis of the ideal equations, one of the problems to be solved is the selection of the reference grid on which the kinematics of the apparatus to a significant degree depends. Here it is necessary to be able to consider the ideal equations directly in those reference grids from which the reference grid which is to be realized is selected. The real problem in the analysis of the error equations is to establish system characteristics such as operational stability, and also to determine how errors in the determination of the coordinates of the moving object depend on the instrument errors of the system elements and the errors in the specification of its initial conditions. These characteristics, clearly, may be obtained by examining the error equations in other coordinates than those in which the actual inertial system operates. The actual instrument errors and the errors in the initial conditions should, of course, be translated into the coordinates in which the error equations are being analyzed.

The possibility of converting the error equation (4.118) from one coordinate system to another derives directly from the fact that these equations are, essentially, variations of the basic inertial navigation equation (1.88), which, clearly, is invariant relative to the coordinate system selected for its solution. Even equations (4.118) have a tensor character and as a result their properties should not depend on the choice of coordinate system

We will show this directly by transforming equations (4.118) into an invariant (vector) form.

Let us consider the case in which the coordinates $\varkappa^s$ are stationary, and there are no errors deriving from the gyroscopic elements, i.e.,

$$\frac{\partial r_s}{\partial t} = 0, \quad \Delta m = 0. \tag{4.155}$$

We introduce the vector

$$\rho = \delta r = \delta\varkappa^s r_s, \tag{4.156}$$

along with its time derivatives

$$p = \frac{d\rho}{dt} \text{ and } q = \frac{d^2\rho}{dt^2}. \tag{4.157}$$

As usual, we will denote the covariant and contravariant components of these vectors in the main basis by ρ_s and ρ^s, p_s and p^s, and q_s and q^s, respectively.

Applying the operation of covariant differentiation to the contravariant components $\rho^s = \delta\varkappa^s$ of the vector $\vec{\rho}$, we find

$$p^s = \delta\dot{\varkappa}^s + \Gamma_{mn}^s \dot{\varkappa}^m \delta\varkappa^n. \tag{4.158}$$

Applying the same operation again, we obtain:

$$q^s = \delta\ddot{\varkappa}^s + 2\Gamma_{mn}^s \dot{\varkappa}^m \delta\dot{\varkappa}^n + \Gamma_{mn}^s \ddot{\varkappa}^m \delta\varkappa^n + \\ + \dot{\varkappa}^m \dot{\varkappa}^\eta \delta\varkappa^n \frac{\partial}{\partial\varkappa^\eta} \Gamma_{mn}^s + \dot{\varkappa}^m \dot{\varkappa}^\eta \delta\varkappa^n \Gamma_{mn}^\theta \Gamma_{\theta\eta}^s. \tag{4.159}$$

According to the definition of the covariant and contravariant components, we have:

$$p = p^s r_s \text{ and } q = q^s r_s. \tag{4.160}$$

On the other hand, for stationary coordinates, the following equalities hold:

$$\left.\begin{aligned}\frac{dr}{dt}&=\dot{x}^{k}r_{k},\\ \frac{d^{2}r}{dt^{2}}&=(\ddot{x}^{k}+\Gamma_{mn}^{k}\dot{x}^{m}\dot{x}^{n})r_{k}.\end{aligned}\right\} \tag{4.161}$$

Varying equalities (4.161), we obtain:

$$\delta\frac{dr}{dt}=\delta\dot{x}^{k}r_{k}+\frac{\partial^{2}r}{\partial x^{k}\partial x^{n}}\dot{x}^{k}\delta x^{n}=(\delta\dot{x}^{k}+\Gamma_{nj}^{k}\dot{x}^{n}\delta x^{j})r_{k}, \tag{4.162}$$

which is equivalent to equalities (4.158), and also

$$\begin{aligned}\delta\frac{d^{2}r}{dt^{2}}&=(\ddot{x}^{k}+\Gamma_{mn}^{k}\dot{x}^{m}\dot{x}^{n})\delta r_{k}+\\ &\quad+r_{k}\left(\delta\ddot{x}^{k}+2\Gamma_{mn}^{k}\dot{x}^{m}\delta\dot{x}^{n}+\dot{x}^{m}\dot{x}^{n}\delta x^{\sigma}\frac{\partial}{\partial x^{\sigma}}\Gamma_{mn}^{k}\right)=\\ &=r_{k}\Big[\delta\ddot{x}^{k}+2\Gamma_{mn}^{k}\dot{x}^{m}\delta\dot{x}^{n}+\dot{x}^{m}\dot{x}^{n}\delta x^{\sigma}\frac{\partial}{\partial x^{\sigma}}\Gamma_{mn}^{k}+\\ &\quad+(\ddot{x}^{s}+\Gamma_{mn}^{s}\dot{x}^{m}\dot{x}^{n})\delta x^{\sigma}\Gamma_{s\sigma}^{k}\Big].\end{aligned} \tag{4.163}$$

Since the variations are isochronic, the operations of variation and differentiation should allow variation in the order of their performance. Therefore, the bracketed expressions in equalities (4.163) should be equal to the expressions in the right sides of relations (4.159). It is easy to show that this is in fact the case.

Indeed, from comparison of these expressions it may be shown that they are equal if the sums

$$\begin{aligned}&\dot{x}^{m}\dot{x}^{\sigma}\delta x^{n}\left(\frac{\partial}{\partial x^{\sigma}}\Gamma_{mn}^{k}+\Gamma_{mn}^{s}\Gamma_{s\sigma}^{k}\right)-\\ &\qquad-\dot{x}^{m}\dot{x}^{n}\delta x^{\sigma}\left(\frac{\partial}{\partial x^{n}}\Gamma_{m\sigma}^{k}+\Gamma_{m\sigma}^{s}\Gamma_{sn}^{k}\right)=\\ &=\dot{x}^{m}\dot{x}^{\sigma}\delta x^{n}\left[\frac{\partial}{\partial x^{\sigma}}\Gamma_{mn}^{k}+\Gamma_{mn}^{s}\Gamma_{s\sigma}^{k}-\frac{\partial}{\partial x^{n}}\Gamma_{m\sigma}^{k}-\Gamma_{m\sigma}^{s}\Gamma_{sn}^{k}\right].\end{aligned} \tag{4.164}$$

are equal to zero.

But the bracketed expressions on the right sides of equalities (4.164) are simply mixed components of a Riemann-Christoffel tensor, i.e., a tensor of spatial curvature defined by coordinates x^{s}:

$$R_{n\sigma m\cdot}^{\cdot\cdot\cdot k}=\frac{\partial}{\partial x^{\sigma}}\Gamma_{mn}^{k}+\Gamma_{mn}^{s}\Gamma_{s\sigma}^{k}-\frac{\partial}{\partial x^{n}}\Gamma_{m\sigma}^{k}-\Gamma_{m\sigma}^{s}\Gamma_{sn}^{k}. \tag{4.165}$$

This three-dimensional space defined by the coordinates x^s is Euclidean. It allows the use of a Cartesian coordinate system. Therefore, the Riemann-Christoffel tensor is identically equal to zero. Consequently,

$$R_{\kappa\alpha m \cdot}^{\cdot\cdot\cdot s} \equiv 0. \tag{4.166}$$

It follows from this that the left sides of equalities (4.164) are also identically equal to zero, which is what we wish to prove.

Considering relations (4.155) and (4.166), the first group of equations (4.118) may be simplified to take the form:

$$\begin{aligned} &\delta\ddot{x}^s + \frac{\mu}{r^3}\delta x^s - \frac{3\mu}{4r^5} a^{s\alpha}\frac{\partial r^2}{\partial x^\alpha}\frac{\partial r^2}{\partial x^\sigma}\delta x^\sigma + 2\Gamma_{mn}^{\cdot\cdot s}\dot{x}^m\delta\dot{x}^n + \\ &+ \left[\dot{x}^m\dot{x}^n\left(\frac{\partial}{\partial x^\sigma}\Gamma_{mn}^{\cdot\cdot s} + \Gamma_{mn}^{\cdot\cdot\alpha}\Gamma_{\alpha\sigma}^{\cdot\cdot s}\right) + \ddot{x}^\alpha\Gamma_{\alpha\sigma}^{\cdot\cdot s}\right]\delta x^\sigma = \Delta n^s. \end{aligned} \tag{4.167}$$

Using equalities (4.166), (4.165), (4.164) and (4.159), we arrive at the following form of equations (4.167):

$$q^s + \frac{\mu}{r^3}\delta x^s - \frac{3\mu}{4r^5}a^{s\eta}\frac{\partial r^2}{\partial x^\eta}\frac{\partial r^2}{\partial x^\sigma}\delta x^\sigma = \Delta n^s. \tag{4.168}$$

But in correspondence with relations (4.111) and (4.109),

$$\frac{\mu}{r^3}\delta x^s - \frac{3\mu}{4r^5}a^{s\eta}\frac{\partial r^2}{\partial x^\eta}\frac{\partial r^2}{\partial x^\sigma}\delta x^\sigma = \delta g^s. \tag{4.169}$$

Therefore, equalities (4.168) take the form:

$$q^s + \delta g^s = \Delta n^s, \tag{4.170}$$

and this is equivalent to the vector equality

$$q + \delta g = \Delta n \tag{4.171}$$

or the equality

$$\frac{d^2\delta r}{dt^2} + \delta g = \Delta n, \tag{4.172}$$

as required.

If the coordinates are non-stationary, then the left side of equation (4.118) may also be reduced to the form of (4.171) or (4.172). In order to demonstrate this, in place of formulas (4.158) and (4.159) the following formulas should be applied:

$$\left.\begin{aligned} p^s &= \delta\dot{x}^s + (\Gamma^s_{mn}\dot{x}^m + \Gamma^s_{0n})\delta x^n,\\ q^s &= \delta\ddot{x}^s + 2(\Gamma^s_{bj}\dot{x}^j + \Gamma^s_{0b})\delta\dot{x}^b +\\ &+ \Big[\Gamma^s_{mn}\ddot{x}^m + (\Gamma^s_{mb}\dot{x}^m + \Gamma^s_{0b})(\Gamma^b_{nj}\dot{x}^j + \Gamma^b_{0n}) +\\ &+ \frac{\partial}{\partial x^j}(\Gamma^s_{mn}\dot{x}^m + \Gamma^s_{0n})\dot{x}^j + \frac{\partial}{\partial t}(\Gamma^s_{mn}\dot{x}^m + \Gamma^s_{0n})\Big]\delta x^n \end{aligned}\right| \tag{4.173}$$

and, in addition to identities (4.164), the identities

$$\begin{aligned} &2\Gamma^k_{0s}\delta\dot{x}^s + \Big[2\dot{x}^s\Big(\frac{\partial}{\partial x^n}\Gamma^k_{0s} + \Gamma^j_{0n}\Gamma^k_{js}\Big) +\\ &\qquad + \Gamma^j_{00}\Gamma^k_{jn} + \frac{\partial}{\partial x^n}\Gamma^k_{00}\Big]\delta x^n \equiv\\ &\equiv 2\Gamma^k_{0s}\delta\dot{x}^s + \Big[\frac{\partial}{\partial x^n}\Gamma^k_{0s}\dot{x}^s + \frac{\partial}{\partial t}(\Gamma^k_{mn}\dot{x}^m + \Gamma^k_{0n}) +\\ &\qquad + \Gamma^k_{mj}\Gamma^j_{0n}\dot{x}^m + \Gamma^k_{0j}(\Gamma^j_{sn}\dot{x}^s + \Gamma^j_{0n})\Big]\delta x^n. \end{aligned} \tag{4.174}$$

should be used, the validity of which follows from the relations

$$\begin{aligned} \frac{\partial}{\partial x^n}\Gamma^k_{0s} &= \frac{\partial}{\partial t}\Gamma^k_{ns} - \Gamma^j_{0n}\Gamma^k_{js} + \Gamma^j_{sn}\Gamma^k_{0j},\\ \frac{\partial}{\partial x^n}\Gamma^k_{00} &= \frac{\partial}{\partial t}\Gamma^k_{0n} + \Gamma^j_{0n}\Gamma^k_{0j} - \Gamma^j_{00}\Gamma^k_{jn}. \end{aligned}$$

Considering that according to equalities (4.197) the terms on the right sides of the first three equations (4.118) containing Δm^n are obtained by expanding the expressions

$$\left(2\Delta m \times \frac{dr}{dt} - \frac{d\,\Delta m}{dt} \times r\right) \cdot r^s, \tag{4.175}$$

we concluded that the first group of equations (4.118) is equivalent to the vector equation

$$\frac{d^2\delta r}{dt^2} + \delta g = \Delta n - 2\,\Delta m \times \frac{dr}{dt} + \frac{d\,\Delta m}{dt} \times r. \tag{4.176}$$

Similarly, the second group of equations (4.118) reduces to the vector equations

$$\frac{d\theta_1}{dt} = \Delta m.$$

(4.177)

Finally, the third and fourth groups of equations (4.118) reduces to the vector equalities

$$\delta r_1 = \theta_1 \times r, \quad \delta r_2 = \delta r + \delta r_1. \tag{4.178}$$

It is evident that equations (4.176) -- (4.178) and (4.81) coincide. In fact, in equations (4.176) -- (4.178) the differentiation is absolute, i.e., carried out in the basic Cartesian coordinate system $O_1\xi^1\xi^2\xi^3$. The differentiation in the first two equations (4.81) was carried out in the coordinate system O_1xyz, which rotates relative to the reference coordinate system $O_1\xi^1\xi^2\xi^3$ with an an angular velocity $\vec{\omega}$. It is evident that the first and second equations (4.81) are the same as equations (4.176) and (4.177), expressed in terms of projections on the axes of the trihedron O_1xyz.

Projecting the vector equalities (4.176) -- (4.178) onto the axes of the basic Cartesian coordinate system and recalling that

$$g = \operatorname{grad}\frac{\mu}{r}, \tag{4.179}$$

we obtain:

$$\left.\begin{aligned}
&\delta\ddot{\xi}^1 + \frac{\mu}{r^5}\left[(\xi^2)^2 + (\xi^3)^2 - 2(\xi^1)^2\right]\delta\xi^1 - \\
&\qquad - \frac{3\mu\xi^1\xi^2}{r^5}\delta\xi^2 - \frac{3\mu\xi^1\xi^3}{r^5}\delta\xi^3 = \\
&= \Delta n_{\xi^1} - 2\left(\Delta m_{\xi^2}\dot{\xi}^3 - \Delta m_{\xi^3}\dot{\xi}^2\right) - \Delta\dot{m}_{\xi^2}\xi^3 + \Delta\dot{m}_{\xi^3}\xi^2, \\
&\delta\ddot{\xi}^2 + \frac{\mu}{r^5}\left[(\xi^3)^2 + (\xi^1)^2 - 2(\xi^2)^2\right]\delta\xi^2 - \\
&\qquad - \frac{3\mu\xi^2\xi^3}{r^5}\delta\xi^3 - \frac{3\mu\xi^2\xi^1}{r^5}\delta\xi^1 = \\
&= \Delta n_{\xi^2} - 2\left(\Delta m_{\xi^3}\dot{\xi}^1 - \Delta m_{\xi^1}\dot{\xi}^3\right) - \Delta\dot{m}_{\xi^3}\xi^1 + \Delta\dot{m}_{\xi^1}\xi^3, \\
&\delta\ddot{\xi}^3 + \frac{\mu}{r^5}\left[(\xi^1)^2 + (\xi^2)^2 - 2(\xi^3)^2\right]\delta\xi^3 - \\
&\qquad - \frac{3\mu\xi^3\xi^1}{r^5}\delta\xi^1 - \frac{3\mu\xi^3\xi^2}{r^5}\delta\xi^2 = \\
&= \Delta n_{\xi^3} - 2\left(\Delta m_{\xi^1}\dot{\xi}^2 - \Delta m_{\xi^2}\dot{\xi}^1\right) - \Delta\dot{m}_{\xi^1}\xi^2 + \Delta\dot{m}_{\xi^2}\xi^1;
\end{aligned}\right\} \tag{4.180}$$

$$\left.\begin{array}{c}\dot{\theta}_{1\xi^1}=\Delta m_{\xi^1},\quad \dot{\theta}_{1\xi^2}=\Delta m_{\xi^2},\\ \dot{\theta}_{1\xi^3}=\Delta m_{\xi^3};\end{array}\right\} \tag{4.181}$$

$$\left.\begin{array}{c}\delta\xi^1_1=\theta_{1\xi^2}\xi^3-\theta_{1\xi^3}\xi^2,\quad \delta\xi^2_1=\theta_{1\xi^3}\xi^1-\theta_{1\xi^1}\xi^3,\\ \delta\xi^3_1=\theta_{1\xi^1}\xi^2-\theta_{1\xi^2}\xi^1,\\ \delta\xi^1_3=\Delta\xi^1+\delta\xi^1_1,\quad \delta\xi^2_3=\delta\xi^2+\delta\xi^2_1,\\ \delta\xi^3_3=\delta\xi^3+\delta\xi^3_1.\end{array}\right\} \tag{4.182}$$

Equations (4.180) -- (4.182) may also be derived from equations (4.83) -- (4.85), if in the latter it is assumed that

$$\omega=0 \tag{4.183}$$

and the correspondence of the coordinates ξ^1, ξ^2, ξ^3 in equations (4.180) -- (4.182) to the coordinates x, y, z in equations (4.83) -- (4.85) is taken into account. Of course, equations (4.180) -- (4.182) may also be obtained directly from equations (4.118). In order to do this, the Cartesian coordinates ξ^1, ξ^2, ξ^3 in the basic Cartesian system should be taken as the coordinates x^s. In this case, in equations (4.118) the symbols Γ^k_{mn}, Γ^k_{0n}, Γ^k_{00} and the quantities a^ℓ_0 vanish, the non-diagonal elements of the metric tensor become zero, the diagonal elements become one, and the Levi-Civita symbols are ± 1 as a function of the order of their indices. Considering all of this, and also the equality

$$\frac{dr^2}{dx^s}=2x^s, \tag{4.184}$$

we immedaitely obtain equations (4.180) -- (4.182).

Thus, analysis of equations (4.118) may be reduced to analysis of the system of vector equations (4.81) or the scaler equations (4.83) -- (4.85) corresponding to this system.

4.4.2. <u>The relation between errors in Cartesian and curvilinear coordinates. Conversion of initial conditions and instrument errors. Examples</u>. Varying equalities (3.89) we arrive at the following equations relating $\delta\xi^s$ to $\delta\varkappa^k$:

$$\delta\xi^s = \frac{\partial\xi^s}{\partial\varkappa^k}\delta\varkappa^k. \tag{4.185}$$

Using the table (3.16) of the direction cosines between the ξ^1, ξ^2, ξ^3 and x, y, z axes, we find

$$\left.\begin{aligned} \delta x = \frac{\partial\xi^s}{\partial\varkappa^k}\delta\varkappa^k\alpha_{s1}, \quad \delta y = \frac{\partial\xi^s}{\partial\varkappa^k}\delta\varkappa^k\alpha_{s2}, \\ \delta z = \frac{\partial\xi^s}{\partial\varkappa^k}\delta\varkappa^k\alpha_{s3}. \end{aligned}\right\} \tag{4.186}$$

In order to find $\delta\dot{x}$, $\delta\dot{y}$, $\delta\dot{z}$ we differentiate equalities (4.186):

$$\left.\begin{aligned} \delta\dot{x} &= \frac{\partial\xi^s}{\partial\varkappa^k}\alpha_{s1}\delta\dot{\varkappa}^k + \\ &+ \left(\frac{\partial^2\xi^s}{\partial\varkappa^k\partial\varkappa^n}\dot{\varkappa}^n + \frac{\partial^2\xi^s}{\partial\varkappa^k\partial t}\right)\delta\varkappa^k\alpha_{s1} + \frac{\partial\xi^s}{\partial\varkappa^k}\delta\varkappa^k\dot{\alpha}_{s1}, \\ \delta\dot{y} &= \frac{\partial\xi^s}{\partial\varkappa^k}\alpha_{s2}\delta\dot{\varkappa}^k + \\ &+ \left(\frac{\partial^2\xi^s}{\partial\varkappa^k\partial\varkappa^n}\dot{\varkappa}^n + \frac{\partial^2\xi^s}{\partial\varkappa^k\partial t}\right)\delta\varkappa^k\alpha_{s2} + \frac{\partial\xi^s}{\partial\varkappa^k}\delta\varkappa^k\dot{\alpha}_{s2}, \\ \delta\dot{z} &= \frac{\partial\xi^s}{\partial\varkappa^k}\alpha_{s3}\delta\dot{\varkappa}^k + \\ &+ \left(\frac{\partial^2\xi^s}{\partial\varkappa^k\partial\varkappa^n}\dot{\varkappa}^n + \frac{\partial^2\xi^s}{\partial\varkappa^k\partial t}\right)\delta\varkappa^k\alpha_{s3} + \frac{\partial\xi^s}{\partial\varkappa^k}\delta\varkappa^k\dot{\alpha}_{s3}. \end{aligned}\right\} \tag{4.187}$$

Here the direction cosines α_{ij} are defined by table (3.16) and equalities (3.21) -- (3.23).

The relation between the θ_1^s components and the projections θ_{1x}, θ_{1y}, θ_{1z} is given by formulas analogous to formulas (4.186):

$$\left.\begin{aligned} \theta_{1x} = \frac{\partial\xi^s}{\partial\varkappa^k}\theta_1^k\alpha_{s1}, \quad \theta_{1y} = \frac{\partial\xi^s}{\partial\varkappa^k}\theta_1^k\alpha_{s2}, \\ \theta_{1z} = \frac{\partial\xi^s}{\partial\varkappa^k}\theta_1^k\alpha_{s3}. \end{aligned}\right\} \tag{4.188}$$

Formulas (4.18), (4.187) and (4.188) permit us to find the initial values $\delta x(0)$, $\delta y(0)$, $\delta z(0)$; $\delta\dot{x}(0)$, $\delta\dot{y}(0)$, $\delta\dot{z}(0)$; $\theta_{1x}(0)$, $\theta_{1y}(0)$, $\theta_{1z}(0)$ in terms of the known initial errors $\varkappa^s(0)$ in the $\varkappa^s$ coordinates, their derivatives $\dot{\varkappa}^s(0)$, and the initial errors $\theta_1^k(0)$.

In addition to converting the initial conditions in making the transition from equations (4.118) to (4.83) -- (4.85), we also need to convert the basic instrument errors. In equations (4.83) -- (4.85) the quantities Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z are the projections of vectors $\Delta\vec{n}$ and $\Delta\vec{m}$ on the axes of the xyz trihedron. The vectors $\Delta\vec{n}$ and $\Delta\vec{m}$ are given by their projections on the sensing axes of the newtonometers and the elements measuring absolute angular velocity. This means that we must obtain formulas for calculating Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z, in terms of the known Δn_{e_s}, Δm_{ξ^s} or $m_{(s)}$, which are related to Δn^s and Δm^s by relations (4.120), (4.121), (4.133).

In accordance with equalities (4.120) we have:

$$\Delta n_{\xi^s} = \Delta n^k r_k \cdot \xi_s = \sqrt{a^{kk}}\,\Delta n_{e_k} r_k \cdot \xi_s. \tag{4.189}$$

From (3.88) and (3.89) it follows that

$$r_k = \xi_s \frac{\partial \xi^s}{\partial x^k}. \tag{4.190}$$

Therefore,

$$\Delta n_{\xi^s} = \sqrt{a^{kk}}\,\Delta n_{e_k} \frac{\partial \xi^s}{\partial x^k}, \tag{4.191}$$

where the summation is carried out over all k.

If we now use table (3.16) of the direction cosines, Δn_x, Δn_y, Δn_z take the form

$$\left.\begin{aligned} \Delta n_x &= \sqrt{a^{kk}}\,\Delta n_{e_k} \frac{\partial \xi^s}{\partial x^k}\alpha_{s1}, \\ \Delta n_y &= \sqrt{a^{kk}}\,\Delta n_{e_k} \frac{\partial \xi^s}{\partial x^k}\alpha_{s2}, \\ \Delta n_z &= \sqrt{a^{kk}}\,\Delta n_{e_k} \frac{\partial \xi^s}{\partial x^k}\alpha_{s3}. \end{aligned}\right\} \tag{4.192}$$

where the summation extends, clearly, over the indices k and s.

Correspondingly,

$$\Delta m_x = \Delta m_\xi a_{s1}, \quad \Delta m_y = \Delta m_\xi a_{s2}, \quad \Delta m_z = \Delta m_\xi a_{s1}. \tag{4.193}$$

Formulas (4.193) give the values of Δm_x, Δm_y, and Δm_z for the case in which the basic inertial navigation system is a free gyro-stablized platform. If the basic kinematic system is a maneuverable platform, and the x, y, z axes coincide with its axes, then the vector $\Delta\vec{m}$ will be given directly by its projections Δm_x, Δm_y, Δm_z. It must be remembered, however, that selection of the rotating coordinate system O_1xyz in which analysis of the error equations (4.83) -- (4.85) is being performed, is not, in general, a function of the orientation of the newtonometer or platform axes. In particular, if the x, y, z axes do not coincide with the axes of the maneuverable platform, the projections Δm_x, Δm_y, Δm_z, the values of which are substituted into equations (4.83) -- (4.85), should be calculated on the basis of the given projections of the vector $\Delta\vec{m}$ on the axes of the maneuverable platform. For this, it is necessary, obviously, to know the position of the platform relative to the x, y, z axes, in terms of the projections on which equations (4.83) -- (4.85) were derived.

Since the goal of the analysis of the error equation is, basically, to study the variation over time of the quantity $|\delta\vec{r}_3|$ between parallel lines as a function of errors in the initial conditions and instrument errors, the inverse transition from the variations δx, δy, δz, to $\delta\varkappa^s$ is, in general, not obligatory. But this transition may be useful in the experimental investigation of an inertial system, since the parameters which may be directly measured in such an experiment will be the errors in the coordinates determined by the system, i.e., the variations of $\delta\varkappa^s$, $\delta\varkappa^s_1$, $\delta\varkappa^s_3$.

In order to effect the reverse transition from δx, δy, δz to $\delta \varkappa^k$, we will use equalities (4.185). They may be regarded as a system of linear algebraic equations in $\delta \varkappa^s$:

$$\left.\begin{aligned}
&\frac{\partial \xi^1}{\partial \varkappa^1}\delta\varkappa^1 + \frac{\partial \xi^1}{\partial \varkappa^2}\delta\varkappa^2 + \frac{\partial \xi^1}{\partial \varkappa^3}\delta\varkappa^3 = \delta\xi^1,\\
&\frac{\partial \xi^2}{\partial \varkappa^1}\delta\varkappa^1 + \frac{\partial \xi^2}{\partial \varkappa^2}\delta\varkappa^2 + \frac{\partial \xi^2}{\partial \varkappa^3}\delta\varkappa^3 = \delta\xi^2,\\
&\frac{\partial \xi^3}{\partial \varkappa^1}\delta\varkappa^1 + \frac{\partial \xi^3}{\partial \varkappa^2}\delta\varkappa^2 + \frac{\partial \xi^3}{\partial \varkappa^3}\delta\varkappa^3 = \delta\xi^3.
\end{aligned}\right| \tag{4.194}$$

The determinant of system (4.194) is the Jacobian J, which according to condition (3.92) is different from zero over the entire range of the operational values of the variables $\varkappa^s$. Therefore, system (4.194) has the unique solution:

$$\left.\begin{aligned}
\delta\varkappa^1 = \frac{1}{J}\Big[&\delta\xi^1\left(\frac{\partial \xi^2}{\partial \varkappa^2}\frac{\partial \xi^3}{\partial \varkappa^3} - \frac{\partial \xi^3}{\partial \varkappa^2}\frac{\partial \xi^2}{\partial \varkappa^3}\right) +\\
&+ \delta\xi^2\left(\frac{\partial \xi^3}{\partial \varkappa^2}\frac{\partial \xi^1}{\partial \varkappa^3} - \frac{\partial \xi^1}{\partial \varkappa^2}\frac{\partial \xi^3}{\partial \varkappa^3}\right) +\\
&+ \delta\xi^3\left(\frac{\partial \xi^1}{\partial \varkappa^2}\frac{\partial \xi^2}{\partial \varkappa^3} - \frac{\partial \xi^2}{\partial \varkappa^2}\frac{\partial \xi^1}{\partial \varkappa^3}\right)\Big],\\
\delta\varkappa^2 = \frac{1}{J}\Big[&\delta\xi^1\left(\frac{\partial \xi^3}{\partial \varkappa^1}\frac{\partial \xi^2}{\partial \varkappa^3} - \frac{\partial \xi^2}{\partial \varkappa^1}\frac{\partial \xi^3}{\partial \varkappa^3}\right) +\\
&+ \delta\xi^2\left(\frac{\partial \xi^1}{\partial \varkappa^1}\frac{\partial \xi^3}{\partial \varkappa^3} - \frac{\partial \xi^3}{\partial \varkappa^1}\frac{\partial \xi^1}{\partial \varkappa^3}\right) +\\
&+ \delta\xi^3\left(\frac{\partial \xi^2}{\partial \varkappa^1}\frac{\partial \xi^1}{\partial \varkappa^3} - \frac{\partial \xi^1}{\partial \varkappa^1}\frac{\partial \xi^2}{\partial \varkappa^3}\right)\Big],\\
\delta\varkappa^3 = \frac{1}{J}\Big[&\delta\xi^1\left(\frac{\partial \xi^2}{\partial \varkappa^1}\frac{\partial \xi^3}{\partial \varkappa^2} - \frac{\partial \xi^2}{\partial \varkappa^2}\frac{\partial \xi^3}{\partial \varkappa^1}\right) +\\
&+ \delta\xi^2\left(\frac{\partial \xi^3}{\partial \varkappa^1}\frac{\partial \xi^1}{\partial \varkappa^2} - \frac{\partial \xi^1}{\partial \varkappa^1}\frac{\partial \xi^3}{\partial \varkappa^2}\right) +\\
&+ \delta\xi^3\left(\frac{\partial \xi^1}{\partial \varkappa^1}\frac{\partial \xi^2}{\partial \varkappa^2} - \frac{\partial \xi^2}{\partial \varkappa^1}\frac{\partial \xi^1}{\partial \varkappa^2}\right)\Big].
\end{aligned}\right| \tag{4.195}$$

In equalities (4.195) the quantities $\delta\xi^s$ should be expressed in terms of δx, δy, δz and α_{ij}. According to table (3.16),

$$\delta\xi^r = \delta x\,\alpha_{r1} + \delta y\,\alpha_{r2} + \delta z\,\alpha_{r3}. \tag{4.196}$$

Henceforth, we will require the explicit expressions deriving from relations (4.186), (4.187), (4.188), (4.189), (4.192), (4.193), (4.195), (4.196), for the curvilinear reference grids considered in §3.3. Let us determine the corresponding relations.

For the first of the reference grids considered in §3.3, the quantities ξ^s were defined in terms of r, φ, λ_1 in formulas (3.246), and their coordinate derivatives and the covariant components of the metric tensor in formulas (3.247) and (3.250). We will assume that the axes of the trihedron O_1xyz coincide with the vectors $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_1$. Then

$$\alpha_{i1} = \frac{\partial \xi^i}{\partial x^2} \frac{1}{\sqrt{a_{22}}}, \quad \alpha_{i2} = \frac{\partial \xi^i}{\partial x^3} \frac{1}{\sqrt{a_{33}}}, \quad \alpha_{i3} = \frac{\partial \xi^i}{\partial x^1} \frac{1}{\sqrt{a_{11}}}. \tag{4.197}$$

From relations (4.192), (4.197), (3.247) and (3.250) we further find:

$$\Delta n_x = \Delta n_{e_2}, \quad \Delta n_y = \Delta n_{e_3}, \quad \Delta n_z = \Delta n_{e_1}. \tag{4.198}$$

From relations (4.193):

$$\left.\begin{aligned} \Delta m_x &= -\Delta m_{\xi^1} \sin\lambda_1 + \Delta m_{\xi^2} \cos\lambda_1, \\ \Delta m_y &= -\Delta m_{\xi^1} \sin\varphi \cos\lambda_1 - \Delta m_{\xi^2} \sin\varphi \sin\lambda_1 + \\ &\qquad + \Delta m_{\xi^3} \cos\varphi, \\ \Delta m_z &= \Delta m_{\xi^1} \cos\varphi \cos\lambda_1 + \Delta m_{\xi^2} \cos\varphi \sin\lambda_1 + \\ &\qquad + \Delta m_{\xi^3} \sin\varphi. \end{aligned}\right\} \tag{4.199}$$

From equalities (4.186):

$$\delta x = r \cos\varphi\, \delta\lambda_1, \quad \delta y = r\, \delta\varphi, \quad \delta z = \delta r. \tag{4.200}$$

From expressions (4.187):

$$\left.\begin{aligned} \delta\dot{x} &= \dot{r} \cos\varphi\, \delta\lambda_1 + r \cos\varphi\, \delta\dot{\lambda}_1 - r\dot{\varphi} \sin\varphi\, \delta\lambda_1, \\ \delta\dot{y} &= \dot{r}\, \delta\varphi + r\, \delta\dot{\varphi}, \quad \delta\dot{z} = \delta\dot{r}. \end{aligned}\right\} \tag{4.201}$$

The substitution of relations (4.197), (3.247) and (3.250) in formulas (4.188) gives:

$$\theta_{1x} = \theta_1^1, \quad \theta_{1y} = \frac{\theta_1^2}{r\cos\varphi}, \quad \theta_{1z} = \frac{\theta_1^3}{r}. \tag{4.202}$$

Finally, from (4.195), (4.196), (4.197), (3.247), (3.248) and (3.250), we find

$$\delta r = \Delta z, \quad \delta\lambda_1 = \frac{\Delta x}{r\cos\varphi}, \quad \delta\varphi = \frac{\Delta y}{r}. \tag{4.203}$$

We note that formulas (3.203) and (4.201) may be obtained immediately from formulas (4.200).

For the second of the reference grids, i.e., for the geocentric coordinates r, λ, φ, formulas (4.198), (4.199), (4.200), (4.201), (4.202) and (4.203) retain their form, with the exception of the substitution of $\delta\lambda$ for $\delta\lambda_1$. In formulas (4.199) the only change required is the substitution of $\lambda + ut$ for λ_1.

Let us now consider the case of the geodetic coordinates r, z, S. The relation between the quantities ξ^s and these coordinates is given by formulas (3.280). The time and coordinate derivatives of ξ^s are determined by formulas (3.283) and the components of the metric tensor by equalities (3.282). Using these formulas, from (4.192) we find:

$$\Delta n_x = \Delta n_{e_2}, \quad \Delta n_y = \Delta n_{e_3}, \quad \Delta n_z = \Delta n_{e_1}. \tag{4.204}$$

These equalities are obvious as a result of the fact that the quantities

$$\Delta n_x \text{ and } \Delta n_{e_2}, \quad \Delta n_y \text{ and } \Delta n_{e_3}, \quad \Delta n_z \text{ and } \Delta n_{e_1}$$

are projections of the vector $\Delta\vec{n}$ on the same direction, since the x, y, z axes and the vectors $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_1$ (and therefore also the vectors $\vec{e}_2$, $\vec{e}_3$, $\vec{e}_1$) coincide.

From expressions (4.193), and using table (3.300), we obtain:

$$\left.\begin{array}{c} \Delta m_x = \Delta m_{\xi^1}\gamma_{21}, \quad \Delta m_y = \Delta m_{\xi^1}\gamma_{31}, \\ \Delta m_z = \Delta m_{\xi^1}\gamma_{11}. \end{array}\right\} \tag{4.205}$$

From (4.186), (4.187), (4.188) and (4.195) formulas corresponding to formulas (4.200), (4.201), (4.202) and (4.203) may be derived, if z and S are substituted for φ and λ_1 in the latter.

For the geographical coordinates h, λ, φ the Cartesian coordinates ξ^s in the basic Cartesian system are given by equalities (3.309) -- (3.311), from which formulas (3.312) and (3.313) for a_{kk} and $\partial\xi^s/\partial x^k$ derive. In the case of a geographical reference grid, which, like the preceding ones, is orthogonal, equalities (4.198) and (4.204) remain valid.

From formulas (4.193) and the table direction cosines obtained from (4.197), (3.311) -- (3.313), it follows for this case that

$$\left.\begin{aligned} \Delta m_x &= -\Delta m_{\xi_1}\sin(\lambda+ut)+\Delta m_{\xi_2}\cos(\lambda+ut),\\ \Delta m_y &= -\Delta m_{\xi_1}\sin\varphi'\cos(\lambda+ut)-\\ &\quad -\Delta m_{\xi_2}\sin\varphi'\sin(\lambda+ut)+\Delta m_{\xi_3}\cos\varphi';\\ \Delta m_z &= \Delta m_{\xi_1}\cos\varphi'\cos(\lambda+ut)+\\ &\quad +\Delta m_{\xi_2}\cos\varphi'\sin(\lambda+ut)+\Delta m_{\xi_3}\sin\varphi'. \end{aligned}\right\} \qquad (4.206)$$

Expressions (4.206) are analogous to expressions (4.199).

Further, from relations (4.186), (4.197), (3.313), (3.312) and (3.311), we obtain:

$$\left.\begin{aligned} \delta x &= \delta\lambda\cos\varphi'\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right),\\ \delta y &= \delta\varphi'\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right], \quad \delta z=\delta h. \end{aligned}\right\} \qquad (4.207)$$

From equalities (4.187) or by direct differentiation of relations (4.207) we find:

$$\left.\begin{aligned} \delta\dot{x} &= \delta\dot{\lambda}\cos\varphi'\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)+\\ &+\delta\lambda\left\{\dot{h}\cos\varphi-\dot{\varphi}'\sin\varphi'\left[h+\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}\right]\right\},\\ \delta\dot{y} &= \delta\dot{\varphi}'\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right]+\\ &\quad +\delta\varphi'\left[\dot{h}+\frac{3a(1-e^2)e^2\dot{\varphi}'\sin\varphi'\cos\varphi'}{(1-e^2\sin^2\varphi')^{5/2}}\right],\\ \delta\dot{z} &= \delta\dot{h}. \end{aligned}\right\} \qquad (4.208)$$

From formulas (4.195) -- (4.197) and (3.311) -- (3.313) direct transformation of equalities (4.207), we obtain:

$$\left.\begin{aligned} \delta h &= \delta z, \\ \delta\lambda &= \frac{\Delta x}{\cos\varphi'}\left(\frac{a}{\sqrt{1 - e^2\sin^2\varphi'}} + h\right)^{-1}, \\ \delta\varphi' &= \Delta y\left[\frac{a(1 - e^2)}{(1 - e^2\sin^2\varphi')^{3/2}} + h\right]^{-1}. \end{aligned}\right| \quad (4.209)$$

Finally, we obtain the formulas deriving from relations (4.192), (4.193), (4.186), (4.187), (4.188), (4.195), and (4.196) for the coordinates r, σ_1, σ_2, the relations of which with ξ^s is given by equalities (3.334).

The reference grid r, σ_1, σ_2 is stationary, but not orthogonal. This latter circumstance makes it less convenient than in the preceding instances to reduce the error equations to the coordinate system O_1xyz; this is due to the fact that previously it was possible to superpose its axes with the vectors $\vec{r}_2$, $\vec{r}_3$, $\vec{r}_1$ (or equivalently for orthgonal coordinates, with the vectors $\vec{r}^1$, $\vec{r}^2$, $\vec{r}^3$). For coordinates r, σ_1, σ_2, we will therefore begin by considering the case of arbitrary direction cosines α_{ij}.

From expressions (4.192), (3.335), (3.341) and table (3.16), it follows that:

$$\left.\begin{aligned} \Delta n_x = {} & \Delta n_{e_r}\left(\alpha_{11}\cos\sigma_1 + \alpha_{21}\cos\sigma_2 + \alpha_{31}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}\right) + \\ & + \Delta n_{e_1}\left(-\alpha_{11}\sin\sigma_1 + \alpha_{31}\frac{\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right) + \\ & + \Delta n_{e_2}\left(-\alpha_{21}\sin\sigma_2 + \alpha_{31}\frac{\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right), \\ \Delta n_y = {} & \Delta n_{e_r}\left(\alpha_{12}\cos\sigma_1 + \alpha_{22}\cos\sigma_2 + \alpha_{32}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}\right) + \\ & + \Delta n_{e_1}\left(-\alpha_{12}\sin\sigma_1 + \alpha_{32}\frac{\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right) + \\ & + \Delta n_{e_2}\left(-\alpha_{22}\sin\alpha_2 + \alpha_{32}\frac{\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right), \\ \Delta n_z = {} & \Delta n_{e_r}\left(\alpha_{13}\cos\sigma_1 + \alpha_{23}\cos\sigma_2 + \alpha_{33}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}\right) + \\ & + \Delta n_{e_1}\left(-\alpha_{13}\sin\sigma_1 + \alpha_{33}\frac{\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right) + \\ & + \Delta n_{e_2}\left(-\alpha_{23}\sin\sigma_2 + \alpha_{33}\frac{\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right). \end{aligned}\right| \quad (4.210)$$

Formulas (4.193), clearly, do not vary as long as specific form of the direction cosines α_{ij} is given.

By analogy with equalities (4.210), from relations (4.186) we find:

$$
\left.\begin{aligned}
\delta x &= \delta r\left(\alpha_{11}\cos\sigma_1 + \alpha_{21}\cos\sigma_2 + \alpha_{31}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}\right) + \\
&\quad + r\delta\sigma_1\left(-\alpha_{11}\sin\sigma_1 + \alpha_{31}\frac{\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right) + \\
&\quad + r\delta\sigma_2\left(-\alpha_{21}\sin\sigma_2 + \alpha_{31}\frac{\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right), \\
\delta y &= \delta r\left(\alpha_{12}\cos\sigma_1 + \alpha_{22}\cos\sigma_2 + \alpha_{32}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}\right) + \\
&\quad + r\delta\sigma_1\left(-\alpha_{12}\sin\sigma_1 + \alpha_{32}\frac{\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right) + \\
&\quad + r\delta\sigma_2\left(-\alpha_{22}\sin\sigma_2 + \alpha_{32}\frac{\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right), \\
\delta z &= \delta r\left(\alpha_{13}\cos\sigma_1 + \alpha_{23}\cos\sigma_2 + \alpha_{33}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}\right) + \\
&\quad + r\delta\sigma_1\left(-\alpha_{13}\sin\sigma_1 + \alpha_{33}\frac{\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right) + \\
&\quad + r\delta\sigma_2\left(-\alpha_{23}\sin\sigma_2 + \alpha_{33}\frac{\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}\right).
\end{aligned}\right| \tag{4.211}
$$

From expressions (4.187) or the direct differentiation of relations (4.211) it is now easy to find $\delta\dot{x}$, $\delta\dot{y}$, $\delta\dot{z}$.

The formulas obtained from relations (4.188) for θ_{1x}, θ_{1y}, θ_{1z}, are fully analogous to formulas (4.211) for δx, δy, δz. It is necessary to substitute in the latter the quantitites θ_1^1, θ_1^2, θ_1^3 for δr, $\delta\sigma_1$, $\delta\sigma_2$.

Let us find, finally, explicit expressions for δr, $\delta\sigma_1$ and $\delta\sigma_2$ in terms of δx, δy, δz. Direct transformation of formulas (4.211) is not obvious, and so we will use the general formulas (4.195) and (4.196). Using relations (3.335) and (3.336), from equalities (4.195) we find:

$$
\left.\begin{aligned}
\delta r &= \delta\xi^1\cos\sigma_1 + \delta\xi^2\cos\sigma_2 + \delta\xi^3\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}, \\
\delta\sigma_1 &= \frac{1}{r}\left(-\delta\xi^1\sin\sigma_1 + \delta\xi^2\cos\sigma_2\cot\sigma_1 + \right. \\
&\qquad \left. + \delta\xi^3\cot\sigma_1\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}\right), \\
\delta\sigma_2 &= \frac{1}{r}\left(\delta\xi^1\cos\sigma_1\cot\sigma_2 - \delta\xi^2\sin\sigma_2 + \right. \\
&\qquad \left. + \delta\xi^3\cot\sigma_2\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}\right).
\end{aligned}\right| \tag{4.212}
$$

where $\delta\xi^s$ are determined by equalities (4.196).

Let us now take the basic trihedron $O_1\xi^1\xi^2\xi^3$ as trihedron O_1xyz. Then, clearly, of the coefficients α_{ij} only the terms of the main diagonal of table (3.16) are different from zero (and equal to one). Therefore, in place of formulas (4.210) we obtain:

$$
\left.\begin{aligned}
&\Delta n_x = \Delta n_{e_1}\cos\sigma_1 - \Delta n_{e_2}\sin\sigma_1,\\
&\Delta n_y = \Delta n_{e_1}\cos\sigma_2 - \Delta n_{e_3}\sin\sigma_2,\\
&\Delta n_z = \Delta n_{e_1}\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2} +\\
&+\Delta n_{e_2}\frac{\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}} + \Delta n_{e_3}\frac{\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}}.
\end{aligned}\right\} \tag{4.213}
$$

Correspondingly, in place of equalities (4.211) we will have:

$$
\left.\begin{aligned}
&\delta x = \delta r\cos\sigma_1 - r\delta\sigma_1\sin\sigma_1,\\
&\delta y = \delta r\cos\sigma_2 - r\delta\sigma_2\sin\sigma_2,\\
&\delta z = \delta r\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2} +\\
&+\frac{r\delta\sigma_1\sin\sigma_1\cos\sigma_1}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}} + \frac{r\delta\sigma_2\sin\sigma_2\cos\sigma_2}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}}.
\end{aligned}\right\} \tag{4.214}
$$

From equalities (4.196)

$$
\delta\xi^1 = \delta x,\quad \delta\xi^2 = \delta y,\quad \delta\xi^3 = \delta z. \tag{4.215}
$$

Substituting equalities (4.215) into formulas (4.212), we obtain expressions for δr, $\delta\sigma_1$, $\delta\sigma_2$ in terms of δx, δy, δz.

For the case under consideration we may select the trihedron O_1xyz for the reduction of the error equations in a somewhat different manner, relating it to the basis vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$. For example, using the fact that the vectors $\vec{r}_1$, $\vec{r}_3$ and $\vec{r}_1$, $\vec{r}_2$ are pairwise perpendicular, we may superpose the x and y axes of the reduction trihedron on the directions $\vec{r}_1$, $\vec{r}_2$ (or the x and z axes on the directions $\vec{r}_1$, $\vec{r}_3$). In this case only the z axis of the trihedron will not be incident with the third vector $\vec{r}_3$ of the fundamental basis, but it will coincide with the vector r^3, the reciprocal of r_1 and r_2.

Let us find the direction cosines α_{ij} of the x, y, z axes relative to the ξ^1, ξ^2, ξ^3 axes for this ease. Using equalities (3.337), the first and second equalities (3.351) and the third line of table (3.353a), we obtain:

$$
\left.\begin{array}{cccc}
 & x & y & z \\
\xi^1 & \cos\sigma_1 & -\dfrac{1}{\sin\sigma_2}\sqrt{\sin^2\sigma_1-\cos^2\sigma_2} & \cos\sigma_1\cot\sigma_2 \\
\xi^2 & \cos\sigma_2 & 0 & -\sin\sigma_2 \\
\xi^3 & \sqrt{\sin^2\sigma_1-\cos^2\sigma_2} & \dfrac{\cos\sigma_1}{\sin\sigma_2} & \cot\sigma_2\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}.
\end{array}\right|
$$

(4.216)

It is easy to see that table (4.216) is orthogonal.

Let us substitute the derived values of α_{ij} into relations (4.210). After grouping and simplifying, we will have:

$$
\left.\begin{array}{l}
\Delta n_x = \Delta n_{e_1}, \\
\Delta n_y = \Delta n_{e_1}\dfrac{\sin\sigma_1\sin\sigma_2}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}} + \Delta n_{e_2}\dfrac{\cos\sigma_1\cos\sigma_2}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}}, \\
\Delta n_z = \Delta n_{e_2}.
\end{array}\right\}
$$

(4.217)

Analogously, substituting the values of α_{ij} into equalities (4.211), we obtain:

$$
\left.\begin{array}{l}
\delta x = \delta r, \\
\delta y = \dfrac{r}{\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}}(\delta\sigma_1\sin\sigma_1\sin\sigma_2 + \delta\sigma_2\cos\sigma_1\cos\sigma_2), \\
\delta z = r\delta\sigma_2.
\end{array}\right\}
$$

(4.218)

Using formulas (4.196) and table (4.216) we find:

$$
\left.\begin{array}{l}
\delta\xi_0^1 = \delta x\cos\sigma_1 - \dfrac{\delta y}{\sin\sigma_2}\sqrt{\sin^2\sigma_1-\cos^2\sigma_2} + \delta z\cos\sigma_1\cot\sigma_2, \\
\delta\xi_0^2 = \delta x\cos\sigma_2 - \delta z\sin\sigma_2, \\
\delta\xi_0^3 = \delta x\sqrt{\sin^2\sigma_1-\cos^2\sigma_2} + \delta y\dfrac{\cos\sigma_1}{\sin\sigma_2} + \\
\qquad + \delta z\cot\sigma_2\sqrt{\sin^2\sigma_1-\cos^2\sigma_2}.
\end{array}\right|
$$

(4.219)

Finally, from relations (4.212) and (4.219), after the required transformations we obtain:

$$\left.\begin{aligned} \delta r &= \delta x, \\ \delta\sigma_1 &= \delta y \frac{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}{r \sin\sigma_1 \sin\sigma_2} - \delta z \frac{\cot\sigma_1 \cot\sigma_2}{r}, \\ \delta\sigma_2 &= \frac{\delta z}{r}. \end{aligned}\right\} \tag{4.220}$$

In the case of the non-orthogonal reference grid r, σ_1, σ_2 in selecting the reduction trihedron O_1xyz we made use, as was indicated, of the perpendicularity of vectors r_1 and r_2, making them incident with the x and y axes. In the more general case, in which the three vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ are not pairwise perpendicular, we may select the reduction trihedron in the following manner: directing one of the x, y, or z axes along any of the vectors $\vec{r}_s$ (or $\vec{r}^s$), we place the second axis in one of the planes defined by the two vectors of the main or reciprocal basis (or one of the vectors of the main basis and of the the vectors of the reciprocal basis).

§4.5. Errors in the Orientation of the Axes of the Sensing Elements. Errors in the Determination of the Orientation of the Object.

4.5.1. Errors in the orientation of the sensing axes of the newtonometers and gyroscopes. Inertial systems must not only determine the coordinates of a moving object, but also determine the parameters characterizing its orientation in space. The relations by means of which these parameters are found were deduced in Chapter 3 in synthesizing the ideal equations.

As we saw, the problem reduces, in the final analysis, to the determination of the orientation of the object relative to the sensing axes of the newtonometers and gyroscopes. The newtonometers and gyroscopes are in turn oriented in a particular way relative to the ξ^1, ξ^2, ξ^3 (ξ_*, η_*, ζ_*) axes of the basic Cartesian coordinate system or relative to the η^1, η^2, η^3 (ξ, η, ζ) axes of the coordinate system associated with the earth. Therefore, determination of the orientation of the objects relative to the axes of sensitivity of the gyroscopes and newtonometers entails simultaneously the determination of its orientation relative to the basic Cartesian coordinate system and relative to the earth.

In the preceding sections of this chapter we obtained the error equations relating the errors deriving from the elements of an inertial system and errors in the specification of the initial conditions to errors in the specification of the coordinates of the object. Let us now obtain the equations defining errors in the determination of orientation. These consist, clearly, of errors in the spatial orientation of the sensing elements of an inertial navigation system and errors in the specification of orientation relative to the axes of the sensing elements. Let us therefore first consider errors in the orientation of the newtonometers and gyroscopes.

Let us return to the system considered in §3.1. This system determines the Cartesian coordinates of the object. The newtonometers in this system are rigidly connected to the platform of the device measuring absolute angular velocity of the gyrostabilized platform (maneuverable or non-maneuverable). The axes of sensitivity of the newtonometers and gyroscopes coincide with the x, y, z axes of the platform. The problem reduces, therefore, to the study of errors in the orientation of the platform.

The simplest case is that in which the spatial gauge of absolute angular velocity is the basic functional diagram . In this case the orientation of the x, y, z axes of the platform does not depend on the coordinates determined by the inertial system. Errors in the orientation of the platform in the basic Cartesian coordinate system are characterized by the angle θ_1, the projections θ_{1x}, θ_{1y}, θ_{1z} of which on the x, y, z axes are given by equation (4.84). Errors in orientation relative to the earth are characterized by angle θ_2, equations for the projection of which on the x, y, z axes, according to (4.88), differ from equations (4.84) only in their right sides. From equations (4.84) it follows that errors in the specification of the orientation of the platform of an inertial system depends in this case only on the initial error in its orientation and on instrument errors deriving from the measurement of angular velocity.

The situation remains the same if the basic functional diagram is a non-maneuverable gyrostablized platform. In this case it is necessary only to set $\omega_x = \omega_y = \omega_z = 0$ in equations (4.84) and (4.88). Analogously, the situation reduces to that of the preceding when the diagram is constructed on the basis of a maneuverable gyrostabilized platform if the controlling moments are developed only as functions of time. The situation is somewhat more complex if the controlling moments are developed as functions not only of time but also of the coordinates of the current location of the object, these also being determined by the inertial system.

In this case equations (4.84) and (4.88) are insufficient for the description of the perturbed position of the trihedron Oxyz associated with the platform. In fact, in this case, the quantities Δm_x, Δm_y, Δm_z are the only instrument errors in the development of the controlling moments. But these moments are developed according to coordinates determined by the inertial system. Errors in the specification of the coordinates give rise to a certain additional deviation in the position of the trihedron Oxyz from the position defined by the ideal equations. This additional deviation is not taken into account in equations (4.84) and (4.88), although it is, of course, explicit contained in equations (4.83) -- (4.85) as a whole. In the instances examined above, in which the orientation of trihedron Oxyz was not a function of the coordinates, the error in the orientation of this trihedron in turn was not a function of the solution of equations (4.83) -- (4.85) as a whole. Now it will depend on them.

Let the orientation of the x, y, z axes relative to the ξ^1, ξ^2, ξ^3 axes be given by the direction cosines α_{ij} [Table (3.16)] or relative to the η^1, η^2, η^3 axes by the direction cosines β_{ij} [Table (3.40)]. It is sufficient, clearly, to examine only the first case. The projections on the x, y, z axes of the absolute angular velocity $\vec{\omega}$ of the rotation of the trihedron Oxyz, corresponding to the given direction cosines α_{ij}, has the form:

$$\left.\begin{aligned}
\omega_x &= \dot{\alpha}_{12}\alpha_{13} + \dot{\alpha}_{22}\alpha_{23} + \dot{\alpha}_{32}\alpha_{33} = -\dot{\alpha}_{13}\alpha_{12} - \dot{\alpha}_{23}\alpha_{22} - \dot{\alpha}_{33}\alpha_{32},\\
\omega_y &= \dot{\alpha}_{13}\alpha_{11} + \dot{\alpha}_{23}\alpha_{21} + \dot{\alpha}_{33}\alpha_{31} = -\dot{\alpha}_{11}\alpha_{13} - \dot{\alpha}_{21}\alpha_{23} - \dot{\alpha}_{31}\alpha_{33},\\
\omega_z &= \dot{\alpha}_{11}\alpha_{12} + \dot{\alpha}_{21}\alpha_{22} + \dot{\alpha}_{31}\alpha_{32} = -\dot{\alpha}_{12}\alpha_{11} - \dot{\alpha}_{22}\alpha_{21} - \dot{\alpha}_{32}\alpha_{31}.
\end{aligned}\right\} \quad (4.221)$$

These formulas follow from the equalities

$$\left.\begin{aligned}
\frac{dy}{dt}\cdot z &= -\frac{dz}{dt}\cdot y = (\omega\times y)\cdot z = -(\omega\times z)\cdot y = \omega_x,\\
\frac{dz}{dt}\cdot x &= -\frac{dx}{dt}\cdot z = (\omega\times z)\cdot x = -(\omega\times x)\cdot z = \omega_y,\\
\frac{dx}{dt}\cdot y &= -\frac{dy}{dt}\cdot x = (\omega\times x)\cdot y = -(\omega\times y)\cdot x = \omega_z.
\end{aligned}\right\} \quad (4.222)$$

Equalities (4.222) derive in turn from the fact that the vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ are the unit vectors of the axes of the orthogonal trihedron Oxyz (or O_1xyz), rotating with an angular velocity $\vec{\omega}$.

The deviation of the position of the platform from that defined by the ideal equations is caused by the instrument errors Δm_x, Δm_y, Δm_z in the development of the controlling moments and by the following quantitites:

$$\left.\begin{aligned}
\delta\omega_x &= \frac{\partial\omega_x}{\partial\xi^1}\delta\xi^1_3 + \frac{\partial\omega_x}{\partial\xi^2}\delta\xi^2_3 + \frac{\partial\omega_x}{\partial\xi^3}\delta\xi^3_3,\\
\delta\omega_y &= \frac{\partial\omega_y}{\partial\xi^1}\delta\xi^1_3 + \frac{\partial\omega_y}{\partial\xi^2}\delta\xi^2_3 + \frac{\partial\omega_y}{\partial\xi^3}\delta\xi^3_3,\\
\delta\omega_z &= \frac{\partial\omega_z}{\partial\xi^1}\delta\xi^1_3 + \frac{\partial\omega_z}{\partial\xi^2}\delta\xi^2_3 + \frac{\partial\omega_z}{\partial\xi^3}\delta\xi^3_3,
\end{aligned}\right\} \quad (4.223)$$

where $\delta\xi^1_3$, $\delta\xi^2_3$, $\delta\xi^3_3$ are the total errors in the determination of the coordinates ξ^1, ξ^2, ξ^3 calculated from expressions (4.16) and (4.20) in the following manner:

$$\left.\begin{array}{c}\delta\xi^s_1 = (\delta r + \theta_1 \times r)\cdot\xi_1, \quad \delta\xi^s_2 = (\delta r + \theta_1 \times r)\cdot\xi_2,\\ \delta\xi^s_3 = (\delta r + \theta_1 \times r)\cdot\xi_3\end{array}\right\} \tag{4.224}$$

[the vector $\vec{\theta}_1$ is determined from the second equations (4.81)].

The deviation of the platform is characterized, clearly, by the variations $\delta\alpha_{ij}$ of the direction cosines defining its orientation relative to the ξ^1, ξ^2, ξ^3 axes:

$$\left.\begin{array}{c}\delta\alpha_{i1} = \delta\xi_i\cdot x + \xi_i\cdot\delta x, \quad \delta\alpha_{i2} = \delta\xi_i\cdot y + \xi_i\cdot\delta y,\\ \delta\alpha_{i3} = \delta\xi_i\cdot z + \xi_i\cdot\delta z.\end{array}\right\} \tag{4.225}$$

On the other hand, in accordance with equalities (4.224)

$$\delta\alpha_{ij} = \frac{\partial\alpha_{ij}}{\partial\xi^s}(\delta r + \theta_1\times r)\cdot\xi_s. \tag{4.226}$$

In formulas (4.226) it is assumed, of course, that summing is taken over s from 1 to 3.

Comparing equalities (4.225) to equalities (4.226), we arrive at the relations

$$\left.\begin{array}{l}\xi_i\cdot\delta x = -\delta\xi_i\cdot x + \frac{\partial\alpha_{i1}}{\partial\xi^s}[(\theta_1\times r)\cdot\xi_s + \delta r\cdot\xi_s],\\ \xi_i\cdot\delta y = -\delta\xi_i\cdot y + \frac{\partial\alpha_{i2}}{\partial\xi^s}[(\theta_1\times r)\cdot\xi_s + \delta r\cdot\xi_s],\\ \xi_i\cdot\delta z = -\delta\xi_i\cdot z + \frac{\partial\alpha_{i3}}{\partial\xi^s}[(\theta_1\times r)\cdot\xi_s + \delta r\cdot\xi_s],\end{array}\right\} \tag{4.227}$$

which enable us to find the variations δx, δy, δz characterizing the deviation of the unit vectors of the x, y, z axes of the platform from their non-perturbed position.

Formulas (4.227) are equivalent to the vector equalities

$$\left.\begin{array}{l}\delta x = [(\theta_1\times\xi_i)\cdot x]\,\xi_i + \frac{\partial\alpha_{i1}}{\partial\xi^s}[(\theta_1\times r)\cdot\xi_s + \delta r\cdot\xi_s]\,\xi_i,\\ \delta y = [(\theta_1\times\xi_i)\cdot y]\,\xi_i + \frac{\partial\alpha_{i2}}{\partial\xi^s}[(\theta_1\times r)\cdot\xi_s + \delta r\cdot\xi_s]\,\xi_i,\\ \delta z = [(\theta_1\times\xi_i)\cdot z]\,\xi_i + \frac{\partial\alpha_{i3}}{\partial\xi^s}[(\theta_1\times r)\cdot\xi_s + \delta r\cdot\xi_s]\,\xi_i.\end{array}\right\} \tag{4.228}$$

In making the transition from relations (4.227) to relations (4.228) we used the first equalities (4.19), which, if we substitute ξ_1, η_2, ζ_3 for ξ_*, η_*, ζ_* respectively, take the form:

$$\delta\xi_i = -\theta_1 \times \xi_i.$$

In relations (4.228) summing is taken over s and i from 1 to 3.

Let us introduce the small rotation vector $\vec{\theta}$ defining the position the trihedron $\vec{x} + \delta\vec{x}$, $\vec{y} + \delta\vec{y}$, $\vec{z} + \delta\vec{z}$ relative to $\vec{x}$, $\vec{y}$, $\vec{z}$. Its projections θ_x, θ_y, θ_z on the x, y, z axes may be represented by analogy with (4.222) in the following form:

$$\left.\begin{array}{c} \theta_x = \delta y \cdot z = -\delta z \cdot y, \quad \theta_y = \delta z \cdot x = -\delta x \cdot z, \\ \theta_z = \delta x \cdot y = -\delta y \cdot x. \end{array}\right\} \qquad (4.229)$$

Substituting the values of δx, δy, δz from formulas (4.228) and using the table (3.16) of the direction cosines we find:

$$\left.\begin{array}{l} \theta_x = \alpha_{i3}\left[(\theta_1 \times \xi_i) \cdot y + \frac{\partial \alpha_{i2}}{\partial \xi^s}(\theta_1 \times r + \delta r) \cdot \xi_s\right], \\ \theta_y = \alpha_{i1}\left[(\theta_1 \times \xi_i) \cdot z + \frac{\partial \alpha_{i3}}{\partial \xi^s}(\theta_1 \times r + \delta r) \cdot \xi_s\right], \\ \theta_z = \alpha_{i2}\left[(\theta_1 \times \xi_i) \cdot x + \frac{\partial \alpha_{i1}}{\partial \xi^s}(\theta_1 \times r + \delta r) \cdot \xi_s\right]. \end{array}\right\} \qquad (4.230)$$

But

$$\alpha_{i3}(\theta_1 \times \xi_i) \cdot y = (\theta_1 \times z) \cdot y = -\theta_{1x},$$
$$\alpha_{i1}(\theta_1 \times \xi_i) \cdot z = -\theta_{1y}, \quad \alpha_{i2}(\theta_1 \times \xi_i) \cdot x = -$$

Substituting these expressions into equalities (4.230) and expanding the second terms in square brackets on the right side of these equalities, we obtain the following formulas for θ_x, θ_y, θ_z:

$$
\left.
\begin{aligned}
\theta_x &= -\theta_{1x} + \alpha_{i3}\Big[\frac{\partial \alpha_{i2}}{\partial \xi^1}(\delta\xi^1 + \theta_{1\xi^2}\xi^3 - \theta_{1\xi^3}\xi^2) + \\
&\quad + \frac{\partial \alpha_{i2}}{\partial \xi^2}(\delta\xi^2 + \theta_{1\xi^3}\xi^1 - \theta_{1\xi^1}\xi^3) + \\
&\qquad + \frac{\partial \alpha_{i2}}{\partial \xi^3}(\delta\xi^3 + \theta_{1\xi^1}\xi^2 - \theta_{1\xi^2}\xi^1)\Big], \\
\theta_y &= -\theta_{1y} + \alpha_{i1}\Big[\frac{\partial \alpha_{i3}}{\partial \xi^1}(\delta\xi^1 + \theta_{1\xi^2}\xi^3 - \theta_{1\xi^3}\xi^2) + \\
&\quad + \frac{\partial \alpha_{i3}}{\partial \xi^2}(\delta\xi^2 + \theta_{1\xi^3}\xi^1 - \theta_{1\xi^1}\xi^3) + \\
&\qquad + \frac{\partial \alpha_{i3}}{\partial \xi^3}(\delta\xi^3 + \theta_{1\xi^1}\xi^2 - \theta_{1\xi^2}\xi^1)\Big], \\
\theta_z &= -\theta_{1z} + \alpha_{i2}\Big[\frac{\partial \alpha_{i1}}{\partial \xi^1}(\delta\xi^1 + \theta_{1\xi^2}\xi^3 - \theta_{1\xi^3}\xi^2) + \\
&\quad + \frac{\partial \alpha_{i1}}{\partial \xi^2}(\delta\xi^2 + \theta_{1\xi^3}\xi^1 - \theta_{1\xi^1}\xi^3) + \\
&\qquad + \frac{\partial \alpha_{i1}}{\partial \xi^3}(\delta\xi^3 + \theta_{1\xi^1}\xi^2 - \theta_{1\xi^2}\xi^1)\Big].
\end{aligned}
\right| \tag{4.231}
$$

Formulas (4.231) define the deviation of the xyz trihedron rigidly bound to the platform from its position as defined by the ideal equation. They define, consequently, the errors in the orientation of the platform.

In the general case, as may be seen from formulas (4.231), the projections θ_x, θ_y, θ_z are a function both of the projections of the vector $\vec{\theta}_1$ and on the errors $\delta\xi^s$ in the determination of the coordinates. The latter are projections on the ξ^1, ξ^2, ξ^3 axes of the vector $\delta\vec{r}$. The projections δx, δy, δz of this vector on the x, y, z axes appear in the first group of the coordinate error equations (4.83). The projections θ_{1x}, θ_{1y}, θ_{1z} of the vector $\vec{\theta}_1$ are found from equations (4.84).

It is easy to see that if the direction cosines α_{ij} are not functions of the coordinates ξ^s, then the angles θ_x, θ_y, θ_z, become identically equal to $-\theta_{1x}$, $-\theta_{1y}$, $-\theta_{1z}$ respectively, as should be the case for α_{ij} as functions of time only.

Formulas (3.231) are valid for the derivatives of α_{ij} (ξ^1, ξ^2, ξ^3, t). For the sake of illustration we will consider the case in which the Oz axis of the trihedron Oxyz in the non-perturbed position is directed along the radius of the earth, i.e., in which the trihedron Oxyz is a moving trihedron on a sphere surrounding the earth. This case is noteworthy in that θ_{1x}, θ_{1y} fall out of the right side of the two first equations (4.231), as we will show.

Formulas (4.231) may be represented in a somewhat different form, if the projections θ_{1x}, θ_{1y}, θ_{1z} are substituted into their right sides. In this case we obtain:

$$\left.\begin{aligned}
\theta_x &= -\theta_{1x} + a_{i3}\Big\{\delta a_{i2} + \frac{\partial a_{i2}}{\partial \xi^t}\big[(\theta_{1y}z - \theta_{1z}y)\,a_{t1} + \\
&\quad + (\theta_{1z}x - \theta_{1x}z)\,a_{t2} + (\theta_{1x}y - \theta_{1y}x)\,a_{t3}\big]\Big\}, \\
\theta_y &= -\theta_{1y} + a_{i1}\Big\{\delta a_{i3} + \frac{\partial a_{i3}}{\partial \xi^t}\big[(\theta_{1y}z - \theta_{1z}y)\,a_{t1} + \\
&\quad + (\theta_{1z}x - \theta_{1x}z)\,a_{t2} + (\theta_{1x}y - \theta_{1y}x)\,a_{t3}\big]\Big\}, \\
\theta_z &= -\theta_{1z} + a_{i2}\Big\{\delta a_{i1} + \frac{\partial a_{i1}}{\partial \xi^t}\big[(\theta_{1y}z - \theta_{1z}y)\,a_{t1} + \\
&\quad + (\theta_{1z}x - \theta_{1x}z)\,a_{t2} + (\theta_{1x}y - \theta_{1y}x)\,a_{t3}\big]\Big\},
\end{aligned}\right\} \tag{4.232}$$

where

$$\delta a_{ij} = \frac{\partial a_{ij}}{\partial \xi^t}\,\delta \xi^t. \tag{4.233}$$

This form permits an easier transition to the accompanying trihedron in equations for θ_x, θ_y, θ_z. In fact, if the Oz axis in the unperturbed position coincides with r, i.e.,

$$z = \frac{r}{r}, \tag{4.234}$$

then, clearly,

$$x = y = 0, \quad z = r. \tag{4.235}$$

If x and y in formulas (4.232) are now set equal to 0, and z equal to the distance r from the origin of trihedron Oxyz to the center of the earth, these formulas simplify significantly:

$$\left.\begin{aligned}
\theta_x &= -\theta_{1x} + a_{i3}\left[\delta a_{i2} + r\,\frac{\partial a_{i2}}{\partial \xi^t}\,(\theta_{1y}a_{t1} - \theta_{1x}a_{t2})\right], \\
\theta_y &= -\theta_{1y} + a_{i1}\left[\delta a_{i3} + r\,\frac{\partial a_{i3}}{\partial \xi^t}\,(\theta_{1y}a_{t1} - \theta_{1x}a_{t2})\right], \\
\theta_z &= -\theta_{1z} + a_{i2}\left[\delta a_{i1} + r\,\frac{\partial a_{i1}}{\partial \xi^t}\,(\theta_{1y}a_{t1} - \theta_{1x}a_{t2})\right].
\end{aligned}\right\} \tag{4.236}$$

Further simplifications of the formulas for θ_x, θ_y, θ_z derive from the relations

$$\alpha_{i3} = \frac{\xi^i}{r}, \tag{4.237}$$

which are a consequence of equality (4.234).

Considering relations (4.237), we find the following sums enter into (4.236):

$$\left.\begin{array}{ll} r\alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi_0^s}\alpha_{s1}, & r\alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi_0^s}\alpha_{s2}, \\ r\alpha_{i1}\frac{\partial\alpha_{i3}}{\partial\xi_0^s}\alpha_{s1}, & r\alpha_{i1}\frac{\partial\alpha_{i3}}{\partial\xi_0^s}\alpha_{s2} \end{array}\right\} \tag{4.238}$$

(the summation is taken over i and s).

In view of the orthogonality of table (3.16)

$$\alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi_0^s} = -\alpha_{i2}\frac{\partial\alpha_{i3}}{\partial\xi_0^s}. \tag{4.239}$$

But according to relations (4.237)

$$\left.\begin{array}{l} \frac{\partial\alpha_{s3}}{\partial\xi_0^s} = \frac{\partial}{\partial\xi_0^s}\left(\frac{\xi^s}{r}\right) = \frac{r^2-(\xi^s)^2}{r^3}, \\ \frac{\partial\alpha_{i3}}{\partial\xi_0^s} = -\frac{\xi^i\xi^s}{r^3} \qquad (i \neq s). \end{array}\right\} \tag{4.240}$$

From equalities (4.235), (4.238), (4.239), (4.240) and table (3.16), we now find:

$$\left.\begin{array}{l} -r\alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi_0^s}\alpha_{s1} = \frac{(\xi^s)^2\alpha_{s2}\alpha_{s1}}{r^3} + \frac{\xi^i\xi^s}{r^3}\alpha_{i2}\alpha_{s1} = -\frac{2xy}{r^3} = 0, \\ r\alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi_0^s}\alpha_{s2} = -1 + \frac{(\xi^s)^2\alpha_{s2}^2}{r^3} - \frac{\xi^i\xi^s\alpha_{i2}\alpha_{s2}}{r^3} = \\ \qquad\qquad = -1 + \frac{2y^2}{r^3} = -1, \\ r\alpha_{i1}\frac{\partial\alpha_{i3}}{\partial\xi_0^s}\alpha_{s1} = 1 - \frac{2x^2}{r^3} = 1, \\ r\alpha_{i1}\frac{\partial\alpha_{i3}}{\partial\xi_0^s}\alpha_{s2} = -\frac{2xy}{r^3} = 0. \end{array}\right\} \tag{4.241}$$

The values (4.241) obtained for sums (4.238) entering into formulas (4.236) immediately significantly simplify the latter.

We obtain:

$$\left.\begin{aligned} \theta_x &= \alpha_{i3}\delta\alpha_{i2}, \\ \theta_y &= \alpha_{i1}\delta\alpha_{i3}, \\ \theta_z &= -\theta_{1z} + \alpha_{i2}\delta\alpha_{i1} + r\alpha_{i2}\frac{\partial\alpha_{i1}}{\partial\xi^3}(\theta_{1y}\alpha_{s1} - \theta_{1x}\alpha_{s2}). \end{aligned}\right\} \tag{4.242}$$

Here the variations $\delta\alpha_{ij}$ of the direction cosines are expressed by the derivatives $\frac{\partial\alpha_{ij}}{\partial\xi^s}$ and the variations $\delta\xi^s$ by formulas (4.233).

Since according to the table of direction cosines (3.16)

$$\delta\xi^s = \alpha_{s1}\delta x + \alpha_{s2}\delta y + \alpha_{s3}\delta z, \tag{4.243}$$

the angles θ_x, θ_y of the deviation of the z axis of the platform from r significantly depend in this case only on the errors in determination of coordinates which derive from equations (4.83).

Let us obtain explicit expressions for θ_x and θ_y through the solutions δx, δy, δz of equations (4.83). From equalities (4.239) (4.240) and the first equality (4.242), we have:

$$\begin{aligned} \theta_x &= \alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi^s}\delta\xi^s = -\alpha_{i2}\frac{\partial\alpha_{i3}}{\partial\xi^s}\delta\xi^s = \\ &= \left[-\alpha_{s2}\frac{r^2-(\xi^s)^2}{r^3} + \alpha_{n2}\frac{\xi^n\xi^s}{r^3}\right]\delta\xi^s. \end{aligned} \tag{4.244}$$

Here the summation is taken over s from 1 to 3 and over all n different from s.

From (4.237) we obtain:

$$\theta_x = -\frac{\alpha_{s2}\delta\xi^s}{r} + \frac{\alpha_{s3}\delta\xi^s}{r}(\alpha_{s3}\alpha_{s1} + \alpha_{n3}\alpha_{n2}) \tag{4.245}$$

But since $n \neq s$, the orthogonality of table (3.16) causes the expression in brackets on the right side of formula (4.245) to be equal to 0. The first term on the right side of the formula, if the $\delta\xi^s$ from equality (4.243) is substituted into it, becomes $\delta y/r$.

Consequently,

$$\theta_x = -\frac{\delta y}{r}. \tag{4.246}$$

Analogously from the second equality (4.242) and expressions (4.237), (4.239), (4.240) and (4.243), we find:

$$\theta_y = \frac{\delta x}{r}. \tag{4.247}$$

Thus, formulas (4.242) take the form:

$$\left.\begin{aligned} &\theta_x = -\frac{\delta y}{r}, \quad \theta_y = \frac{\delta x}{r}, \\ &\theta_z = -\theta_{1z} + a_{l2}\delta a_{l1} + r a_{l2}\frac{\partial a_{l1}}{\partial \xi^s}(\theta_{1y}a_{s1} - \theta_{1x}a_{s2}). \end{aligned}\right\} \tag{4.248}$$

The following sums enter into the third equality (4.248):

$$\left.\begin{aligned} &a_{l2}\delta a_{l1} = a_{l2}\frac{\partial a_{l1}}{\partial \xi^s}\delta\xi^s, \quad r a_{l2}\frac{\partial a_{l1}}{\partial \xi^s}a_{s1}, \\ &r a_{l2}\frac{\partial a_{l1}}{\partial \xi^s}a_{s2}. \end{aligned}\right\} \tag{4.249}$$

In order expand these sums, it is necessary to specify the orientation of the x and y axes relative to the ξ^1, ξ^2, ξ^3 axes. As yet only the direction of the z axis is fully determined, and the x and y axes are known only to be located in a plane perpendicular to the vector r. This is, clearly, insufficient.

The simplest way to supplement the definition of the position of the xyz trihedron is to take as the unperturbed position

$$m_z = \omega_z = 0. \tag{4.250}$$

In this case, from the third equality of (4.221) it follows that

$$\alpha_{i2}\frac{\partial\alpha_{i1}}{\partial\xi^r}=0, \quad \alpha_{i1}\frac{\partial\alpha_{i1}}{\partial\xi^r}=0. \tag{4.251}$$

Then, in the right side of the third formula (4.248) all terms with the exception of the third vanish, and the expressions for θ_x, θ_y, θ_z take the form:

$$\theta_x=-\frac{\delta y}{r}, \quad \theta_y=\frac{\delta x}{r}, \quad \theta_z=-\theta_{12}. \tag{4.252}$$

As a second example of the determination of the directions of the x and y axes of the moving trihedron, we will consider the case in which the y axis lies in the plane containing the axis of rotation of the earth, i.e., the ξ^3. In this case, of course, the x axis is parallel to the plane of the equator. For the sake of precision we will assume further that the point O is located in the first octant of the $O_1\xi^1\xi^2\xi^3$ coordinate system, and that the y axis forms acute angles with the ξ^1 and ξ^2 axes. It is evident that under these conditions the trihedron Oxyz becomes the moving trihedron of a geocentric coordinate system, and its y axis points to the north.

In this case we may write the following expressions for the direction cosines α_{s1} and α_{s2} of the x and y axes in terms of the coordinates ξ^s:

$$\left.\begin{aligned}
&\alpha_{11}=-\frac{\xi^2}{[r^2-(\xi^3)^2]^{1/2}}, && \alpha_{21}=\frac{\xi^1}{[r^2-(\xi^3)^2]^{1/2}},\\
&\alpha_{12}=-\frac{\xi^1\xi^3}{r[r^2-(\xi^3)^2]^{1/2}}, && \alpha_{22}=-\frac{\xi^2\xi^3}{r[r^2-(\xi^3)^2]^{1/2}},\\
&\alpha_{31}=0, && \alpha_{32}=\frac{[r^2-(\xi^3)^2]^{1/2}}{r}.
\end{aligned}\right\} \tag{4.253}$$

Differentiating these direction cosines with respect to ξ^s and forming the sums (4.249), we obtain:

$$\left.\begin{aligned}
r\alpha_{i2}\frac{\partial\alpha_{i1}}{\partial\xi^s}\alpha_{s2}&=0,\\
\alpha_{i2}\delta\alpha_{i1}&=\frac{\xi^3}{r[r^2-(\xi^3)^2]^{1/2}}(\alpha_{11}\delta\xi^1+\alpha_{21}\delta\xi^2),\\
r\alpha_{i1}\frac{\partial\alpha_{i2}}{\partial\xi^s}\alpha_{s1}&=\frac{\xi^3}{[r^2-(\xi^3)^2]^{1/2}}.
\end{aligned}\right\} \tag{4.254}$$

where

$$\left.\begin{array}{c} a_{11}\,\delta\xi^1 + a_{21}\,\delta\xi^2 = \delta x, \\ \dfrac{\xi^3}{[r^2-(\xi^3)^2]^{1/2}} = -\dfrac{a_{12}}{a_{13}} = -\dfrac{a_{22}}{a_{23}} = \tan\varphi \end{array}\right\} \tag{4.255}$$

is the geocentric latitude).

Considering relations (4.254) and (4.255), the third equality (4.248) takes the form:

$$\theta_z = -\theta_{1z} + \tan\varphi\,\frac{\delta x + r\theta_{1y}}{r}. \tag{4.256}$$

Noting that

$$\delta x + r\theta_{1y} = \delta x_3,$$

and combining the first two equalities (4.248) with (4.256), we obtain the following formulas for the case under consideration:

$$\left.\begin{array}{l} \theta_x = -\dfrac{\delta y}{r}, \quad \theta_y = \dfrac{\delta x}{r}, \\ \theta_z = -\theta_{1z} + \dfrac{\delta x_3}{r}\tan\varphi. \end{array}\right\} \tag{4.257}$$

Finally, if the x and y axes are oriented tangentially to the coordinate lines z = constant and S = constant of the geodetic reference grid, then, proceeding in the same way as in the derivation of formulas (4.257), we arrive at the equalities

$$\left.\begin{array}{l} \theta_x = -\dfrac{\delta y}{r}, \quad \theta_y = \dfrac{\delta x}{r}, \\ \theta_z = -\theta_{1z} + \dfrac{\delta x_3}{r}\tan z \end{array}\right\} \tag{4.258}$$

Let us continue to consider questions associated with errors in the orientation of the sensing elements of an inertial system and turn to the specification of arbitrary curvilinear coordinates.

Let the unit vector $\vec{e}_s$ of the axes of sensitivity of the newtonometers coincide with the unit vectors of the common basis

$$e_s = \frac{r^s}{\sqrt{a^{ss}}}. \tag{4.259}$$

The position of these directions relative to the stablized platform, i.e., relative to the ξ^1, ξ^2, ξ^3 axes is given by the table of direction cosines (3.173). Let us denote the elements of this table by ℓ_{sk}. Then,

$$l_{sk} = e_s \cdot \xi_k = \xi_k \cdot \frac{r^s}{\sqrt{a^{ss}}}. \tag{4.260}$$

Varying these relations, we obtain:

$$\delta l_{sk} = \delta e_s \cdot \xi_k + e_s \cdot \delta \xi_k. \tag{4.261}$$

On the other hand,

$$\delta l_{sk} = \frac{\partial l_{sk}}{\partial \varkappa^\sigma} \delta \varkappa_3^\sigma. \tag{4.262}$$

Comparing equalities (4.261) and (4.262), we find

$$\delta e_s \cdot \xi_k = -e_s \cdot \delta \xi_k + \frac{\partial l_{sk}}{\partial \varkappa^\sigma} \delta \varkappa_3^\sigma. \tag{4.263}$$

or

$$\delta e_s = \left(-e_s \cdot \delta \xi_k + \frac{\partial l_{sk}}{\partial \varkappa^\sigma} \delta \varkappa_3^\sigma \right) \xi_k. \tag{4.264}$$

where on the right side the summation is taken over σ and k from 1 to 3.

The three formulas (4.264) are analogous to equalities (4.228), but in the former the quantities e_s and ℓ_{sk} are functions not of the coordinates ξ^s, but of curvilinear coordinates $\varkappa^s$. Formulas (4.264) define the vectors $\vec{e}_s$ characterizing the deviations of the axes of sensitivity of the newtonometers from their position defined by the ideal equation.

Since the unit vectors $\vec{\xi}_k$ of the ξ^k axes do not change their directions in space, i.e., do not depend on the coordinates $\varkappa^\sigma$, in equalities (4.264)

$$\frac{\partial l_{sk}}{\partial \varkappa^\sigma} = \xi_k \cdot \frac{\partial}{\partial \varkappa^\sigma}\left(\frac{r^s}{\sqrt{a^{ss}}}\right). \tag{4.265}$$

According to the definition of the vector θ_1 and the errors $\delta\varkappa_3^\sigma$,

$$\delta\xi_k = -\theta_1 \times \xi_k, \quad \delta\varkappa_3^\sigma = \delta\varkappa^\sigma + \delta\varkappa_1^\sigma, \tag{4.266}$$

where

$$\delta\varkappa_1^\sigma = (\theta_1 \times r) \cdot r^\sigma. \tag{4.267}$$

In formulas (4.266) and (4.267) the small rotation vector $\vec{\theta}_1$ is defined by the second group of equations (4.188), and the magnitudes of $\delta\eta^\sigma$ are defined by the first group of these same equations.

Let us find the error in the specification of the directions of $\vec{e}_s$ caused by the deviations $\delta\vec{e}_s$. We introduce a small rotation vector $\vec{\theta}$ such that

$$\delta e_s = \theta \times e_s, \tag{4.268}$$

and obtain the equations for this vector.

Substituting expressions (4.264) into formulas (4.268) and taking into account equalities (4.265) -- (4.267), we arrive at the following equation:

$$\theta \times e_s = \xi_k \left\{ e_s \cdot (\theta_1 \times \xi_k) + \frac{\partial l_{sk}}{\partial \varkappa^\sigma} [\delta\varkappa^\sigma + (\theta_1 \times r) \cdot r^\sigma] \right\} \tag{4.269}$$

or after obvious transformations, at the equations

$$(\theta + \theta_1) \times e_s = \frac{\partial e_s}{\partial \varkappa^\sigma} [\delta\varkappa^\sigma + (\theta_1 \times r) \cdot r^\sigma]. \tag{4.270}$$

If the unit vectors $\vec{e}_s$, as before, are not a function of the coordinates $\varkappa^{\sigma}$, it follows from equations (4.270) that

$$\theta = -\theta_1. \tag{4.271}$$

It is also easily shown that if the coordinates $\varkappa^{\sigma}$ are Cartesian, equalities (4.231) and (4.232) are obtained from equations (4.270) as a special case.

In conclusion let us consider one more question.

In considering in §4.4 the error equations for arbitrary curvilinear coordinates, we reduced equations (4.118) derived for this case to the error equations of a system determining Cartesian coordinates, i.e., equations (4.83) -- (4.85). The procedure which was used to accomplish this permits the conclusion that an analogous reduction may be carried out with regard to equations (4.270), i.e., that equations (4.270) may be reduced to equations (4.231) or (4.232). We will show that this is the case using orthogonal curvilinear coordinates as an example.

Equalities (4.231) may be replaced with the single vector equality

$$\begin{aligned} \theta + \theta_1 = x(z \cdot \delta y) + y(x \cdot \delta z) + z(y \cdot \delta x) + \\ + \xi_s \cdot (\theta_1 \times r)\left[x\left(z \cdot \frac{\partial y}{\partial \xi^s}\right) + y\left(x \cdot \frac{\partial z}{\partial \xi^s}\right) + z\left(y \cdot \frac{\partial x}{\partial \xi^s}\right)\right]. \end{aligned} \tag{4.272}$$

Multiplying this equality by x, we find:

$$(\theta + \theta_1) \times x = \delta x + [(\theta_1 \times r) \cdot \xi_s] \frac{\partial x}{\partial \xi^s}. \tag{4.273}$$

For the sake of simplicity let the trihedron xyz coincide with the trihedron $r_1 r_2 r_3$. It is then evident that

$$x = e_1, \quad y = e_2, \quad z = e_3. \tag{4.274}$$

and from equations (4.270) we obtain for s = 1:

$$(\theta+\theta_1)\times x = \delta x + \frac{\partial x}{\partial x^\sigma}[(\theta_1\times r)\cdot r^\sigma]. \tag{4.275}$$

But

$$\xi_s = \frac{\partial \xi^s}{\partial x^\sigma} r^\sigma, \quad \frac{\partial x}{\partial \xi^s} = \frac{\partial x}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial \xi^s}. \tag{4.276}$$

Substituting these values into equation (4.273), we arrive at th uality

$$[(\theta_1\times r)\cdot \xi_s]\frac{\partial x}{\partial \xi^s} = \frac{\partial x}{\partial x^\sigma}[(\theta_1\times r)\cdot r^\sigma], \tag{4.277}$$

which shows the equivalence of formulas (4.232) and (4.270) for the case of orthogonal curvilinear coordinates.

For oblique-angled curvilinear coordinates the equivalents of formulas (4.232) and (4.270) is also easily demonstrated, by taking the unit vectors $\vec{r}_1$, $\vec{r}^2$ and $\vec{r}_1 \times \vec{r}^2$ as the $\vec{x}$, $\vec{y}$, $\vec{z}$ unit vectors.

4.5.2. <u>Errors in the specification of the orientation of the object. Errors in the specification of given directions in space</u>. The orientation of the object relative to the trihedron Oxyz associated with the platform of the gauge of absolute angular velocity for the gyrostabilized platform is characterized by the angles α, β, γ of the rotations of the rings of the gimbal mount of the platform. Errors in the specification of the orientation of the object relative to the axes of the platform will be characterized by instrument errors Δα, Δβ, Δγ in the measurement of the angles α, β, γ. For small values of these errors the error in the specification of orientation may be given by a small rotation vector $\vec{\theta}_3$, the components of which will be functions of the instrument error Δα, Δβ, Δγ.

Let us find the projections of vector $\vec{\theta}_3$ on the x, y, z axes of of a platform. We will introduce ε_{ij} to designate the direction cosines between the angles X, Y, Z of the object and the x, y, z angles of the platform. The table direction cosines will have the form:

$$
\begin{array}{cccc}
 & x & y & z \\
X & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
Y & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
Z & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{array}
\tag{4.278}
$$

The quantities ε_{ij} as functions of the angles α, β and γ are obtained from the comparison of tables (4.278) and (3.66). The errors $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$ reduce to the errors $\delta\varepsilon_{ij}$ in the specification of the direction cosines ε_{ij}. The orientation found of the X, Y, Z axes will therefore correspond not to their actual position relative to the x, y, z axes, but rather to their perturbed position X', Y', Z' relative to the x, y, z axes, characterized by the following table of direction cosines:

$$
\begin{array}{cccc}
 & x & y & z \\
X' & \varepsilon_{11}+\delta\varepsilon_{11} & \varepsilon_{12}+\delta\varepsilon_{12} & \varepsilon_{13}+\delta\varepsilon_{13} \\
Y' & \varepsilon_{21}+\delta\varepsilon_{21} & \varepsilon_{22}+\delta\varepsilon_{22} & \varepsilon_{23}+\delta\varepsilon_{23} \\
Z' & \varepsilon_{31}+\delta\varepsilon_{31} & \varepsilon_{32}+\delta\varepsilon_{32} & \varepsilon_{33}+\delta\varepsilon_{33}.
\end{array}
\tag{4.279}
$$

From (4.278) and (4.279) we find that the deviation of the calculated (perturbed) orientation from the actual orientation is characterized by the following table of direction cosines:

$$
\begin{array}{cccc}
 & X & Y & Z \\
X' & 1 & \varepsilon_{21}\delta\varepsilon_{11}+\varepsilon_{22}\delta\varepsilon_{12}+\varepsilon_{23}\delta\varepsilon_{13} & \varepsilon_{31}\delta\varepsilon_{11}+\varepsilon_{32}\delta\varepsilon_{12}+\varepsilon_{33}\delta\varepsilon_{13} \\
Y' & \varepsilon_{11}\delta\varepsilon_{21}+\varepsilon_{12}\delta\varepsilon_{22}+\varepsilon_{13}\delta\varepsilon_{23} & 1 & \varepsilon_{31}\delta\varepsilon_{21}+\varepsilon_{32}\delta\varepsilon_{22}+\varepsilon_{33}\delta\varepsilon_{23} \\
Z' & \varepsilon_{11}\delta\varepsilon_{31}+\varepsilon_{12}\delta\varepsilon_{32}+\varepsilon_{13}\delta\varepsilon_{33} & \varepsilon_{21}\delta\varepsilon_{31}+\varepsilon_{22}\delta\varepsilon_{32}+\varepsilon_{23}\delta\varepsilon_{33} & 1.
\end{array}
\tag{4.280}
$$

As a result of the orthogonality of trihedra XYZ and X'Y'Z', the following equalities obtain:

$$
\left.
\begin{aligned}
&\varepsilon_{11}\delta\varepsilon_{21}+\varepsilon_{12}\delta\varepsilon_{22}+\varepsilon_{13}\delta\varepsilon_{23}= \\
&\qquad =-\varepsilon_{21}\delta\varepsilon_{11}-\varepsilon_{22}\delta\varepsilon_{12}-\varepsilon_{23}\delta\varepsilon_{13}, \\
&\varepsilon_{31}\delta\varepsilon_{11}+\varepsilon_{32}\delta\varepsilon_{12}+\varepsilon_{33}\delta\varepsilon_{13}= \\
&\qquad =-\varepsilon_{11}\delta\varepsilon_{31}-\varepsilon_{12}\delta\varepsilon_{32}-\varepsilon_{13}\delta\varepsilon_{33}, \\
&\varepsilon_{31}\delta\varepsilon_{21}+\varepsilon_{32}\delta\varepsilon_{22}+\varepsilon_{33}\delta\varepsilon_{23}= \\
&\qquad =-\varepsilon_{21}\delta\varepsilon_{31}-\varepsilon_{22}\delta\varepsilon_{32}-\varepsilon_{23}\delta\varepsilon_{33}.
\end{aligned}
\right\}
\tag{4.281}
$$

indicating that the table of direction cosines (4.280) is skew-symmetric.

This fact permits determination of the mutual position of the trihedra XYZ and X'Y'Z' by the small rotation vector $\vec{\theta}_3$, the projections of which are:

$$\left.\begin{aligned} \theta_{3X} &= c_{31}\delta c_{21} + c_{32}\delta c_{22} + c_{33}\delta c_{23}, \\ \theta_{3Y} &= c_{11}\delta c_{31} + c_{12}\delta c_{32} + c_{13}\delta c_{33}, \\ \theta_{3Z} &= c_{21}\delta c_{11} + c_{22}\delta c_{12} + c_{23}\delta c_{13}. \end{aligned}\right\} \tag{4.282}$$

Equalities (4.282) are analogous to equalities (4.52), (4.54), and (4.65) introduced earlier in order to define the small rotation vectors $\vec{\theta}_1$ and $\vec{\theta}_2$. In accordance with equalities (4.282) the errors in the specification of the orientation of the XYZ axes, the variations in the unit vectors of these axes, are equal to:

$$\delta X = \theta_3 \times X, \quad \delta Y = \theta_3 \times Y, \quad \delta Z = \theta_3 \times Z. \tag{4.283}$$

Let us find explicit expressions for the projections θ_{3X}, θ_{3Y}, θ_{3Z} in terms of α, β, γ and $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$. From tables (3.66) and (4.278) we have:

$$\left.\begin{aligned} \delta c_{11} &= -\Delta\beta \sin\beta \cos\gamma - \Delta\gamma \sin\gamma \cos\beta, \\ \delta c_{12} &= \Delta\beta \sin\beta \sin\gamma - \Delta\gamma \cos\beta \cos\gamma, \\ \delta c_{13} &= \Delta\beta \cos\beta, \\ \delta c_{21} &= \Delta\alpha (\cos\alpha \sin\beta \cos\gamma - \sin\alpha \sin\gamma) + \\ &\quad + \Delta\gamma (-\sin\alpha \sin\beta \sin\gamma + \cos\alpha \cos\gamma) + \\ &\quad\quad + \Delta\beta \sin\alpha \cos\beta \cos\gamma, \\ \delta c_{22} &= \Delta\alpha (-\cos\alpha \sin\beta \sin\gamma - \sin\alpha \cos\gamma) - \\ &\quad - \Delta\gamma (\sin\alpha \sin\beta \cos\gamma - \cos\alpha \sin\gamma) - \\ &\quad\quad - \Delta\beta \sin\alpha \cos\beta \sin\gamma, \\ \delta c_{23} &= -\Delta\alpha \cos\alpha \cos\beta + \Delta\beta \sin\alpha \sin\beta. \end{aligned}\right| \tag{4.284}$$

Substituting the derived values δc_{ij} and the values of c_{ij} [from table (2.66)] into formulas (4.282) and making use of the second equality (4.281), we obtain the following equations:

$$\left.\begin{aligned} \theta_{3X} &= -\Delta\alpha - \Delta\gamma\sin\beta, \\ \theta_{3Y} &= -\Delta\beta\cos\alpha + \Delta\gamma\sin\alpha\cos\beta, \\ \theta_{3Z} &= -\Delta\beta\sin\alpha - \Delta\gamma\cos\alpha\cos\beta. \end{aligned}\right\} \tag{4.285}$$

Now, projecting the vector $\vec{\theta}_3$ on the x, y, z axes, from the derived values of θ_{3X}, θ_{3Y}, θ_{3Z} and the table of direction cosines (3.66) we obtain:

$$\left.\begin{aligned} \theta_{3x} &= -\Delta\alpha\cos\beta\cos\gamma - \Delta\beta\sin\gamma, \\ \theta_{3y} &= \Delta\alpha\cos\beta\sin\gamma - \Delta\beta\cos\gamma, \\ \theta_{3z} &= -\Delta\alpha\sin\beta - \Delta\gamma. \end{aligned}\right\} \tag{4.286}$$

The total error in the specification of the orientation of the object in space is composed of errors in the orientation of the platform and errors in the specification of the orientation of the object relative to the platform. Thus, if we designate the total error by $\vec{\theta}_4$,

$$\theta_4 = \theta + \theta_3, \tag{4.287}$$

where vector $\vec{\theta}_3$ is defined by its projections (4.286), and the vector $\vec{\theta}$, by projections (4.231) or (4.232).

Let us consider the following circumstance.

The vector $\vec{\theta}$ characterizes the deviation of the trihedron xyz bound to the platform of the inertial system from its unperturbed position. As was shown above, the expressions for the vector $\vec{\theta}$ vary as a function of the means of specifying the unperturbed orientation of the inertial system platform relative to the ξ^1, ξ^2, ξ^3 axes. Thus, in superposing the x, y, z axes on the ξ^1, ξ^2, ξ^3 axes for θ_x, θ_y, θ_z, the following formulas obtain:

$$\theta_x = -\dot{\theta}_{1x}, \quad \theta_y = -\dot{\theta}_{1y}, \quad \theta_z = -\dot{\theta}_{1z}; \tag{4.288}$$

if the z axis of trihedron xyz coincides with the vector $\vec{r}$, formulas (4.248) apply; if the z axis is superposed on the vector $\vec{r}$, and the x axis lies in the plane of the meridian, formulas (4.257) etc. are valid. Expressions for vector $\vec{\theta}_4$ will vary depending on how the vector $\vec{\sigma}$ is expressed.

Formula (4.287) characterizes the error in the specification of the orientation of the object relative to the unperturbed position of the platform. At the same time it may be necessary to define the orientation of the object relative to axes not bound to the platform.

Thus, if the basic system is a non-maneuverable gyrostablized platform, the unperturbed orientation of the x, y, z, axes will be invariant in inertial space, the x, y, z axes, for example, being superposed on the ξ^1, ξ^2, ξ^3 axes. In this case formula (4.287) is the error in the orientation of the object relative to these axes. At the same time the conditions under which an inertial system is used may make it necessary to define the orientation of the object not relative to the ξ^1, ξ^2, ξ^3 axes, but relative to other axes, for example, relative to countries of the world, i.e., relative to a geocentric moving trihedron. In this case, there arises the problem of finding the orientation errors. Formula (4.287) does not as yet supply a solution to this problem.

It can be shown, however, that formula (4.287) can be extended to this case as well. If we follow once again the derivation of formulas (4.231) and (4.232), we will see that they are also valid for cases other than the xyz trihedron bound to the platform.

In fact, let a trihedron x^0, y^0, z^0 be defined relative to the ξ^1, ξ^2, ξ^3 axes of the basic Cartesian coordinate system, and let its position be defined by the direction cosines α^0_{ij} forming a table analogous to table (3.16), and let α^0_{ij} be functions of time and the coordinates of the object.

We substitute in formulas (4.231) and (4.232) α^0_{ij} for the direction cosines α_{ij}, and θ_{1x^0}, θ_{1y^0}, θ_{1z^0} for the projections θ_{1x}, θ_{1y}, θ_{1z}. The quantities θ_{x^0}, θ_{y^0}, θ_{z^0} will clearly characterize the errors in the specification of the orientation of trihedron x^0, y^0, z^0.

Therefore the vector equality

$$\theta_1 = \theta + \theta_3 \quad (\theta = \theta_{x^0} x^0 + \theta_{y^0} y^0 + \theta_{z^0} z^0). \tag{4.289}$$

which is fully analogous to equality (4.287), will define the errors in the specification of the orientation of the object relative to the x^0, y^0, z^0 trihedron not bound to the platform of the inertial system.

We note that the extension of formulas (4.231) and (4.232) to the case of an arbitrary trihedron xyz not rigidly bound to the platform permits us to find the errors in the specification of any given directions in space. These directions may be, for example those by which a projectile fired from a moving object is oriented. They may be directions to terrestrial orienting points and celestial bodies being used for correction of an inertial system. In particular, it follows from formulas (4.231) that, if a bearing on a distant star is defined, i.e., a bearing invariantly oriented in the $\xi^1\xi^2\xi^3$ coordinate system and characterized by the unit vector $\vec{\rho}$, then the error $\delta\vec{\rho}$ in the specification of this bearing is a function only of the instrument errors of the gyroscopic elements

$$\delta\rho = \theta_1 \times \rho \tag{4.290}$$

and is not a function of the instrument errors of the newtonometers.

§4.6. The Reduction of Instrument Errors to Equivalent Sensing Element Errors.

4.6.1. Reduction formulas. In the preceding sections in deriving the error equations of an inertial system it was assumed that the only instrument errors occurring in the system occurred in the sensing elements: the newtonometer error Δn and the gyroscope sensing element Δm.

Equations (4.283) -- (4.85), to which, as was shown, the coordinate error equations for any fully independent inertial navigation system reduce, contain the components Δn_x, Δn_y, Δn_z, and Δm_x, Δm_y, Δm_z of the vectors $\Delta\vec{n}$ and $\Delta\vec{m}$ along the x, y, z, axes, in terms of the projections on which equations (4.83) -- (4.85) were compiled. In the solution of the problem of reducing error equations (4.118) of an arbitrary interial system determining the curvilinear coordinates of an object, formulas (4.192) and (4.193) were obtained for equations (4.83) -- (4.85) in order to find the corresponding values of Δn_x, Δn_y, Δn_z and Δm_x, Δm_y, and Δm_z.

As was noted in §4.1, the retention of only the instrument errors of the sensing elements in the error equations is justified by the fact that under certain conditions other errors may be reduced to equivalent sensing element errors.

The conditions permitting this possibility ultimately reduce to the absence in the system of bugs giving rise to distortions of its operational algorithm, i.e., functional disturbances in the operation of the system. An example of such a bug is the non-correspondence of the functional arrangement of the system to the selected algorithm describing the system's ideal functioning, i.e., incorrect connections in the system. Bugs of this sort usually cause breakdowns.

If defects in the elements and devices of the system do not disturb its operational algorithm, the homogeneous error equations retain the form which they have in the absence of instrument errors, when the cause of the perturbed motion of the inertial system is only incorrectly specified initial conditions. The presence of instrument errors changes only the right sides of the error equations. This makes possible the reduction of the instrument errors to equivalent sensing element errors.

In order to demonstrate this, we turn to the first two groups of the error equations, i.e., to equations (4.83) and (4.84). Their right sides are functions of Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, and Δm_z characterizing the sensing element errors.

Introducing F_x, F_y, F_z for the right sides of equations (4.83), we will have:

$$\left.\begin{aligned}
F_x &= \Delta n_x - 2(\Delta m_y \dot{z} - \Delta m_z \dot{y}) - \Delta \dot{m}_y z + \Delta \dot{m}_z y - \\
&\quad - \omega_x (\Delta m_y y + \Delta m_z z) - \Delta m_x (\omega_y y + \omega_z z) + \\
&\quad + 2x(\omega_y \Delta m_y + \omega_z \Delta m_z). \\
F_y &= \Delta n_y - 2(\Delta m_z \dot{x} - \Delta m_x \dot{z}) - \Delta \dot{m}_z x + \Delta \dot{m}_x z - \\
&\quad - \omega_y (\Delta m_z z + \Delta m_x x) - \Delta m_y (\omega_z z + \omega_x x) + \\
&\quad + 2y(\omega_z \Delta m_z + \omega_x \Delta m_x). \\
F_z &= \Delta n_z - 2(\Delta m_x \dot{y} - \Delta m_y \dot{x}) - \Delta \dot{m}_x y + \Delta \dot{m}_y x - \\
&\quad - \omega_z (\Delta m_x x + \Delta m_y y) - \Delta m_z (\omega_x x + \omega_y y) + \\
&\quad + 2z(\omega_x \Delta m_x + \omega_y \Delta m_y)
\end{aligned}\right\} \tag{4.291}$$

Analogously, for the right sides of equations (4.84) we introduce the designations

$$f_x = \Delta m_x, \quad f_y = \Delta m_y, \quad f_z = \Delta m_z. \tag{4.292}$$

Let us assume that some group of errors other than sensing element instrument errors are to be taken into account. Then, in place of the functions F_x, F_y, F_z, f_x, f_y, f_z in the right sides of error equations (4.83) and (4.84) other functions F'_x, F'_y, F'_z, f'_x, f'_y, f'_z depending on the group of instrument errors under consideration will appear.

If we now transform the equalities

$$\left.\begin{array}{lll} F_x = F'_x, & F_y = F'_y, & F_z = F'_z, \\ f_x = f'_x, & f_y = f'_y, & f_z = f'_z \end{array}\right\} \tag{4.293}$$

and, using expressions (4.291) and (4.292), we solve them for Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, and Δm_z, we will obtain $\Delta n'_x$, $\Delta n'_y$, $\Delta n'_z$, $\Delta m'_x$, $\Delta m'_y$, and $\Delta m'_z$, the substitution of which into the right sides of relations (4.291) and (4.292) in place of Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z transforms these right sides into the functions F'_x, F'_y, F'_z, f'_x, f'_y, and f'_z.

The quantities $\Delta n'_x$, $\Delta n'_y$, $\Delta n'_z$, $\Delta m'_x$, $\Delta m'_y$, $\Delta m'_z$ will be the equivalent sensing element errors for the group of instrument errors under consideration.

Since it follows from equalities (4.292) and (4.293) that

$$\Delta m'_x = f'_x, \quad \Delta m'_y = f'_y, \quad \Delta m'_z = f'_z, \tag{4.294}$$

we obtain from (4.291), (4.293) and (4.294):

$$\left.\begin{array}{l} \Delta n'_x = F'_x + 2(f'_y \dot{z} - f'_z \dot{y}) + \dot{f}'_y z - \dot{f}'_z y + \\ \quad + \omega_x (f'_y y + f'_z z) + f'_x (\omega_y y + \omega_z z) - 2x(\omega_y f'_y + \omega_z f'_z), \\ \Delta n'_y = F'_y + 2(f'_z \dot{x} - f'_x \dot{z}) + \dot{f}'_z x - \dot{f}'_x z + \\ \quad + \omega_y (f'_z z + f'_x x) + f'_y (\omega_z z + \omega_x x) - 2y(\omega_z f'_z + \omega_x f'_x), \\ \Delta n'_z = F'_z + 2(f'_x \dot{y} - f'_y \dot{x}) + \dot{f}'_x y - \dot{f}'_y x + \\ \quad + \omega_z (f'_x x + f'_y y) + f'_z (\omega_x x + \omega_y y) - 2z(\omega_x f'_x + \omega_y f'_y). \end{array}\right| \tag{4.295}$$

Formulas (4.294) and (4.295) define the equivalent sensing element errors for some i-th group of instrument errors in the elements and devices of the system. The functions F_x, F_y, F_z, f_x, f_y, f_z as is evident from equalities (4.291) and (4.292), are linear with regard to the instrument errors. Therefore the total equivalent errors $\Delta m'^{\Sigma}_x$, $\Delta m'^{\Sigma}_y$, $\Delta m'^{\Sigma}_z$, $\Delta n'^{\Sigma}_x$, $\Delta n'^{\Sigma}_y$, and $\Delta n'^{\Sigma}_z$ may be represented in the form of sums of the equivalent errors of all of the groups and of the actual instrument errors in the sensing elements:

$$\left.\begin{array}{c}\Delta m_x'^{\Sigma} = \Delta m_x + \sum_i \Delta m_x'^{i}, \quad \Delta m_y'^{\Sigma} = \Delta m_y + \sum_i \Delta m_y'^{i}, \\ \Delta m_z'^{\Sigma} = \Delta m_z + \sum_i \Delta m_z'^{i}; \\ \Delta n_x'^{\Sigma} = \Delta n_x + \sum_i \Delta n_x'^{i}, \quad \Delta n_y'^{\Sigma} = \Delta n_y + \sum_i \Delta n_y'^{i}, \\ \Delta n_z'^{\Sigma} = \Delta n_z + \sum_i \Delta n_z'^{i}. \end{array}\right\} \tag{4.296}$$

4.6.2. Examples of use of the reduction formulas. We will give several examples of the reduction of instrument errors to equivalent sensing element errors (basic errors). For this purpose we will consider errors in the specification of the intensity of the gravitational field and the earth rate, integration errors, and errors in the positioning of newtonometers and gyroscopes on the platform.

In §4.2, in the derivation of the error equations, in addition to the sensing element instrument errors, the errors Δg_x, Δg_y, Δg_z in the specification (or formation) of the characteristics of the gravitational field and also the errors Δu_x, Δu_y, Δu_z in the specification of the earth rate, were retained. These errors were retained in order to demonstrate how they reduce to the basic errors. At the end of §4.2, certain preliminary considerations in this regard were expressed which may now be elaborated upon.

For the errors Δg_x, Δg_y, Δg_z we have:

$$f'_x = f'_y = f'_z = 0, \quad F'_x = \Delta g_x, \quad F'_y = \Delta g_y, \quad F'_z = \Delta g_z. \tag{4.297}$$

Therefore

$$\left.\begin{array}{l}\Delta m'_x = \Delta m'_y = \Delta m'_z = 0, \\ \Delta n'_x = \Delta g_x, \quad \Delta n'_y = \Delta g_y, \quad \Delta n'_z = \Delta g_z. \end{array}\right\} \tag{4.298}$$

If we retain only the errors Δu_x, Δu_y, Δu_z, then

$$\left.\begin{array}{c} f'_x = -\Delta u_x, \quad f'_y = -\Delta u_y, \quad f'_z = -\Delta u_z, \\ F'_x = F'_y = F'_z = 0 \end{array}\right\} \tag{4.299}$$

and in accordance with equalties (4.294) we obtain:

$$\Delta m'_x = -\Delta u_x, \quad \Delta m'_y = -\Delta u_y, \quad \Delta m'_z = -\Delta u_z. \tag{4.300}$$

From formulas (4.295) in turn, we find:

$$\left.\begin{array}{l} \Delta n'_x = -2(\Delta u_y \dot{z} - \Delta u_z \dot{y}) - \Delta \dot{u}_y z + \Delta \dot{u}_z y - \\ \quad - \omega_x (\Delta u_y y + \Delta u_z z) - \Delta u_x (\omega_y y + \omega_z z) + \\ \qquad + 2x (\omega_y \Delta u_y + \omega_z \Delta u_z), \\ \Delta n'_y = -2(\Delta u_z \dot{x} - \Delta u_x \dot{z}) - \Delta \dot{u}_z x + \Delta \dot{u}_x z - \\ \quad - \omega_y (\Delta u_z z + \Delta u_x x) - \Delta u_y (\omega_z z + \omega_x x) + \\ \qquad + 2y (\omega_z \Delta u_z + \omega_x \Delta u_x), \\ \Delta n'_z = -2(\Delta u_x \dot{y} - \Delta u_y \dot{x}) - \Delta \dot{u}_x y + \Delta \dot{u}_y x - \\ \quad - \omega_z (\Delta u_x x + \Delta u_y y) - \Delta u_z (\omega_x x + \omega_y y) + \\ \qquad + 2z (\omega_x \Delta u_x + \omega_y \Delta u_y). \end{array}\right| \tag{4.301}$$

In taking account of errors g_x, g_y, g_z and u_x, u_y, u_z simultaneously, it is necessary, obviously, to sum the corresponding equalities (4.298) and (4.300), (4.301).

Let us see how several characteristic errors are reduced to basic errors.

We will first derive the error due to the counter of the inertial system to equivalent sensing element errors. We will do this using as a example a system which determines the Cartesian coordinates of the object.

Turning to equations (3.59) -- (3.65) defining the ideal operation of this system, we note that the error in the specification of time, i.e., the discrepancy between timer signals and Newtonian time, reduces to errors in the computation of the integrals in equations (3.59)--(3.65).

Here it must be noted that the character of these errors depends on the way the integrals are computed in the system. However, this remark has a wider significance. It is evident that errors in a concrete system realized by concrete elements are always to a great extent determined by these elements. In other words, the reduced errors of various units of an inertial system may be different for various concrete system elements which differ in the principle of their functioning, even if the functions of the latter in the system are identical.

Let us assume, for example, that integration in the system is performed numerically, i.e., by the method of constructing integral sums. This is possible when the system's computational device operates on the principle of a digital computer. Let us assume, further, that, the timer, instead of the true time t, emits the quantity

$$t' = t + \tau(t), \tag{4.302}$$

where $\tau(t)$ is the error in the specificaion of time $[\tau(0) = 0]$. Let us assume that the integral

$$y = \int_0^t x(t)\,dt. \tag{4.303}$$

is computed.

Then, taking equality (4.302) into account, we obtain:

$$y' = \int_0^t x[t + \tau(t)]\,d[t + \tau(t)]. \tag{4.304}$$

This means, therefore, that in the formation of the integral sum the value of the integrand at the moment of time t' is taken in place of t and is multiplied by the time interval $\Delta t'$ instead of by Δt. The upper limit of integration remains the same (t), since in an inertial system integration is performed, as a rule, not up to some moment of time specified by the timer, but up to a moment of time at which the coordinates determined by the system attain specified values. This permits, incidentally, derivation of the error equations of an inertial system by isochronic variation of the ideal equations.

From equalities (4.302), (4.303) and (4.304), performing a change of variables in equality (4.304) and using the mean-value theorem, we obtain:

$$\delta y = y' - y = \tau(t)\,x(t) = \tau(t)\,\dot{y}(t). \tag{4.305}$$

Applying formula (4.305) to equations (3.59), we arrive at the following expressions for F'_x, F'_y, F'_z:

$$\left.\begin{aligned} F'_x &= \frac{d}{dt}\left[\tau\frac{dv_x}{dt} + \frac{d}{dt}\left(\tau\frac{dx}{dt}\right)\right], \\ F'_y &= \frac{d}{dt}\left[\tau\frac{dv_y}{dt} + \frac{d}{dt}\left(\tau\frac{dy}{dt}\right)\right], \\ F'_z &= \frac{d}{dt}\left[\tau\frac{dv_z}{dt} + \frac{d}{dt}\left(\tau\frac{dz}{dt}\right)\right]. \end{aligned}\right\} \tag{4.306}$$

Analogous, from equations (3.60), we find:

$$\left.\begin{aligned} f'_x = \tau\omega_y\omega_z + \frac{d}{dt}(\tau\omega_x), \quad f'_y = \tau\omega_z\omega_x + \frac{d}{dt}(\tau\omega_y), \\ f'_z = \tau\omega_x\omega_y + \frac{d}{dt}(\tau\omega_z). \end{aligned}\right\} \tag{4.307}$$

Formulas (4.294) and (4.295) may now be used to find the equivalent errors.

We have considered the case in which integration is carried out numerically, i.e., by the method of constructing integral sums. But the operations of integration may also be performed by special continuous integrating devices. These devices are based on the use of some physical process, two parameters of which are interrelated in such a way that one of them is proportioned to the time integral of the other. An example of such a device is the gyroscopic integrator, which makes use of the fact that the angle of precession of the gyroscope is proportional to the integral of the applied moment. In this case, integration is performed in natural newtonian time, and the integrator error may be reduced to an error in the coefficient of proportionality which may, in the general case, be a function of time or even a function of the quantity being integrated (for example, range of sensitivity).

Denoting the normalized errors in the scale of integration by k_x^1, k_y^1, k_z^1, k_x^2, k_y^2, k_z^2, we find from equations (3.59):

$$\left.\begin{aligned} F'_x &= \frac{d}{dt}\left[k_x^1 \int_0^t \dot{v}_x\,dt + \frac{d}{dt}\left(k_x^2 \int_0^t \dot{x}\,dt\right)\right], \\ F'_y &= \frac{d}{dt}\left[k_y^1 \int_0^t \dot{v}_y\,dt + \frac{d}{dt}\left(k_y^2 \int_0^t \dot{y}\,dt\right)\right], \\ F'_z &= \frac{d}{dt}\left[k_z^1 \int_0^t \dot{v}_z\,dt + \frac{d}{dt}\left(k_z^2 \int_0^t \dot{z}\,dt\right)\right]. \end{aligned}\right\} \quad (4.308)$$

Analogously, we can use equations (3.60) to find f'_x, f'_y, f'_z, after which formulas (4.294) and (4.295) may be used to determine the corresponding values of the equivalent newtonometer and gyroscopic sensing element errors.

Let us now turn to a characteristic group of errors in inertial navigation systems, namely errors arising as a result of the non-coincidence of the axes of corresponding newtonometers and gyroscopic elements, and also as a result of the axes of sensitivity of the newtonometers and gyroscopes not forming orthogonal trihedra. Errors of this sort arise both as a result of engineering errors in the installation of the newtonometers and gyroscopes on the platform and as a result of deformations in the structural elements of the platform, the newtonometer and the gyroscopes.

As before, let xyz be an orthogonal trihedron along the axes of which the axes of sensitivity of the newtonometers and gyroscopic elements are aligned in the absence of the above-mentioned errors.

The directions along which the axes of sensitivity of the newtonometers are in fact aligned will be designated by x', y', z' (the trihedron $O_{x'y'z'}$ is non-orthogonal). Let us introduce the direction cosines characterizing the directions of the x'y'z' axes relative to the x, y, z axes:

$$\begin{array}{cccc} & x' & y' & z' \\ x & 1 & \varepsilon_{12} & \varepsilon_{13} \\ y & \varepsilon_{21} & 1 & \varepsilon_{23} \\ z & \varepsilon_{31} & \varepsilon_{32} & 1. \end{array} \tag{4.309}$$

where ε_{ij} are small, with $\varepsilon_{ij} \neq \varepsilon_{ji}$.

Analogously, the directions along which the axes of sensitivity of the gyroscopic elements are aligned will be designated by x", y", z" and specified relative to the x, y, z axes by the direction cosines

$$\begin{array}{cccc} & x'' & y'' & z'' \\ x & 1 & e_{12} & e_{13} \\ y & e_{21} & 1 & e_{23} \\ z & e_{31} & e_{32} & 1. \end{array} \tag{4.310}$$

where e_{ij} are small, with $e_{ij} \neq e_{ji}$.

The relative positions of the x', y', and z', and x", y", z" axes are specified therefore by the following direction cosines:

$$\begin{array}{cccc} & x'' & y'' & z'' \\ x' & 1 & e_{12}+\varepsilon_{21} & e_{13}+\varepsilon_{31} \\ y' & \varepsilon_{12}+e_{21} & 1 & e_{23}+\varepsilon_{32} \\ z' & \varepsilon_{13}+e_{31} & \varepsilon_{23}+e_{32} & 1. \end{array} \tag{4.311}$$

Table (4.311) is obtained from tables (4.309) and (4.310) under the assumption that e_{ij} and ε_{ij} are small, such that their squares and products may be ignored.

According to table (4.309) the quantities measuring by the newtonometers aligned along the x', y', z' axes are:

$$\left.\begin{array}{c} n_{x'} = n_x + \varepsilon_{21} n_y + \varepsilon_{31} n_z, \quad n_{y'} = n_y + \varepsilon_{12} n_x + \varepsilon_{32} n_z, \\ n_{z'} = n_z + \varepsilon_{13} n_x + \varepsilon_{23} n_y. \end{array}\right\} \tag{4.312}$$

In accordance with table (4.310) we have by analogy with (4.312):

$$\left.\begin{aligned} m_{x'} &= m_x + e_{21}m_y + e_{31}m_z, \\ m_{y'} &= m_y + e_{12}m_x + e_{32}m_z, \\ m_{z'} &= m_z + e_{13}m_x + e_{23}m_y. \end{aligned}\right| \qquad (4.313)$$

Referring now to the ideal equations (3.59), we find that in the right sides of the coordinate error equations corresponding to them the following additional quantities appear:

$$\left.\begin{aligned} F'_x &= e_{21}n_y + e_{31}n_z - v_z(e_{12}\omega_x + e_{32}\omega_z) + v_y(e_{13}\omega_x + e_{23}\omega_y) - \\ &\quad - \frac{d}{dt}[z(e_{12}\omega_x + e_{32}\omega_z) - y(e_{13}\omega_x + e_{23}\omega_y)], \\ F'_y &= e_{12}n_x + e_{32}n_z - v_z(e_{13}\omega_x + e_{23}\omega_y) + v_x(e_{21}\omega_y + e_{31}\omega_z) - \\ &\quad - \frac{d}{dt}[x(e_{13}\omega_x + e_{23}\omega_y) - z(e_{21}\omega_y + e_{31}\omega_z)], \\ F'_z &= e_{13}n_x + e_{23}n_y - v_y(e_{21}\omega_y + e_{31}\omega_z) + v_x(e_{12}\omega_x + e_{32}\omega_z) - \\ &\quad - \frac{d}{dt}[y(e_{21}\omega_y + e_{31}\omega_z) - x(e_{12}\omega_x + e_{32}\omega_z)], \end{aligned}\right| \qquad (4.314)$$

where according to relations (3.59)

$$\left.\begin{aligned} n_x &= \dot{v}_x + \omega_y v_z - \omega_z v_y - g_x, \\ n_y &= \dot{v}_y + \omega_z v_x - \omega_x v_z - g_y, \\ n_z &= \dot{v}_z + \omega_x v_y - \omega_y v_x - g_z \end{aligned}\right| \qquad (4.315)$$

and

$$\left.\begin{aligned} v_x = \dot{x} + \omega_y z - \omega_z y, \quad v_y = \dot{y} + \omega_z x - \omega_x z, \\ v_z = \dot{z} + \omega_x y - \omega_y x. \end{aligned}\right| \qquad (4.316)$$

Analogously, from equations (3.60) it follows that the following quantities are added to the right sides of equations (4.51):

$$\left.\begin{aligned} &a_{12}(e_{13}\omega_x + e_{23}\omega_y) - a_{13}(e_{12}\omega_x + e_{32}\omega_z), \\ &a_{13}(e_{21}\omega_y + e_{31}\omega_z) - a_{11}(e_{13}\omega_x + e_{23}\omega_y), \\ &a_{11}(e_{12}\omega_x + e_{32}\omega_z) - a_{12}(e_{21}\omega_y + e_{31}\omega_z); \\ &a_{22}(e_{13}\omega_x + e_{23}\omega_y) - a_{23}(e_{12}\omega_x + e_{32}\omega_z), \\ &a_{23}(e_{21}\omega_y + e_{31}\omega_z) - a_{21}(e_{13}\omega_x + e_{23}\omega_y), \\ &a_{21}(e_{12}\omega_x + e_{32}\omega_z) - a_{22}(e_{21}\omega_y + e_{31}\omega_z); \\ &a_{32}(e_{13}\omega_x + e_{23}\omega_y) - a_{33}(e_{12}\omega_x + e_{32}\omega_z), \\ &a_{33}(e_{21}\omega_y + e_{31}\omega_z) - a_{31}(e_{13}\omega_x + e_{23}\omega_y), \\ &a_{31}(e_{12}\omega_x + e_{32}\omega_z) - a_{32}(e_{21}\omega_y + e_{31}\omega_z). \end{aligned}\right| \qquad (4.317)$$

Therefore, the expressions added to the right sides of the error equations (4.84), in accordance with the procedure for deriving equations (4.84) from equalities (4.51), take the form:

$$\left.\begin{array}{c} f'_x = e_{21}\omega_y + e_{31}\omega_z, \quad f'_y = e_{12}\omega_x + e_{32}\omega_z, \\ f'_z = e_{13}\omega_x + e_{23}\omega_y. \end{array}\right\} \qquad (4.318)$$

Comparing expressions (4.318) and (4.313), it will be noticed that expressions (4.318) are a direct consequence of relations (4.313). In the right sides of equalities (4.318) there appear the differences $m_{x''} - m_x$ $m_{y''} - m_y$ $m_{z''} - m_z$ deriving from relations (4.313).

Formulas (4.294) and (4.295) now give the possibility of determing $\Delta n'_x$ $\Delta n'_y$ $\Delta n'_z$ $\Delta m'_x$ $\Delta m'_y$ $\Delta m'_z$ for the case in question.

We have considered examples of the reduction of errors in an inertial navigation system to equivalent sensing element errors for a system determining Cartesian coordinates. The derivation of errors for systems determining curvilinear coordinates does not differ in principle from the procedure used in these examples.

Chapter 5

THE ANALYSIS OF ERROR EQUATIONS. THE RELATION BETWEEN ERRORS IN THE SPECIFICATION OF COORDINATES AND ORIENTATION AND INSTRUMENT ERRORS AND ERRORS IN INITIAL CONDITIONS.

§5.1. General Properties of Error Equations. Possible Means of Investigating Them.

5.1.1. <u>The general properties of error equations</u>. The error equations for an inertial navigation system derived in the preceding chapter include equations (4.83) -- (4.85) which define errors in the determination of the coordinates of the object and equations (4.232), (4.286) and (4.287), defining errors in the specification of the parameters of its orientation in space.

The coordinate error equations reduce to two groups of differential equations (4.83) and (4.84) and to the algebraic relations (4.85).

The first group differential coordinate error equations has the form:

$$
\begin{aligned}
&\delta\ddot{x} + \left[\frac{\mu}{r^3}(y^2 + z^2 - 2x^2) - \omega_y^2 - \omega_z^2\right]\delta x + \\
&\quad + \left(\omega_x\omega_y - \dot{\omega}_z - \frac{3\mu xy}{r^5}\right)\delta y - 2\omega_z\delta\dot{y} + \\
&\quad + \left(\omega_x\omega_z + \dot{\omega}_y - \frac{3\mu xz}{r^5}\right)\delta z + 2\omega_y\delta\dot{z} = \\
&\quad = \Delta n_x - 2(\Delta m_y\dot{z} - \Delta m_z\dot{y}) - \Delta\dot{m}_y z + \Delta\dot{m}_z y - \\
&\quad - \omega_x(\Delta m_y y + \Delta m_z z) - \Delta m_x(\omega_y y + \omega_z z) + \\
&\qquad\qquad + 2x(\omega_y\Delta m_y + \omega_z\Delta m_z). \\
&\delta\ddot{y} + \left[\frac{\mu}{r^3}(z^2 + x^2 - 2y^2) - \omega_z^2 - \omega_x^2\right]\delta y + \\
&\quad + \left(\omega_y\omega_z - \dot{\omega}_x - \frac{3\mu yz}{r^5}\right)\delta z - 2\omega_x\delta\dot{z} + \\
&\quad + \left(\omega_y\omega_x + \dot{\omega}_z - \frac{3\mu yx}{r^5}\right)\delta x + 2\omega_z\delta\dot{x} = \\
&\quad \Delta n_y - 2(\Delta m_z\dot{x} - \Delta m_x\dot{z}) - \Delta\dot{m}_z x + \Delta\dot{m}_x z - \\
&\quad - \omega_y(\Delta m_z z + \Delta m_x x) - \Delta m_y(\omega_z z + \omega_x x) + \\
&\qquad\qquad + 2y(\omega_z\Delta m_z + \omega_x\Delta m_x). \qquad (5.1) \\
&\delta\ddot{z} + \left[\frac{\mu}{r^3}(x^2 + y^2 - 2z^2) - \omega_x^2 - \omega_y^2\right]\delta z + \\
&\quad + \left(\omega_z\omega_x - \dot{\omega}_y - \frac{3\mu zx}{r^5}\right)\delta x - 2\omega_y\delta\dot{x} + \\
&\quad + \left(\omega_z\omega_y + \dot{\omega}_x - \frac{3\mu zy}{r^5}\right)\delta y + 2\omega_x\delta\dot{y} = \\
&\quad = \Delta n_z - 2(\Delta m_x\dot{y} - \Delta m_y\dot{x}) - \Delta\dot{m}_x y + \Delta\dot{m}_y x - \\
&\quad - \omega_z(\Delta m_x x + \Delta m_y y) - \Delta m_z(\omega_x x + \omega_y y) + \\
&\qquad\qquad + 2z(\omega_x\Delta m_x + \omega_y\Delta m_y). \\
&\qquad r^2 = x^2 + y^2 + z^2.
\end{aligned}
$$

Here by analogy with equations (4.83) the errors Δg_x, Δg_y, Δg_z which are reduced to the equivalent newtonometer errors, are omitted from the right sides.

In accordance with expressions (4.86) and (4.87) the initial conditions for the equations of this group are the following:

$$\left.\begin{aligned}
&\delta x(0)=\delta x^0, \quad \delta y(0)=\delta y^0, \quad \delta z(0)=\delta z^0,\\
&\delta\dot{x}(0)=\delta\dot{x}^0=\delta\dot{x}_0^0+(\delta\omega_y^0-\Delta m_y^0)z^0-(\delta\omega_z^0-\Delta m_z^0)y^0,\\
&\delta\dot{y}(0)=\delta\dot{y}^0=\delta\dot{y}_0^0+(\delta\omega_z^0-\Delta m_z^0)x^0-(\delta\omega_x^0-\Delta m_x^0)z^0,\\
&\delta\dot{z}(0)=\delta\dot{z}^0=\delta\dot{z}_0^0+(\delta\omega_x^0-\Delta m_x^0)y^0-(\delta\omega_y^0-\Delta m_y^0)x^0.
\end{aligned}\right\} \quad (5.2)$$

The second group of differential error equations is the following:

$$\left.\begin{aligned}
&\dot{\theta}_{1x}+\omega_y\theta_{1z}-\omega_z\theta_{1y}=\Delta m_x,\\
&\dot{\theta}_{1y}+\omega_z\theta_{1x}-\omega_x\theta_{1z}=\Delta m_y,\\
&\dot{\theta}_{1z}+\omega_x\theta_{1y}-\omega_y\theta_{1x}=\Delta m_z
\end{aligned}\right\} \quad (5.3)$$

with the initial conditions

$$\theta_{1x}(0)=\theta_{1x}^0, \quad \theta_{1y}(0)=\theta_{1y}^0, \quad \theta_{1z}(0)=\theta_{1z}^0. \quad (5.4)$$

Finally, the coordinate error equations include the algebraic relations:

$$\left.\begin{aligned}
&\delta x_1=\theta_{1y}z-\theta_{1z}y, \quad \delta y_1=\theta_{1z}x-\theta_{1x}z,\\
&\qquad \delta z_1=\theta_{1x}y-\theta_{1y}x;\\
&\delta x_3=\delta x+\delta x_1, \quad \delta y_3=\delta y+\delta y_1,\\
&\qquad \delta z_3=\delta z+\delta z_1.
\end{aligned}\right\} \quad (5.5)$$

The error equations defining the parameters of the orientation of the object in space include the set of three systems of algebraic equalities (4.232), (4.286) and (4.287).

The first system consists of three equalities of the form:

$$\left.\begin{aligned}
\theta_x &= -\theta_{1x} + a_{i1}\left\{\delta a_{i1} + \frac{\partial a_{i1}}{\partial \xi^s}\left[(\theta_{1y} z - \theta_{1z} y) a_{s1} + \right.\right.\\
&\qquad \left.\left. + (\theta_{1z} x - \theta_{1x} z) a_{s2} + (\theta_{1x} y - \theta_{1y} x) a_{s3}\right]\right\}, \\
\theta_y &= -\theta_{1y} + a_{i2}\left\{\delta a_{i2} + \frac{\partial a_{i2}}{\partial \xi^s}\left[(\theta_{1y} z - \theta_{1z} y) a_{s1} + \right.\right.\\
&\qquad \left.\left. + (\theta_{1z} x - \theta_{1x} z) a_{s2} + (\theta_{1x} y - \theta_{1y} x) a_{s3}\right]\right\}, \\
\theta_z &= -\theta_{1z} + a_{i3}\left\{\delta a_{i3} + \frac{\partial a_{i3}}{\partial \xi^s}\left[(\theta_{1y} z - \theta_{1z} y) a_{s1} + \right.\right.\\
&\qquad \left.\left. + (\theta_{1z} x - \theta_{1x} z) a_{s2} + (\theta_{1x} y - \theta_{1y} x) a_{s3}\right]\right\},
\end{aligned}\right| \tag{5.6}$$

where

$$\delta a_{ij} = \frac{\partial a_{ij}}{\partial \xi^s}\,\delta \xi^s, \tag{5.7}$$

and the summation over s and i is performed from 1 to 3.

The second system of equalities defining orientation errors includes the relations

$$\left.\begin{aligned}
\theta_{3x} &= -\Delta\alpha\cos\beta\cos\gamma - \Delta\beta\sin\gamma, \\
\theta_{3y} &= \Delta\alpha\cos\beta\sin\gamma - \Delta\beta\cos\gamma, \\
\theta_{3z} &= -\Delta\alpha\sin\beta - \Delta\gamma.
\end{aligned}\right\} \tag{5.8}$$

Finally, the orientation error equations include the following relations deriving from formula (4.287):

$$\left.\begin{aligned}
\theta_{4x} = \theta_x + \theta_{3x}, \quad \theta_{4y} = \theta_y + \theta_{3y}, \\
\theta_{4z} = \theta_z + \theta_{3z}.
\end{aligned}\right\} \tag{5.9}$$

Equations (5.1) define the errors δx, δy, δz in the Cartesian coordinates x, y, z in the trihedron 0_1xyz with its origin at the center of the earth and which rotates relative to the basic Cartesian coordinate system $0_1\xi_*\eta_*\zeta_*$ $(0_1\xi^1\xi^2\xi^3)$ with an angular velocity $\vec{\omega}$. In the left sides of these equations the quantities ω_x, ω_y, ω_z are the projections of the vector $\vec{\omega}$ on the x, y, z axes, and μ is the product of the gravitational constant and the mass of the earth.

The right sides of equations (5.1) contain Δn_x, Δn_y, Δn_z, the newtonometer instrument errors, and Δm_x, Δm_y, Δm_z, the instrument errors deriving from the gyroscopic elements. The latter are errors in the measurement or formation of the projections ω_x, ω_y, ω_z of the absolute rate of rotation of the xyz trihedron about its axis.

Equations (5.3) characterize the error, caused by the instrument errors Δm_x, Δm_y, Δm_z, in the specification in the orientation of trihedron 0_1xyz relative to trihedron $0_1\xi^1\xi^2\xi^3$, fixed invariantly to the bearings the center of the earth to distant stars. The quantities θ_{1x}, θ_{1y}, θ_{1z} are projections on the x, y, z axes of the small rotation vector $\vec{\theta}_1$ designating this error. The initial conditions θ^0_{1x}, θ^0_{1y}, θ^0_{1z} of angles θ_{1x}, θ_{1y}, θ_{1z} refer, as do the initial values (5.2), to the moment at which the inertial system begins to operate.

The first three equalities (5.5) are expressions for the errors δx_1, δy_1, δz_1 in the specification of the coordinates of the object relative to trihedron $0_1\xi^1\xi^2\xi^3$. The errors θx_1, θy_1, θz_1 result from the orientation errors θ_{1x}, θ_{1y}, θ_{1z} and characterize the errors in the specification of the coordinates of the object in the coordinate system $0_1\xi^1\xi^2\xi^3$ given in terms of projections on the x, y, z axes.

The final three equalities (5.5) are expressions for the total errors δx_3, δy_3, δz_3 in the specification of the coordinates of the object. The total errors, as is evident from these equalities, are the sum of the errors deriving from equations (5.1) of the group, and the errors δx_1, δy_1, δz_1, deriving from equations (5.3) of the second group.

Equations (5.6) -- (5.9) give the possibility of finding the errors in the orientation of the object.

According to equations (5.9), angle θ_4 is composed of angles θ and θ_3. Angle θ characterizes the total error in the specification of the position of the x, y, z axes relative to which the orientation of the object is determined. The projections θ_x, θ_y, θ_z of the vector $\vec{\theta}$ on the x, y, z axes are computed according to formula (5.6). These same formulas serve to determine the errors in the orientation of the inertial system platform, if its unperturbed position is a function of the coordinates. In this case the trihedron O_1xyz is considered as rigidly bound to the platform.

The first terms θ_{1x}, θ_{1y}, θ_{1z} of the right sides of formula (5.6) are the solutions to equations (5.3) and characterize that portion of the error in the orientation of trihedron O_1xyz which would occur if the operational algorithm of the inertial system did not presuppose that the orientation of this trihedron was a function of the coordinates determined by the system. The second terms of the right sides of formulas (5.6) define the orientation error caused by errors in the specification of the coordinates of the object for the general case in which the operation algorithm of the inertial system defines the orientation of trihedron O_1xyz as a function of the coordinates being determined. The quantities α_{ij} (ξ^1, ξ^2, ξ^3) which enter into the right sides of formulas (5.6) are the direction cosines of the x, y, z axes relative to the ξ^1, ξ^2, ξ^3 axes.

It is evident that the second terms of the right sides of formulas (5.6) are functions both of δx, δy, δz and δx_1, δy_1, δz_1. In fact, $\delta\alpha_{ij}$ in the brackets are, according to equalities (5.7), functions of $\delta\xi^s$, i.e., of δx, δy, δz. The expressions in parentheses are equal to δx_1, δy_1, δz_1, respectively [see relations (5.5)].

The projections θ_{3x}, θ_{3y}, θ_{3z} of the angle θ_3 appear in formulas (5.8). The angle θ_3 characterizes the error in the specification of the orientation of the object relative to the x, y, z axes of the inertial system platform, caused by the instrument errors $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$ in sampling angles α, β, γ of rotation of the wheels of the platform gimbal rings on the object.

We will now consider several general properties of the error equations (5.1) -- (5.9).

Let us first recall those of their properties which were examined in the preceding chapter in the process of deriving and transforming the error equations.

The error equations of any inertial navigation system determining the position of an object in arbitrary curvilinear coordinates, in general non-orthogonal and non-stationary, reduce to equations (5.1) -- (5.9).

Equations (5.1) -- (5.9) permit us to take into account the instrument errors of any element or device in the system, since these errors may be reduced to equivalent basic instrument errors, i.e., to newtonometer errors Δn_x, Δn_y, Δn_z and the errors Δm_x, Δm_y, Δm_z deriving from the gyroscopic measuring elements.

The homogeneous equations (5.1) are exact equations for the perturbations δx, δy, δz (with the exception of the terms of these equations resulting from variation in the strength of the gravitational field, which contain only the linear portion of the corresponding increments). The homogeneous equations (5.3) are first order approximations. When it is necessary to consider exact homogeneous equations of the second group, the homogeneous equations (4.51), to which equations (5.3) correspond to within the second order of smallness are taken.

Continuing the discussion of equations (5.1) -- (5.9), let us now consider the following properties of them.

The first group of equations (5.1) permit an interesting analogy: they are, essentially, perturbation equations for the motion of a mass point in the earth's gravitational field under the influence of external forces, i.e., these equations are perturbation equations (to a first approximation -- equations in variations) for Newton's general equations of motion written in terms of projections on the movable x, y, z axes.

This analogy is entirely valid and easily predictable. It derives rom the fact that the ideal equations of an inertial system, by variation f which the error equations (5.1) were obtained, are, essentially, he equations of motion of a mass point (the sensitive mass of the ewtonometer) under the influence of gravitational forces and some ystem of surface forces. This statement derives from the form of the asic inertial navigation equation (1.88). Therefore, as has already een noted, the ideal operational equations for an inertial system in n arbitrary (curvilinear) reference grid, which were obtained in Chapter 3, may simultaneously be treated as the Newtonian equations of motion of a mass point in this reference grid.

Equations (5.3) are analogous in form to the well known Poisson equations, to which reduces the problem of determining the orientation of a moving (rotating) trihedron relative to an immobile (invariantly oriented) trihedron using the well-known projections of the absolute rate of rotation of a moving trihedron on its axes. This results from the fact that equations (5.3) were obtained by varying the Poisson equations (3.60).

The coordinate and orientation error equations (5.1) -- (5.9) were tained in terms of projections on the x, y, z axes, which in the se of a system determining Cartesian coordinates were rigidly bound the gyroscopic platform or, equivalently, to the directions of the s of sensitivity of the newtonometers. Equations (5.1) -- (5.9) , however, essentially vector (invariant) equations and therefore also, if necessary, be written in terms of projections on the axes any other coordinate system.

Equations (5.1) -- (5.5) retain their form if the transition is made to their projections on the axes of any other orthogonal trihedron having a common origin with trihedron O_1xyz and freely rotating relative to it. Equations (5.1) -- (5.5) allow, consequently, a group of rotations. The existence of a group of rotations follows from the arbitrary specification of the vector $\vec{\omega}$. It may also be demonstrated with the aid of the corresponding change of variables.

In passing from trihedron O_1xyz to another which may be designated as $O_1x'y'z'$, $\delta x'$, $\delta y'$, $\delta z'$ in equations (5.1) -- (5.5) should be substituted for δx, δy, δz, and $\theta_{1x'}$, $\theta_{1y'}$, $\theta_{1z'}$ for θ_{1x}, θ_{1y}, θ_{1z}, and projections ω_x, ω_y, ω_z should be replaced by the projections $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ of the absolute angular velocity of trihedron $O_1x'y'z'$ on its axes. Moreover, it is necessary to replace Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z by the projection $\Delta n_{x'}$, $\Delta n_{y'}$, $\Delta n_{z'}$, $\Delta m_{x'}$, $\Delta m_{y'}$, $\Delta m_{z'}$ of the vectors $\Delta\vec{n}$ and $\Delta\vec{m}$ on the x', y', z' axes. Analogously, x, y, z should be replaced by the projections x', y', z' of the vector $\vec{r}$ on the x', y', z' axes.

The relation between δx, δy, δz, x, y, z, θ_{1x}, θ_{1y}, θ_{1z}, Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z with $\delta x'$, $\delta y'$, $\delta z'$, x', y', z', θ_{1x}', θ_{1y}', θ_{1z}', $\Delta n_x'$, $\Delta n_y'$, $\Delta n_z'$, $\Delta m_x'$, $\Delta m_y'$, $\Delta m_z'$ is determined by the table of the direction cosines between the x, y, z axes and the x', y', z' axes:

$$\begin{array}{cccc} & x' & y' & z' \\ x & \beta'_{11} & \beta'_{12} & \beta'_{13} \\ y & \beta'_{21} & \beta'_{22} & \beta'_{23} \\ z & \beta'_{31} & \beta'_{32} & \beta'_{33} \end{array} \qquad (5.10)$$

In order to obtain ω'_x, ω'_y, ω'_z this table and relations of the form (4.221) may be used. Thus, for example,

$$\omega'_x = \omega_x\beta'_{11} + \omega_y\beta'_{21} + \omega_z\beta'_{31} + \dot{\beta}'_{12}\beta'_{13} + \dot{\beta}'_{22}\beta'_{23} + \dot{\beta}'_{32}\beta'_{33}.$$

In accordance with what was said above, we will henceforth consider that equations (5.1) -- (5.5) are written in terms of projections on the axes of the arbitrary trihedron xyz, rotating relative to the trihedron $\xi^1\xi^2\xi^3$ with an angular velocity $\vec{\omega}$.

Let us consider equations (5.6). They were derived in terms of projections on the x, y, z axes, which are either rigidly bound to the platform of the inertial system (when errors in the orientation of the platform are being considered), or are axes relative to which the orientation of the object is determined (when errors in this orientation are being considered).

The relative position of the xyz and $\xi^1\xi^2\xi^3$ trihedra is characterized by the direction cosines $\alpha_{ij}(\xi^1, \xi^2, \xi^3)$.

The projections of equations (5.6) on the axes of the freely rotating trihedron x', y', z' are obtained using table (5.10). After obvious transformations the expressions for θ_x', θ_y', θ_z' take the form:

$$\left.\begin{aligned}
\theta_{x'} = -\theta_{1x'} &+ \left(\alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi^s}\beta'_{11} + \alpha_{i1}\frac{\partial\alpha_{i3}}{\partial\xi^s}\beta'_{21} + \right.\\
&\left. + \alpha_{i2}\frac{\partial\alpha_{i1}}{\partial\xi^s}\beta'_{31}\right)\left[\delta\xi^{s\prime} + (\theta_{1y'}z' - \theta_{1z'}y')\alpha'_{s1} + \right.\\
&\left. + (\theta_{1z'}x' - \theta_{1x'}z')\alpha'_{s2} + (\theta_{1x'}y' - \theta_{1y'}x')\alpha'_{s3}\right].
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\theta_{y'} = -\theta_{1y'} &+ \left(\alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi^s}\beta'_{12} + \alpha_{i1}\frac{\partial\alpha_{i3}}{\partial\xi^s}\beta'_{22} + \right.\\
&\left. + \alpha_{i2}\frac{\partial\alpha_{i1}}{\partial\xi^s}\beta'_{32}\right)\left[\delta\xi^{s\prime} + (\theta_{1y'}z' - \theta_{1z'}y')\alpha'_{s1} + \right.\\
&\left. + (\theta_{1z'}x' - \theta_{1x'}z')\alpha'_{s2} + (\theta_{1x'}y' - \theta_{1y'}x')\alpha'_{s3}\right].\\
\theta_{z'} = -\theta_{1z'} &+ \left(\alpha_{i3}\frac{\partial\alpha_{i2}}{\partial\xi^s}\beta'_{13} + \alpha_{i1}\frac{\partial\alpha_{i3}}{\partial\xi^s}\beta'_{23} + \right.\\
&\left. + \alpha_{i2}\frac{\partial\alpha_{i1}}{\partial\xi^s}\beta'_{33}\right)\left[\delta\xi^{s\prime} + (\theta_{1y'}z' - \theta_{1z'}y')\alpha'_{s1} + \right.\\
&\left. + (\theta_{1z'}x' - \theta_{1x'}z')\alpha'_{s2} + (\theta_{1x'}y' - \theta_{1y'}x')\alpha'_{s3}\right].
\end{aligned}\right\} \tag{5.11a}$$

Here, as in equations (5.6), summation over the indices i and s is taken from 1 to 3.

In the right sides of these formulas α'_{sj} designates the direction cosines of the x', y', z' axes relative to the ξ^1, ξ^2, ξ^3 axes. The meaning of the indicies s and j is the same as that of i and j in table (3.16) for α_{ij}.

As may be seen from relations (5.11a), equations (5.6) do not retain their form in the conversion to the arbitrary trihedron x', y', z'. This is easily foreseen, since equations (5.6) clearly contain the direction cosines α_{ij} (ξ^1, ξ^2, ξ^3) characterizing the position of the axes of the platform (or the directions of the axes of sensitivty of the newtonometers) as a function of coordinates ξ^s.

Substituting α'_{ij} for α_{ij} in equalities (5.11a), we may rewrite them as follows:

$$
\left.
\begin{aligned}
\theta_x = -\theta_{1x} &+ \left(\alpha'_{i3}\frac{\partial\alpha'_{i2}}{\partial\xi^s}\beta'_{11} + \alpha'_{i1}\frac{\partial\alpha'_{i3}}{\partial\xi^s}\beta'_{21} + \right.\\
&\left. + \alpha'_{i2}\frac{\partial\alpha'_{i1}}{\partial\xi^s}\beta'_{31}\right)\left[\delta\xi^s + (\theta_{1y}z - \theta_{1z}y)\alpha_{s1} + \right.\\
&\left. + (\theta_{1z}x - \theta_{1x}z)\alpha_{s2} + (\theta_{1x}y - \theta_{1y}x)\alpha_{s3}\right].\\
\theta_y = -\theta_{1y} &+ \left(\alpha'_{i3}\frac{\partial\alpha'_{i2}}{\partial\xi^s}\beta'_{12} + \alpha'_{i1}\frac{\partial\alpha'_{i3}}{\partial\xi^s}\beta'_{22} + \right.\\
&\left. + \alpha'_{i2}\frac{\partial\alpha'_{i1}}{\partial\xi^s}\beta'_{32}\right)\left[\delta\xi^s + (\theta_{1y}z - \theta_{1z}y)\alpha_{s1} + \right.\\
&\left. + (\theta_{1z}x - \theta_{1x}z)\alpha_{s2} + (\theta_{1x}y - \theta_{1y}x)\alpha_{s3}\right].
\end{aligned}
\right|
$$

$$
\left.
\begin{aligned}
\theta_z = -\theta_{1z} &+ \left(\alpha'_{i3}\frac{\partial\alpha'_{i2}}{\partial\xi^s}\beta'_{13} + \alpha'_{i1}\frac{\partial\alpha'_{i3}}{\partial\xi^s}\beta'_{23} + \right.\\
&\left. + \alpha'_{i2}\frac{\partial\alpha'_{i1}}{\partial\xi^s}\beta'_{33}\right)\left[\delta\xi^s + (\theta_{1y}z - \theta_{1z}y)\alpha_{s1} + \right.\\
&\left. + (\theta_{1z}x - \theta_{1x}z)\alpha_{s2} + (\theta_{1x}y - \theta_{1y}x)\alpha_{s3}\right].
\end{aligned}
\right|
$$

(5.11b)

Equations (5.11b) may be considered as projected on the axes of some arbitrarily oriented trihedron xyz. We will henceforth consider this trihedron as coinciding with the one on which equations (5.1) -- (5.5) are projected. In equations (5.11b) the direction cosines β'_{ij} characterize the relative position of this trihedron and the trihedron in terms of projections on the axes of which equations (5.6) are written. The position of the latter trihedron relative to the ξ^1, ξ^2, ξ^3 axes is characterized by the direction cosine β'_{ij}, and the position of the xyz trihedron relative to the ξ^1, ξ^2, ξ^3 axes is characterized by the direction cosines α_{ij}.

Let us consider equations (5.8). They are written in terms of projections on the axes of the trihedron bound to the platform of the inertial system, but they may also be projected on the axes of an arbitrary trihedron xyz. As in equations (5.6), they do not retain their form in projection. Using table (5.10), we obtain from relations (5.8):

$$\left.\begin{aligned}
0_{3x} &= -(\Delta\alpha\cos\beta\cos\gamma + \Delta\beta\sin\gamma)\beta'_{11} + \\
&\quad + (\Delta\alpha\cos\beta\sin\gamma - \Delta\beta\cos\gamma)\beta'_{12} - (\Delta\alpha\sin\beta + \Delta\gamma)\beta'_{13}, \\
0_{3y} &= -(\Delta\alpha\cos\beta\cos\gamma + \Delta\beta\sin\gamma)\beta'_{21} + \\
&\quad + (\Delta\alpha\cos\beta\sin\gamma - \Delta\beta\cos\gamma)\beta'_{22} - (\Delta\alpha\sin\beta + \Delta\gamma)\beta'_{23}, \\
0_{3z} &= -(\Delta\alpha\cos\beta\cos\gamma + \Delta\beta\sin\gamma)\beta'_{31} + \\
&\quad + (\Delta\alpha\cos\beta\sin\gamma - \Delta\beta\cos\gamma)\beta'_{32} - (\Delta\alpha\sin\beta + \Delta\gamma)\beta'_{33}.
\end{aligned}\right| \qquad (5.12)$$

Now equations (5.9) may also be written in terms of projections on the axes of an arbitrarily oriented trihedron.

Thus, equations (5.1) -- (5.5), (5.11b), (5.7), (5.12) and (5.9) are error equations projected on the axes of the same trihedron xyz. Since the orientation of trihedron xyz relative to the basic Cartesian coordinate system is arbitrary, in the analysis of the equations in question it may be selected in various ways. Careful selection of this trihedron can in many ways facilitate analysis of the error equations.

5.1.2. Various representations of the error equations. Henceforth it will be convenient to use, in addition to the arbitrary position of the xyz trihedron, the following alternatives for selection of its orientation.

In one of these alternatives the trihedron on whose axes the error equations are projected is taken to be fixed in space, as, for example, trihedron $\xi^1\xi^2\xi^3$ might be. In this case

$$\omega_x = \omega_y = \omega_z = 0, \qquad (5.13)$$

should be substituted into equations (5.1), causing them to take the form:

$$\left.\begin{aligned}
&\delta\ddot{x}+\frac{\mu}{r^5}[(y^2+z^2-2x^2)\delta x-3xy\,\delta y-3xz\,\delta z]=\\
&\quad=\Delta n_x-2(\Delta m_y\dot{z}-\Delta m_z\dot{y})-\Delta\dot{m}_y z+\Delta\dot{m}_z y,\\
&\delta\ddot{y}+\frac{\mu}{r^5}[(z^2+x^2-2y^2)\delta y-3yz\,\delta z-3yx\,\delta x]=\\
&\quad=\Delta n_y-2(\Delta m_z\dot{x}-\Delta m_x\dot{z})-\Delta\dot{m}_z x+\Delta\dot{m}_x z,\\
&\delta\ddot{z}+\frac{\mu}{r^5}[(x^2+y^2-2z^2)\delta z-3zx\,\delta x-3zy\,\delta y]=\\
&\quad=\Delta n_z-2(\Delta m_x\dot{y}-\Delta m_y\dot{x})-\Delta\dot{m}_x y+\Delta\dot{m}_y x,\\
&r^2=x^2+y^2+z^2.
\end{aligned}\right| \tag{5.14}$$

For equations (5.3) under conditions (5.13) we obtain:

$$\dot{\theta}_{1x}=\Delta m_x,\quad \dot{\theta}_{1y}=\Delta m_y,\quad \dot{\theta}_{1z}=\Delta m_z. \tag{5.15}$$

The projections of the absolute angular velocity $\vec{\omega}$ do not enter into equations (5.5), and so the form of these equations remains unchanged. In equations (5.11b) α_{sj} should be equal to 0, if $s \neq j$, and, in addition, α'_{ij} should be set equal to β'_{ij}. The other equations do not change.

We note that if the vector equations (4.81), i.e., the equations

$$\left.\begin{aligned}
&\delta\ddot{r}+2\omega\times\delta\dot{r}+\dot{\omega}\times\delta r+\omega\times(\omega\times\delta r)+\\
&\quad+\frac{\mu}{r^3}\delta r-\frac{\mu r}{r^3}\frac{3(r\cdot\delta r)}{r^2}=\\
&\quad=\Delta n-2\Delta m\times\dot{r}-\Delta\dot{m}\times r-\\
&\quad-\Delta m\times(\omega\times r)-\omega\times(\Delta m\times r),\\
&\dot{\theta}_1+\omega\times\theta_1=\Delta m,\\
&\delta r_1=\theta_1\times r,\quad \delta r_3=\delta r+\delta r_1,
\end{aligned}\right| \tag{5.16}$$

correspond to equations (5.1), (5.3), and (5.5), then, clearly, the vector equations

$$\left.\begin{aligned}
&\delta\ddot{r}+\frac{\mu}{r^3}\delta r-\frac{\mu r}{r^3}\frac{3(r\cdot\delta r)}{r^2}=\\
&\quad=\Delta n-2\Delta m\times\dot{r}-\Delta\dot{m}\times r,\\
&\dot{\theta}_1=\Delta m,\\
&\delta r_1=\theta_1\times r,\quad \delta r_3=\delta r+\delta r_1
\end{aligned}\right| \tag{5.17}$$

correspond to equations (5.14) and (5.15).

Another trihedron which is convenient in the analysis of the error equations is often one having one of its axes directed along the radius vector $\vec{r}$. In this case the xyz trihedron becomes a Darboux trihedron on a sphere of radius $\vec{r}$ surrounding the earth.

Let the z axis of trihedron xyz be directed along the radius vector $\vec{r}$. Then,

$$z = r, \quad x = y = 0. \tag{5.18}$$

Taking this into account, we obtain from equations (5.1):

$$\left.\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y - \\
&\quad - 2\omega_z\delta\dot{y} + (\omega_x\omega_z + \dot{\omega}_y)\delta z + 2\omega_y\delta\dot{z} = \\
&\quad = \Delta n_x - 2\Delta m_y\dot{r} - \Delta\dot{m}_y r - \omega_x\Delta m_z r - \omega_z\Delta m_x r.
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + (\omega_y\omega_z - \dot{\omega}_x)\delta z - \\
&\quad - 2\omega_x\delta\dot{z} + (\omega_y\omega_x + \dot{\omega}_z)\delta x + 2\omega_z\delta\dot{x} = \\
&\quad = \Delta n_y + 2\Delta m_x\dot{r} + \Delta\dot{m}_x r - \omega_y\Delta m_z r - \omega_z\Delta m_y r. \\
&\delta\ddot{z} - \left(\frac{2\mu}{r^3} + \omega_x^2 + \omega_y^2\right)\delta z + (\omega_z\omega_x - \dot{\omega}_y)\delta x - \\
&\quad - 2\omega_y\delta\dot{x} + (\omega_z\omega_y + \dot{\omega}_x)\delta y + 2\omega_x\delta\dot{y} = \\
&\qquad\qquad = \Delta n_z + 2r(\omega_x\Delta m_x + \omega_y\Delta m_y).
\end{aligned}\right\} \tag{5.19}$$

Since xyz is a moving trihedron, in equations (5.19)

$$r\omega_x = -v_y, \quad r\omega_y = v_x, \quad \dot{r} = v_z, \tag{5.20}$$

where v_x, v_y, v_z are the projection of the absolute angular velocity of the point O (i.e., of the object) on the x, y, z axes.

Equations (5.3) do not change, and so x and y do not occur in them. Equations (5.5) take the form:

$$\left.\begin{aligned}
&\delta x_1 = \theta_{1y} r, \quad &&\delta y_1 = -\theta_{1x} r, \quad &&\delta z_1 = 0, \\
&\delta x_3 = \delta x + \delta x_1, \quad &&\delta y_3 = \delta y + \delta y_1, \quad &&\delta z_3 = \delta z.
\end{aligned}\right\} \tag{5.21}$$

Equations (5.11b) also simplify significantly in this case. They may now be written in the following form:

$$\left.\begin{aligned}
\Theta_x &= -\Theta_{1x} + \left(\alpha'_{i3}\frac{\partial\alpha'_{i2}}{\partial\xi^s}\beta'_{11} + \alpha'_{i1}\frac{\partial\alpha'_{i3}}{\partial\xi^s}\beta'_{21} + \right. \\
&\qquad \left. + \alpha'_{i2}\frac{\partial\alpha'_{i1}}{\partial\xi^s}\beta'_{31}\right)\delta\xi^s + r(\Theta_{1y}\alpha_{31} - \Theta_{1z}\alpha_{21}), \\
\Theta_y &= -\Theta_{1y} + \left(\alpha'_{i3}\frac{\partial\alpha'_{i2}}{\partial\xi^s}\beta'_{12} + \alpha'_{i1}\frac{\partial\alpha'_{i3}}{\partial\xi^s}\beta'_{22} + \right. \\
&\qquad \left. + \alpha'_{i2}\frac{\partial\alpha'_{i1}}{\partial\xi^s}\beta'_{32}\right)\delta\xi^s + r(\Theta_{1y}\alpha_{31} - \Theta_{1z}\alpha'_{21}), \\
\Theta_z &= -\Theta_{1z} + \left(\alpha'_{i3}\frac{\partial\alpha'_{i2}}{\partial\xi^s}\beta'_{13} + \alpha'_{i1}\frac{\partial\alpha'_{i3}}{\partial\xi^s}\beta'_{23} + \right. \\
&\qquad \left. + \alpha'_{i2}\frac{\partial\alpha'_{i1}}{\partial\xi^s}\beta'_{33}\right)\delta\xi^s + r(\Theta_{1y}\alpha_{31} - \Theta_{1x}\alpha_{32})
\end{aligned}\right\} \tag{5.22}$$

As a result of the fact that $\vec{z} = \vec{r}/r$, in equations (5.22)

$$\alpha_{i3} = \frac{\xi^i}{r}. \tag{5.23}$$

Of special interest below will be the case in which $\alpha_{ij} = \beta'_{ij}$, i.e., in which the trihedron bound to the platform of the inertial system will also be a moving trihedron. In this case $\beta'_{ii} = 1$, and $\beta'_{ij} = 0$ (with $i \neq j$) and equations (5.22) reduce, as expected, to equations (5.6). The further simplification of these equations is based on relations (5.23). The corresponding transformation was carried out in §4.5 and the final result expressed in formulas (4.248), which we cite here:

$$\left.\begin{aligned}
&\Theta_x = -\frac{\delta y}{r}, \quad \Theta_y = \frac{\delta x}{r}, \\
&\Theta_z = -\Theta_{1z} + \alpha_{i2}\delta\alpha_{i1} + r\alpha_{i3} \quad \ldots{}_y\alpha_{31} - \Theta_{1x}\alpha_{32}).
\end{aligned}\right\} \tag{5.24}$$

As was shown above, if α_{ij} are not functions of the coordinates ξ^s, but only functions of time, i.e., if

$$\alpha_{ij} = \alpha_{ij}(t), \tag{5.25}$$

then we obtain from equations (5.22) in terms of projections on the axes of the moving trihedron where θ_{1x}, θ_{1y}, θ_{1z} are the solutions to equations (5.3).

In equalities (5.12) the x, y, z, coordinates do not appear, and condition (5.18) does not effect them.

The Darboux trihedron on a sphere surrounding the earth, in terms of projections on whose axes equations (5.19), (5.21), (5.22) and (5.24) were written, is not yet fully defined relative to the basic Cartesian coordinate system. Only the direction of its z axis is determined. The moving trihedron may be fully determined in the same way as in §3.5. We may, for example, set

$$\theta_x = -\theta_{1x}, \quad \theta_y = -\theta_{1y}, \quad \theta_z = -\theta_{1z}, \tag{5.27}$$

and then we will obtain a so-called free-azimuth trihedron, or place the y axis in a plane containing the ξ^3 axis, i.e., the axis of rotation of the earth. In the latter case the xyz trihedron becomes a geocentric moving trihedron oriented to the points of the compass (the y axis pointing to the north).

If $\omega_z = 0$, then from equations (5.3), (5.19) and (5.21) we obtain the following error equations:

$$\left.\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x + \omega_x\omega_y\,\delta y + \dot{\omega}_y\,\delta z + 2\omega_y\,\delta\dot{z} = \\
&\qquad = \Delta n_x - 2\Delta m_y\dot{r} - \Delta\dot{m}_y r - \omega_x\,\Delta m_z r, \\
&\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_x^2\right)\delta y - \dot{\omega}_x\,\delta z - 2\omega_x\,\delta\dot{z} + \omega_x\omega_y\,\delta x = \\
&\qquad = \Delta n_y + 2\Delta m_x\dot{r} + \Delta\dot{m}_x r - \omega_y\,\Delta m_z r, \\
&\delta\ddot{z} - \left(\frac{2\mu}{r^3} + \omega_x^2 + \omega_y^2\right)\delta z - \dot{\omega}_y\,\delta x - 2\omega_y\,\delta\dot{x} + \\
&\quad + \dot{\omega}_x\,\delta y + 2\omega_x\,\delta\dot{y} = \Delta n_z + 2r\,(\omega_x\,\Delta m_x + \omega_y\,\Delta m_y);
\end{aligned}\right| \tag{5.28}$$

$$\left.\begin{aligned}
\dot{\theta}_{1x} + \omega_y\theta_{1z} = \Delta m_x, \quad \dot{\theta}_{1y} - \omega_x\theta_{1z} = \Delta m_y, \\
\dot{\theta}_{1z} + \omega_x\theta_{1y} - \omega_y\theta_{1x} = \Delta m_z;
\end{aligned}\right\} \tag{5.29}$$

$$\left.\begin{aligned}
\delta x_1 = \theta_{1y} r, \qquad \delta y_1 = -\theta_{1x} r, \\
\delta x_2 = \delta x + \delta x_1, \quad \delta y_2 = \delta y + \delta y_1, \quad \delta z_2 = \delta z.
\end{aligned}\right\} \tag{5.30}$$

Equation (5.24) converts to equations (4.252):

$$\left.\begin{array}{c}\theta_x = -\frac{\delta y}{r}, \quad \theta_y = \frac{\delta x}{r}, \\ \theta_z = -\theta_{1z}.\end{array}\right\} \qquad (5.31)$$

Equations (5.12) do not change.

Finally, if the moving trihedron xyz is oriented by the cardinal points, the direction cosines of its axes relative to the ξ^1, ξ^2, ξ^3 axes are expressed in terms of the coordinates ξ^1, ξ^2, ξ^3 by formulas (4.253) and (5.23).

In this case the coordinate error equations will be equations (5.3), (5.19) and (5.21), and the orientation error equations will be relations (5.12), (5.9) and the equalities

$$\left.\begin{array}{c}\theta_x = -\frac{\delta y}{r}, \quad \theta_y = \frac{\delta x}{r}, \\ \theta_z = -\theta_{1z} + \frac{\delta x}{r}\tan\varphi.\end{array}\right\} \qquad (5.32)$$

to which relations (5.24) reduce.

If trihedron xyz is a moving trihedron of a geodetic reference grid, then equations (5.3), (5.19) and (5.21) remain valid, as do the first two equations (5.32). In the last equation (5.32), in accordance with the last equality (4.258), φ should be replaced by z.

We have considered several possible alternatives for selecting the xyz trihedron. In a number of instances a trihedron, one of whose axes coincides with the direction of the absolute velocity vector of the object, and another coinciding with the direction of the principal normal to its trajectory (a so-called moving trihedron of trajectory), may prove to be more convenient. A moving trihedron of trajectory in an earth body-axis coordinate system, i.e., a moving trihedron not of the absolute, but of the relative (to the earth) trajectory of the object, may also be used.

Analogously, the orientation of the x and y axes of the moving trihedron on the sphere may be selected such that one of the axes, for example the x axis, lies in a plane containing the vector of the absolute or relative velocity of the object.

5.1.3. Possible ways of analyzing the error equations. Let us turn to equations (5.1) -- (5.9) and the relations deriving from them. The basic problem in the analysis of these equations is, clearly, analysis of the systems of differential equations (5.1) and (5.3). The remaining relations are of finite algebriac equalities, and their analysis causes no difficulty.

The systems of equations (5.1) and (5.3) are independent of one another and may therefore be considered separately. They are systems of linear differential equations with variable coefficients. The right sides of these equations may be either determined or random functions of time.

The systems of differential equations (5.1) and (5.3) determine the operational stability of the inertial system as a whole. Their solutions, moreover, relate errors in the specification of coordinates to the instrument errors of the elements and devices of the system. It is impossible to determine the functional accuracy of an inertial system and to formulate requirements on the precision of its elements without analyzing these equations. Analysis of these equations is also necessary for the selection of means of correcting the operation of an inertial system.

It must, however, be said that analysis of differential equations (5.1) and (5.3), that is, their analytical analysis in a sufficiently general form, gives rise to insuperable mathematical difficulties. Especially difficult in this regard is system (5.1).

Equations (5.3), as was noted above, are variations of the well-known Poisson equations, which reduce to the Riccati equation and which are in the general case not soluble.[1] Under closer examination, however, it proves to be the case that equations (5.3) are nevertheless much simpler to deal with than equations (5.1). The Poisson equations have a first integral, and this completes solution of the problem of analyzing the stability of the solutions to equations (5.3). Because θ_{1x}, θ_{1y}, θ_{1z} are so small, their squares and products may be ignored, and the solution to equation (5.3) may be constructed in quadratures.

With regard to ways of analyzing equation (5.1), the following possibilities also exist.

To begin with we note that there exists several special cases of the motion of an object in which equations (5.1) or equations (5.14), (5.19) and (5.28) deriving from them, reduce to equations with constant coefficients.

The simplest case is that of a basis fixed in the coordinate system $O_1\xi^1\xi^2\xi^3$, in which

$$\omega_x = \omega_y = \omega_z = 0, \quad r = \text{const}$$

Taking this into account, we obtain from equations (5.19):

$$\left.\begin{aligned} &\delta\ddot{x} + \frac{\mu}{r^3}\delta x = \Delta n_x - \Delta\dot{m}_y r, \\ &\delta\ddot{y} + \frac{\mu}{r^3}\delta y = \Delta n_y + \Delta\dot{m}_x r, \quad \delta\ddot{z} - \frac{2\mu}{r^3}\delta z = \Delta n_z. \end{aligned}\right\} \tag{5.33}$$

The equations for x, y, z have separated. Since

$$\frac{\mu}{r^3} = \text{const},$$

the solution to equations (5.33) is obvious.

The second case, in which the coefficients of equations (5.19) become constant, will be the case of the motion of an object at a constant distance r from the center O_1 of the earth at a constant velocity in a plane, fixed relative to the trihedron $O_1\xi^1\xi^2\xi^3$, passing through the center of the earth. If the x axis is placed in the plane of motion of the object,

$$\omega_x = \omega_z = 0, \quad \omega_y = \text{const}, \quad r = \text{const}. \tag{5.34}$$

Substituting expressions (5.34) into (5.19) or (5.28), we arrive at the equation

$$\left.\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x + 2\omega_y\,\delta\dot{z} = \Delta n_x - \Delta\dot{m}_y r, \\
&\delta\ddot{y} + \frac{\mu}{r^3}\,\delta y = \Delta n_y + \Delta\dot{m}_x r - \omega_y\,\Delta m_z r, \\
&\delta\ddot{z} - \left(\frac{2\mu}{r^3} + \omega_y^2\right)\delta z - 2\omega_y\,\delta\dot{x} = \Delta n_z + 2r\omega_y\,\Delta m_y.
\end{aligned}\right| \tag{5.35}$$

The second equation (5.35) stands out and has an obvious solution. The second and third equations (5.35) form a fourth order system of differential equations with constant coefficients. As we will see below, the characteristic equation of this system reduces to a biquadratic equation and equations (5.35) prove to be fully soluble in their general form.

Special cases of equations (5.35) will be equations for the case of a fixed space, i.e., equations (5.33); equations for the case of motion at constant velocity along the equator, with the y axis coinciding with the axis of rotation of the earth; and equations for the case of motion of a satellite in a circular orbit, with

$$\omega_y^2 = \frac{\mu}{r^3}.$$

The final case of the reduction of coordinate error equations (5.1) to equations with constant coefficients is the case of an object which is fixed in relation to the earth or an object moving at constant velocity along the parallel of latitude. This case includes, clearly, all of the preceding ones.

If the object moves along the parallel with a velocity v, then, placing the y axis of the moving trihedron in a plane containing the earth's axis of rotation, i.e., in the plane of the meridian, and directing it to the north, we obtain the following expressions for the projections of the rate of rotation of trihedron xyz around its axis:

$$\left.\begin{aligned} &\omega_x = 0, \\ &\omega_y = u\cos\varphi + \frac{v}{r} = \text{const}, \\ &\omega_z = u\sin\varphi + \frac{v}{r}\,\mathrm{tg}\,\varphi = \text{const}, \end{aligned}\right\} \tag{5.36}$$

where φ is the geocentric latitude of the parallel along which the object moves, r is the constant distance to the center of the earth, and u is the earth rate.

The equations of the first group for this case are obtained from equations (5.19), in the latter $\omega_x = 0$, and ω_y and ω_z are replaced by their values (5.36). The characteristic equation of the system (5.19) reduces in this case to a bicubic equation.

In addition to the above-mentioned cases of the reduction of equations (5.1) to equations with constant coefficients, the first group of the error equations may also be completely analyzed in certain cases in which the coefficients are variable, namely in those cases in which the general integral of the equations of motion of the object is known. In these cases, it is possible to construct (on the basis of the analogy described at the beginning of this section) the solution to equations (5.1) in quadratic forms, using the well-known Poincare theorem for solving equations in variations.[2] The specific case to which we apply this approach is that of Keplerian motion of the object.

Thus, from the point of view of practical applications there exist a number of interesting cases of motion of the object in which the error equations of the first group, i.e., equations (5.1), may be completely analyzed. Using these cases as examples, it is possible to elucidate the basic properties of the solutions of these equations. Moreover, the exact solutions which result may be used as first approximations in the construction of approximate solutions for those cases of the motion of the object in which equations (5.5) do not permit exact integration.

§5.2. Stability Analysis and Integration of the Error Equations of the Second Group.

5.2.1. Stability analysis. Let us examine the second group of the differential coordinate error equations, i.e., the system of equations (5.3)

$$\left.\begin{aligned} &\dot{\theta}_{1x}+\omega_y\theta_{1z}-\omega_z\theta_{1y}=\Delta m_x,\\ &\dot{\theta}_{1y}+\omega_z\theta_{1x}-\omega_x\theta_{1z}=\Delta m_y,\\ &\dot{\theta}_{1z}+\omega_x\theta_{1y}-\omega_y\theta_{1x}=\Delta m_z \end{aligned}\right\} \tag{5.37}$$

and the homogeneous system cooresponding to it

$$\left.\begin{aligned} &\dot{\theta}_{1x}+\omega_y\theta_{1z}-\omega_z\theta_{1y}=0,\\ &\dot{\theta}_{1y}+\omega_z\theta_{1x}-\omega_x\theta_{1z}=0,\\ &\dot{\theta}_{1z}+\omega_x\theta_{1y}-\omega_y\theta_{1x}=0. \end{aligned}\right\} \tag{5.38}$$

The homogeneous equations (5.38) have a first integral. In order to obtain it, we multiply the first equation (5.38) by θ_{1x}, the second by θ_{1y}, the third by θ_{1z} and add. As a result we arrive at the equality

$$\theta_{1x}\dot{\theta}_{1x}+\theta_{1y}\dot{\theta}_{1y}+\theta_{1z}\dot{\theta}_{1z}=0, \tag{5.39}$$

which may, clearly, be integrated. Taking the initial conditions into account, we have:

$$\theta_{1x}^2+\theta_{1y}^2+\theta_{1z}^2=\theta_{1x}^{0\,2}+\theta_{1y}^{0\,2}+\theta_{1z}^{0\,2}. \tag{5.40}$$

As applied to a gyrostabilized platform, the first integral (5.40) has a simple mechanical significance. In the absence of perturbing moments (free drift) the gyrostabilized platform does not change its position in space, and, consequently, the intial error in its orientation remains.

The first integral (5.40) is a positive definite coordinate function. It may therefore be taken as a Lyapunov function. The total derivative of the quadratic form (5.40) vanishes by virtue of equations (5.38), whence the stability of the Lyapunov solutions.

The following circumstance should be noted. Equations (5.38) are first-approximation equations. Therefore, on the basis of integral (5.40) of these equations, we can arrive at a final judgment as to the stability of the inertial system with regard to the errors θ_{1x}, θ_{1y}, θ_{1z}. If we return to equations (4.51), however, from which equations (5.3) were derived, we see that the homogeneous equations (5.41) also allow first integrals. Multiplying the first of these equations by $\delta\alpha_{11}$, the second by $\delta\alpha_{12}$, the third by $\delta\alpha_{13}$ and adding, we find:

$$\delta\alpha_{11}\delta\dot{\alpha}_{11} + \delta\alpha_{12}\delta\dot{\alpha}_{12} + \delta\alpha_{13}\delta\dot{\alpha}_{13} = 0, \tag{5.41}$$

from which it follows that

$$(\delta\alpha_{11})^2 + (\delta\alpha_{12})^2 + (\delta\alpha_{13})^2 = \text{const.} \tag{5.42}$$

Analogously, from the fourth, fifth and sixth equations (4.51) we obtain:

$$(\delta\alpha_{21})^2 + (\delta\alpha_{22})^2 + (\delta\alpha_{23})^2 = \text{const.} \tag{5.43}$$

Finally, the first three equations (4.51) give:

$$(\delta\alpha_{31})^2 + (\delta\alpha_{32})^2 + (\delta\alpha_{33})^2 = \text{const} \tag{5.44}$$

The stability of the homogeneous Lyapunov equations (4.51) follows from expressions (5.42) -- (5.44). But these equations are exact equations for perturbations $\delta\alpha_{ij}$, since equations (3.60), by variation of which equations (4.51) were obtained, are linear in α_{ij}. Thus, the exact equations give the same answer to the question of stability as the first-approximation equation (5.38).

5.2.2. <u>Integration of the equations of the second group in quadratic forms for free motion of the object.</u> Let us consider the integration of equations (5.37). The general solution to the homogeneous equations (5.38) will be the expressions

$$\left.\begin{aligned} \theta_{1x} &= c_1\alpha_{11} + c_2\alpha_{21} + c_3\alpha_{31}, \\ \theta_{1y} &= c_1\alpha_{12} + c_2\alpha_{22} + c_3\alpha_{32}, \\ \theta_{1z} &= c_1\alpha_{13} + c_2\alpha_{23} + c_3\alpha_{33}, \end{aligned}\right\} \tag{5.45}$$

where α_{ij} are elements of table (3.16) of the direction cosines between the x, y, z axes and the ξ_*, η_*, ζ_* (ξ^1, ξ^2, ξ^3) axes, and c_1, c_2, c_3 are arbitrary constants. In order to verify that the right sides of equalities (5.45) satisfy equations (5.38), it is sufficient to substitute them into equations (5.38) and to take equalities (4.221) into account, by means of which ω_x, ω_y, ω_z are expressed in terms of α_{ij} and $\dot{\alpha}_{ij}$

In order to find the general solution to the homogeneous equations (5.37), we may now use the Lagrange method of variation of arbitrary parameters. Assuming, in accordance with this method, that the parameters c_1, c_2, c_3 are functions of time and substituting expression (5.45) into equations (5.37), we arrive at the following system of equations:

$$\left.\begin{aligned} \dot{c}_1\alpha_{11} + \dot{c}_2\alpha_{21} + \dot{c}_3\alpha_{31} &= \Delta m_x, \\ \dot{c}_1\alpha_{12} + \dot{c}_2\alpha_{22} + \dot{c}_3\alpha_{32} &= \Delta m_y, \\ \dot{c}_1\alpha_{13} + \dot{c}_2\alpha_{23} + \dot{c}_3\alpha_{33} &= \Delta m_z. \end{aligned}\right\} \tag{5.46}$$

Solving this system relative to $\dot{c}_1$, $\dot{c}_2$, $\dot{c}_3$ and integrating the resulting expressions, we find:

$$\left.\begin{aligned} c_1 &= \int_0^t (\Delta m_x\alpha_{11} + \Delta m_y\alpha_{12} + \Delta m_z\alpha_{13})\,dt + c_1^0, \\ c_2 &= \int_0^t (\Delta m_x\alpha_{21} + \Delta m_y\alpha_{22} + \Delta m_z\alpha_{23})\,dt + c_2^0, \\ c_3 &= \int_0^t (\Delta m_x\alpha_{31} + \Delta m_y\alpha_{32} + \Delta m_z\alpha_{33})\,dt + c_3^0. \end{aligned}\right\} \tag{5.47}$$

Substituting relations (5.47) into (5.45) and taking account of the initial conditions, we obtain a general solution to system (5.37) in the following form:

$$
\left.
\begin{aligned}
\theta_{1x} = a_{11}&\left[\int_0^t (\Delta m_x a_{11} + \Delta m_y a_{12} + \Delta m_z a_{13})\,dt + \right. \\
&\left. + \theta^0_{1x} a^0_{11} + \theta^0_{1y} a^0_{12} + \theta^0_{1z} a^0_{13}\right] + \\
&+ a_{21}\left[\int_0^t (\Delta m_x a_{21} + \Delta m_y a_{22} + \Delta m_z a_{23})\,dt + \right. \\
&\left. + \theta^0_{1x} a^0_{21} + \theta^0_{1y} a^0_{22} + \theta^0_{1z} a^0_{23}\right] + a_{31}\left[\int_0^t (\Delta m_x a_{31} + \right. \\
&\left. + \Delta m_y a_{32} + \Delta m_z a_{33})\,dt + \theta^0_{1x} a^0_{31} + \theta^0_{1y} a^0_{32} + \theta^0_{1z} a^0_{33}\right]. \\
\theta_{1y} = a_{12}&\left[\int_0^t (\Delta m_x a_{11} + \Delta m_y a_{12} + \Delta m_z a_{13})\,dt + \right. \\
&\left. + \theta^0_{1x} a^0_{11} + \theta^0_{1y} a^0_{12} + \theta^0_{1z} a^0_{13}\right] + a_{22}\left[\int_0^t (\Delta m_x a_{21} + \right. \\
&\left. + \Delta m_y a_{22} + \Delta m_z a_{23})\,dt + \theta^0_{1x} a^0_{21} + \theta^0_{1y} a^0_{22} + \theta^0_{1z} a^0_{23}\right] + \\
&+ a_{32}\left[\int_0^t (\Delta m_x a_{31} + \Delta m_y a_{32} + \Delta m_z a_{33})\,dt + \right. \\
&\qquad\qquad \left. + \theta^0_{1x} a^0_{31} + \theta^0_{1y} a^0_{32} + \theta^0_{1z} a^0_{33}\right]. \\
\theta_{1z} = a_{13}&\left[\int_0^t (\Delta m_x a_{11} + \Delta m_y a_{12} + \Delta m_z a_{13})\,dt + \right. \\
&\left. + \theta^0_{1x} a^0_{11} + \theta^0_{1y} a^0_{12} + \theta^0_{1z} a^0_{13}\right] + a_{23}\left[\int_0^t (\Delta m_x a_{21} + \right. \\
&\left. + \Delta m_y a_{22} + \Delta m_z a_{23})\,dt + \theta^0_{1x} a^0_{21} + \theta^0_{1y} a^0_{22} + \theta^0_{1z} a^0_{23}\right] + \\
&+ a_{33}\left[\int_0^t (\Delta m_x a_{31} + \Delta m_y a_{32} + \Delta m_z a_{33})\,dt + \right. \\
&\qquad\qquad \left. + \theta^0_{1x} a^0_{31} + \theta^0_{1y} a^0_{32} + \theta^0_{1z} a^0_{33}\right].
\end{aligned}
\right|
\tag{5.48}
$$

Formulas (5.48) give the solution in quadratic forms. They contain the quantities a_{ij} under the integral sign, these being known functions of time if the motion of the object with which the inertial system is associated is specified. It must, of course, be kept in mind that the intended motion of the object may be defined not only by the explicit specification of the coordinates as functions of time, but also by differential relations which are, in the general case, non-integrable. In this case it may be more convenient not to numerically integrate the equations defining the intended motion of the object and to substitute the results of the integration in quadratic forms (5.48), but rathe to determine ω_x, ω_y, ω_z and to numerically integrate the initial system (5.37).

Solution (5.48) to equations (5.37) may also be obtained in a somewhat different manner. It is possible to project equations (5.37) on the fixed ξ_*, η_*, ζ_* axes. Then in place of equations (5.37) we obtain equations

$$\dot{\theta}_{1\xi_*} = \Delta m_{\xi_*}, \quad \dot{\theta}_{1\eta_*} = \Delta m_{\eta_*}, \quad \dot{\theta}_{1\zeta_*} = \Delta m_{\zeta_*}. \tag{5.49}$$

Integrating these equations and then again passing to θ_{1x}, θ_{1y}, θ_{1z} and Δm_x, Δm_y, Δm_z, we obtain formulas (5.48) once again.

From equations (5.49) the following evaluation derives:

$$\sqrt{\theta_{1x}^2 + \theta_{1y}^2 + \theta_{1z}^2} \leq \sqrt{\theta_{1x}^{0\,2} + \theta_{1y}^{0\,2} + \theta_{1z}^{0\,2}} + \int_0^t \sqrt{(\Delta m_x)^2 + (\Delta m_y)^2 + (\Delta m_z)^2}\, dt. \tag{5.50}$$

In order to obtain this evaluation, we return to the vector equation

$$\frac{d\theta_1}{dt} = \Delta m, \tag{5.51}$$

which is equivalent to equations (5.49). From equations (5.51) we find:

$$\theta_1 = \theta_1^0 + \int_0^t \Delta m\, dt. \tag{5.52}$$

Consequently,

$$|\theta_1| \leq |\theta_1^0| + \left|\int_0^t \Delta m\, dt\right|, \tag{5.53}$$

but

$$\left|\int_0^t \Delta m\, dt\right| \leq \int_0^t |\Delta m|\, dt, \tag{5.54}$$

from which inequality (5.50) is obtained.

We note that an evaluation analogous to inequality (5.50) may also be given for the modulus of vector θ_4. From relations (5.8) we find:

$$\sqrt{\theta_{4x}^2+\theta_{4y}^2+\theta_{4z}^2} \leqslant \sqrt{(\Delta\alpha)^2+(\Delta\beta)^2+(\Delta\gamma)^2+2\Delta\alpha\Delta\gamma\sin\beta}. \tag{5.55}$$

Apropos of solution (5.48) to equations (5.37) it is useful, in order to avoid misunderstanding, to present the following clarification. Equations (5.37) are obtained by variation of the Poisson equations and are themselves analogous in form to these equations. However, between equations (5.37) and the Poisson equations

$$\left.\begin{aligned} &\dot{\alpha}_{11}+\omega_y\alpha_{13}-\omega_z\alpha_{12}=0,\\ &\dot{\alpha}_{12}+\omega_z\alpha_{11}-\omega_x\alpha_{13}=0,\\ &\dot{\alpha}_{13}+\omega_x\alpha_{12}-\omega_y\alpha_{11}=0 \end{aligned}\right\} \tag{5.56}$$

there is a profound difference, as a result of which solution (5.48) to system (5.37) has no relation to the solution of the Poisson equations (5.56). The difference between equations (5.37) and (5.36) consists in the fact that α_{ij}, entering into equations (5.56) and related to θ_x, θ_y, θ_z by equalities (4.221), are unknown, while at the same time in equations (5.37) the quantities θ_x, θ_y, θ_z, of course, are not expressed in terms of θ_{1x}, θ_{1y}, θ_{1z}. In the case of equations (5.37) the problem reduces, essentially, to that of finding the small deflection of the moving trihedron in terms of the projections of the absolute angular velocity of this deflection on the axes of the moving trihedron, while in the case of the Poisson equations the final deflection is sought.

5.2.3. Special cases: an object fixed in absolute space; motion at constant velocity on the arc of a large circle and on a parallel; Keplerian motion. Below we will require exact expressions for θ_{1x}, θ_{1y}, θ_{1z} for the cases enumerated in the preceding section, when equations (5.1) are integrated. We will need them so that, when we obtained δx, δy, δz from equations (5.1), we will then be able to find

δx_3, δy_3, δz_3, and θ_x, θ_y, θ_z. Let us therefore write out the values of θ_{1x}, θ_{1y}, θ_{1z} for the cases in question.

If the object with which the inertial system is associated is ...ed in the $O_1\xi^1\xi^2\xi^3$ coordinate system, then in formulas (5.48) ... may set

$$\left.\begin{array}{l} a_{ij}=1 \text{ when } i=j, \\ a_{ij}=0 \text{ when } i\neq j. \end{array}\right\} \quad (5.57)$$

Then from formulas (5.48) we find the following values of θ_{1x}, θ_{1y}, θ_{1z}:

$$\left.\begin{array}{c} \theta_{1x}=\int_0^t \Delta m_x dt+\theta_{1x}^0, \quad \theta_{1y}=\int_0^t \Delta m_y dt+\theta_{1y}^0, \\ \theta_{1z}=\int_0^t \Delta m_z dt+\theta_{1z}^0, \end{array}\right| \quad (5.58)$$

which are obtained immediately and directly from equations (5.37) by setting $\omega_x = \omega_y = \omega_z = 0$, which is the case when the object is fixed in the $O_1\xi^1\xi^2\xi^3$ coordinate system.

The second case, in which equations (5.51) are integrated is that of motion of the object at a constant velocity v at a constant distance from the center of the earth in a fixed plane containing the center O_1 of the earth, i.e., the case of motion at a constant velocity along a fixed large circle of a sphere of constant radius concentric ... the earth. We may, without lose of generality, superpose the plane ...otion with the $\xi^1\xi^2$ plane, and take as the initial position of the ...ct its position on the ξ^1 axis. The x and y axes of the O_1xyz t...hedron will then lie in the $\xi^1\xi^2$ plane, and the y axis will coincide w...h the ξ^3 axis. As a result the following elements of table (3.16) ...l be different from 0:

$$\left.\begin{array}{lll} a_{11}=-\sin\omega_y t, & a_{13}=\cos\omega_y t, & \\ a_{31}=\cos\omega_y t, & a_{33}=\sin\omega_y t, & a_{22}=1, \end{array}\right\} \quad (5.59)$$

where $\omega_y = v/r$.

Substituting expressions (5.59) into solutions (5.48), we obtain after obvious transformations:

$$
\left.\begin{aligned}
\theta_{1x} &= \theta_{1x}^{0}\cos\omega_y t - \theta_{1z}^{0}\sin\omega_y t + \\
&+ \int_0^t [\Delta m_x(\tau)\cos\omega_y(t-\tau) - \Delta m_z(\tau)\sin\omega_y(t-\tau)]\,d\tau, \\
\theta_{1y} &= \theta_{1y}^{0} + \int_0^t \Delta m_y(\tau)\,d\tau, \\
\theta_{1z} &= \theta_{1x}^{0}\sin\omega_y t + \theta_{1z}^{0}\cos\omega_y t + \\
&+ \int_0^t [\Delta m_x(\tau)\sin\omega_y(t-\tau) + \Delta m_z(\tau)\cos\omega_y(t-\tau)]\,d\tau.
\end{aligned}\right| \tag{5.60}
$$

As with expressions (5.58), formulas (5.60) may also be obtained directly from equations (5.37), since the coefficients of the system (5.37) are in this case constant: $\omega_x = 0$, $\omega_z = 0$, $\omega_y = \text{const}$. System (5.37) takes the form:

$$
\dot{\theta}_{1x} + \omega_y\theta_{1z} = \Delta m_x, \quad \dot{\theta}_{1y} = \Delta m_y, \quad \dot{\theta}_{1z} - \omega_y\theta_{1x} = \Delta m_z. \tag{5.61}
$$

The second equation is singled out. The second formula (5.60) immediately follows from it. The characteristic equations of the remaining second order system has the roots $\pm\omega_y$. Representing its solution as a Duhamel integral, we arrive at the first and second formulas (5.60).

Let us turn to the third case -- the motion of the object along a parallel. As before, we will consider the ξ^3 as coinciding with the earth's axis of rotation, and the xyz trihedron as moving on a sphere surrounding the earth, and oriented to the points of the compass (with the y axis directed towards the north). The values of the direction cosines α_{ij} for this case may be obtained from table (3.260), if we note that the unit vectors of the x, y, z axes correspond to the unit vectors $\vec{e}_2$, $\vec{e}_3$, $\vec{e}_1$ and that according to expressions (5.36)

$$
\lambda_1 = u + \frac{v}{r\cos\varphi}t. \tag{5.62}
$$

Equality (5.62) assumes, of course, that at the beginning of its motion the object is located in the $O_1\xi^1\xi^2\xi^3$ plane.

Therefore, introducing for the sake of simplicity the notation

$$u + \frac{v}{r\cos\varphi} = \omega, \qquad (5.63)$$

we find:

$$\left.\begin{aligned}
&\alpha_{11} = -\sin\omega t, \quad \alpha_{12} = -\sin\varphi\cos\omega t, \quad \alpha_{13} = \cos\varphi\cos\omega t,\\
&\alpha_{21} = \cos\omega t, \quad \alpha_{22} = -\sin\varphi\sin\omega t, \quad \alpha_{23} = \cos\varphi\sin\omega t,\\
&\alpha_{31} = 0, \quad \alpha_{32} = \cos\varphi, \quad \alpha_{33} = \sin\varphi.
\end{aligned}\right\} \qquad (5.64)$$

We recall that in relations (5.62) -- (5.64), φ is the geocentric latitude of the parallel along which the object is moving, and v is the velocity of its motion relative to the earth.

Substituting the values (5.64) of the direction cosines in solutions (5.48) and performing the required transformations, we arrive at the following formulas:

$$\left.\begin{aligned}
\theta_{1x} = {} & \theta^0_{1x}\cos\omega t + (\theta^0_{1y}\sin\varphi - \theta^0_{1z}\cos\varphi)\sin\omega t + \\
& + \int_0^t [\Delta m_x(\tau)\cos\omega(t-\tau) + \\
& + [\Delta m_y(\tau)\sin\varphi - \Delta m_z(\tau)\cos\varphi]\sin\omega(t-\tau)]\,d\tau,\\
\theta_{1y} = {} & -\theta^0_{1x}\sin\varphi\sin\omega t + \theta^0_{1y}(\sin^2\varphi\cos\omega t + \cos^2\varphi) + \\
& + \theta^0_{1z}\sin\varphi\cos\varphi(1-\cos\omega t) + \\
& + \int_0^t \{-\Delta m_x(\tau)\sin\varphi\sin\omega(t-\tau) + \\
& + \Delta m_y(\tau)[\cos\omega(t-\tau)\sin^2\varphi + \cos^2\varphi] - \\
& - \Delta m_z(\tau)\sin\varphi\cos\varphi[\cos\omega(t-\tau) - 1]\}\,d\tau,\\
\theta_{1z} = {} & \theta^0_{1x}\cos\varphi\sin\omega t + \theta^0_{1y}\sin\varphi\cos\varphi(1-\cos\omega t) + \\
& + \theta^0_{1z}(\cos^2\varphi\cos\omega t + \sin^2\varphi) + \\
& + \int_0^t \{\Delta m_x(\tau)\cos\varphi\sin\omega(t-\tau) + \\
& + \Delta m_y(\tau)\sin\varphi\cos\varphi[1-\cos\omega(t-\tau)] + \\
& + \Delta m_z(\tau)[\cos^2\varphi\cos\omega(t-\tau) + \sin^2\varphi]\}\,d\tau.
\end{aligned}\right\} \qquad (5.65)$$

Formulas (5.65) can, of course, be obtained directly from equations (5.37), rather than from the general solution (5.48); taking equalities

(5.36) and (5.63) into account, equations (5.37) take the form:

$$\left.\begin{aligned}
&\dot{\theta}_{1x} + \theta_{1z}\omega\cos\varphi - \theta_{1y}\omega\sin\varphi = \Delta m_x,\\
&\dot{\theta}_{1y} + \theta_{1z}\omega\sin\varphi = \Delta m_y,\\
&\dot{\theta}_{1z} - \theta_{1x}\omega\cos\varphi = \Delta m_z.
\end{aligned}\right\} \tag{5.66}$$

The characteristic equation of the system is

$$p(p^2 + \omega^2) = 0, \tag{5.67}$$

the roots of which are

$$p_1 = 0, \quad p_{2,3} = \pm j\omega \tag{5.68}$$

Putting the solution to the system (5.66) in the form of the Duhamel integral immediately gives formulas (5.65).

We note that with $\varphi = 0$ formulas (5.65) reduce to formulas (5.60), and with $\varphi = \frac{\pi}{2}$ they reduce to formulas (5.58) by conversion to a free-azimuth trihedron.

Keplerian motion presents a somewhat more difficult situation than the previous cases; in this case the direction cosines α_{ij} entering into formulas (5.48) cannot be expressed as simply as in the examples considered above. In the case of Keplerian motion it is possible, without sacrificing generality, to take the $O_1\xi^1\xi^2$ plane as the plane of motion. Then, placing the x and z axes in this plane and directing the z axis along the vector $\vec{r}$, we obtain, as in the case of plane motion at a constant distance from the center of the earth:

$$\alpha_{12} = \alpha_{22} = \alpha_{31} = \alpha_{33} = 0, \quad \alpha_{32} = 1 \tag{5.69}$$

If we further assume that at the beginning of its motion the object is located on the ξ^1 axis, then in addition to equalities (5.69) we will have:

$$\alpha^0_{11} = 0, \quad \alpha^0_{23} = 0, \quad \alpha^0_{21} = \alpha^0_{13} = 1. \tag{5.70}$$

Taking relations (5.69) and (5.70) into account, formulas (5.48) simplify and take the form:

$$
\left.
\begin{aligned}
\theta_{1x} &= \alpha_{11}\left[\int\limits_0^t (\Delta m_x \alpha_{11} + \Delta m_z \alpha_{13})\,dt + \theta_{1x}^0\right] + \\
&\quad + \alpha_{21}\left[\int\limits_0^t (\Delta m_x \alpha_{21} + \Delta m_z \alpha_{23})\,dt + \theta_{1z}^0\right], \\
\theta_{1y} &= \int\limits_0^t \Delta m_y\,dt + \theta_{1y}^0, \\
\theta_{1z} &= \alpha_{13}\left[\int\limits_0^t (\Delta m_x \alpha_{11} + \Delta m_z \alpha_{13})\,dt + \theta_{1x}^0\right] + \\
&\quad + \alpha_{23}\left[\int\limits_0^t (\Delta m_x \alpha_{21} + \Delta m_z \alpha_{23})\,dt + \theta_{1z}^0\right].
\end{aligned}
\right\}
\qquad (5.71)
$$

We will denote the angle between the ξ^1 and z axes by σ. Then,

$$
\left.
\begin{aligned}
\alpha_{11} &= -\sin\alpha, & \alpha_{13} &= \cos\sigma, \\
\alpha_{21} &= \cos\alpha, & \alpha_{23} &= \sin\sigma,
\end{aligned}
\right\}
\qquad 5.72)
$$

where, obviously,

$$
\sigma = \int\limits_0^t \omega_y\,dt. \qquad (5.73)
$$

In order to obtain explicit expressions for θ_{1x} and θ_{1z} we have only to substitute into formulas (5.71) the values of σ which, for the case of Keplerian motion, may be found as functions of time.

We note that as in the preceding cases, the solution (5.71) for the values of α_{ij} determined by equalities (5.72) and (5.73), may also be obtained directly from equations (5.37), which in this case take the form:

$$
\left.
\begin{aligned}
\dot{\theta}_{1x} + \omega_y \theta_{1z} &= \Delta m_x, \\
\dot{\theta}_{1y} &= \Delta m_y, \\
\dot{\theta}_{1z} - \omega_y \theta_{1x} &= \Delta m_z.
\end{aligned}
\right\}
\qquad (5.74)
$$

The second formula (5.71) follows directly from the second equation (5.74). The homogeneous equations corresponding to the first and third equations (5.74), after elimination of one of the variables, for example θ_{1z}, reduce to the second-order equation:

$$\ddot{\theta}_{1x} - \frac{\dot{\omega}_y}{\omega_y}\dot{\theta}_{1x} + \omega_y^2\theta_{1x} = 0. \tag{5.75}$$

The functions

$$\sin\int_0^t \omega_y\,dt \quad и \quad \cos\int_0^t \omega_y\,dt. \tag{5.76}$$

will be partial solutions to this equation, as can easily be shown by direct substitution. Using the method of variation of arbitrary parameters, the general solution to the system of corresponding non-homogeneous equations we find in the form

$$\left.\begin{aligned}
\theta_{1x} &= \theta_{1x}^0\cos\int_0^t \omega_y\,dt - \theta_{1z}^0\sin\int_0^t \omega_y\,dt + \\
&+ \int_0^t\left[\Delta m_x(\tau)\cos\int_0^{t-\tau}\omega_y\,d\tau - \Delta m_z(\tau)\sin\int_0^{t-\tau}\omega_y\,d\tau\right]d\tau, \\
\theta_{1z} &= \theta_{1x}^0\sin\int_0^t \omega_y\,dt + \theta_{1z}^0\cos\int_0^t \omega_y\,dt + \\
&+ \int_0^t\left[\Delta m_x(\tau)\sin\int_0^{t-\tau}\omega_y\,d\tau + \Delta m_z(\tau)\cos\int_0^{t-\tau}\omega_y\,d\tau\right]d\tau.
\end{aligned}\right| \tag{5.77}$$

Using equalities (5.72) and (5.73), we can easily show that formulas (5.77) and (5.71) coincide.

§5.3. Stability Analysis and the Solution to the First Group of Error Equations for Cases in which They Reduce to Equations with Constant Coefficients.

5.3.1. Stability analysis. Let us consider the homogeneous equations corresponding to the system (5.19), which is obtained from the system (5.1) when the z axis of the xyz trihedron is directed along the vector $\vec{r}$. In §5.1 it was shown that the coeffficients of the left side of equations (5.19) become constant in three cases: when the object is stationary in the $O_1\xi^1\xi^2\xi^3$ coordinate system, when it moves at constant velocity at a constant distance from the center of the earth in a plane containing the center of the earth, and when motion occurs at a constant velocity along a parallel.

The latter case is the most general. Analysis of the stability of the error equations of the first group is therefore most conveniently carried out for the case of motion of the object along a parallel.

In this case, if the xyz trihedron is oriented according to the points of the compass, we obtain the following system of homogeneous equations:

$$\left.\begin{aligned} \delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x - 2\omega_z\delta\dot{y} + 2\omega_y\delta\dot{z} = 0, \\ \delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_z^2\right)\delta y + \omega_y\omega_z\delta z + 2\omega_z\delta\dot{x} = 0, \\ \delta\ddot{z} + \left(-\frac{2\mu}{r^3} - \omega_y^2\right)\delta z - 2\omega_y\delta\dot{x} + \omega_z\omega_y\delta y = 0. \end{aligned}\right\} \tag{5.78}$$

where μ/r^3, ω_y, ω_z are constant and ω_y and ω_z are related to the velocity of motion of the object, its distance r from the center of the earth and the latitude φ of the parallel along with the object is moving by formulas (5.36).

The characteristic equation of system (5.78) reduces to a complete cubic equation relative to the square of the unknown ($p^2 = q$):

$$\begin{gathered} q^3 + 2q^2(\omega_y^2 + \omega_z^2) + q\left[-3\omega_0^4 + 3\omega_0^2(\omega_y^2 - 2\omega_z^2) + (\omega_y^2 + \omega_z^2)^2\right] - \\ - \omega_0^2(\omega_0^2 - \omega_y^2 - \omega_z^2)(2\omega_0^2 + \omega_y^2 - 2\omega_z^2) = 0, \end{gathered} \tag{5.79}$$

where we have introduced for convenience the notation

$$\frac{\mu}{r^3} = \omega_0^2 \tag{5.80}$$

Since the characteristic equation lacks the odd powers of p, it cannot satisfy the conditions of asymptotic stability (the Hurwitz conditions).

For asymptotic stability equation (5.79) must, of course, have negative or zero roots, and the linear elements of the denominator of the characterstic matrix

$$\begin{vmatrix} p^2 + \omega_0^2 - \omega_y^2 - \omega_z^2 & -2\omega_z p & 2\omega_y p \\ 2\omega_z p & p^2 + \omega_0^2 - \omega_z^2 & \omega_y\omega_z \\ -2\omega_y p & \omega_z\omega_y & p^2 - 2\omega_0^2 + \omega_y^2 \end{vmatrix}$$

o system (5.78) should correspond to the multiple root of the characteristic equation of this system.

Direct analysis by this means of the characteristic equation and the characteristic matrix of system (5.78) with arbitrary ω_y and ω_z requires, however, unwieldy calculations. Therefore we will proceed in a somewhat different way.

We will make use of the fact that the system of differential equations (5.78) may be regarded as a system describing the motion of a point of unit mass under the influence of only potential and gyroscopic forces.

The expression for the force function of the potential forces may be written, clearly, in the following form:

$$U = -\frac{1}{2}\left[(\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x^2 + (\omega_0^2 - \omega_y^2)\delta y^2 - \right.$$
$$\left. - (2\omega_0^2 + \omega_y^2)\delta z^2 + 2\omega_y\omega_z\delta y\,\delta z\right]. \tag{5.81}$$

The gyroscopic forces are represented by the terms

$$-2\omega_z\delta\dot{y} + 2\omega_y\delta\dot{z}, \quad 2\omega_z\delta\dot{x}, \quad -2\omega_y\delta\dot{x}. \tag{5.82}$$

since they are proportional to the velocities $\dot{x}$, $\dot{y}$, $\dot{z}$, the coefficients of proportionality form the anti-symmetric matrix[3]

$$\begin{Vmatrix} 0 & 2\omega_z & -2\omega_y \\ -2\omega_z & 0 & 0 \\ 2\omega_y & 0 & 0 \end{Vmatrix}.$$

Equations (5.78) for the case under consideration have the energy integral

$$\delta\dot{x}^2 + \delta\dot{y}^2 + \delta\dot{z}^2 - 2U = \text{const} \tag{5.83}$$

In order to obtain this integral from equations (5.78), we must multiply the first of these equations by $\delta\dot{x}$, the second by $\delta\dot{y}$, the third by $\delta\dot{z}$, and then add. Integration of the sum will then give equality (5.83). We note that the gyroscopic forces do not enter into the energy integral (5.83), since the influence of these forces on real displacement is zero.

If we now ignore the gyroscopic forces in equations (5.78), only the potential forces defined by force function (5.81) will remain. For stability of equilibrium under the influence of only potential forces, the force function should, according to the well known Lagrange theorem[4], have a maximum at the point of equilibrium.

Since the force function (5.81) is a quadratic form, its maximum is determined by the well known Sylvester conditions[5] of positive definiteness of the quadratic form. For this case they reduce to the equalities

$$\left.\begin{aligned} \omega_0^2-\omega_y^2-\omega_z^2>0,\\ 2\omega_0^2-2\omega_z^2+\omega_y^2<0.\end{aligned}\right\} \qquad (5.84)$$

It is evident that the areas defined by each of the inequalities (5.84) do not intersect (Figure 5.1), and therefore the force function does not have a maximum at the equilibrium point. Since for this case the force function is a homogeneous function of the second-order, according to the well known Lyapunov theorem the absence of a maximum for this function implies that the system is unstable without any need to consider terms of higher orders of smallness.

But we have not yet considered the gyroscopic forces. Let us now turn to them.

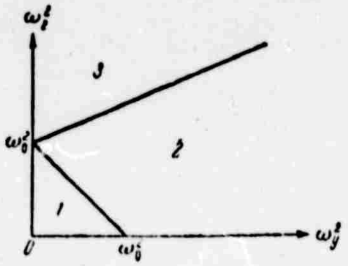

Figure 5.1.

In regions 1 and 3 (Figure 5.1), where the degree of instability (the number of negative Poincare coefficients of instability) is odd, the gyroscopic forces, according to the theorem of Thomson and Tait,[6] cannot stabilize the equilibrium.

In region 2, where the degree of instability is even, the theoretical possibility of stabilization by gyroscopic forces remains. This stabilization, as is well known, has an intermittent character and is disturbed by forces of overall internal dissipation.

Stabilization by gyroscopic forces is effected, if, for example,

$$\omega_y^2 = \omega_0^2, \quad \omega_z^2 = \varepsilon^2, \tag{5.85}$$

were ε^2 is some sufficiently small magnitude.

It can be shown that polynominal (5.79) (i.e., a cubic polynomial in q, not the characteristic equation of the system) in this case satisfies the Hurwitz conditions. The descriminant D of the cubic equation obtained from equation (5.79) by substituting

$$y = q + \frac{2(\omega_0^2 + \varepsilon^2)}{3}, \tag{5.86}$$

is negative:

$$D = -\omega_0^{10}\varepsilon^2 \frac{1}{9} < 0. \tag{5.87}$$

Therefore, all of the roots of the characteristic equation of system (5.78) with conditions (5.85) are prime and purely imaginary, and so stabilization of the system by gyroscopic forces is possible.

Stabilization by forces of a gyroscopic nature, expressions for which enter into equations (5.78), is possible, as has already been noted, only in region 2, where

$$\left.\begin{aligned} 2\omega_0^2 - 2\omega_z^2 + \omega_y^2 > 0, \\ \omega_0^2 - \omega_y^2 - \omega_z^2 < 0. \end{aligned}\right\} \tag{5.88}$$

Here the free term in the characteristic equation is positive, and so the degree of instability is even.

For practical applications the most interesting region is region 1, where

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0. \tag{5.89}$$

In this region the inertial system is unstable during motion of the object along a parallel. It should, however, be noted that instability of an inertial system does not imply the impossibility of its practical realization and application.

Instability implies that the operating time of the system may be small relative to the duration of the transient processes defined by the error equations. Therefore an inertial system may not be stable in the rigorous sense of the term, but the divergence of the amplitudes of the solutions to the error equations may be small relative to their initial values during some limited operational time interval.[7] In order to determine the divergence of the amplitudes of the solutions relative to the initial conditions, it is necessary to develop either a solution to the error equations or some set of upper bound evaluations of these solutions.

The solution to the error equations depends on their initial conditions and the right sides of these equations, i.e., on the instrument errors. Even if the solution diverges it is always possible, within the limits of a given operational time interval and the maximum allowable system error, to define requirements on the magnitudes of the instrument errors and the errors in the initial conditions in such a way as to guarantee a given level of operational accuracy in the system. If these requirements on the accuracy of the elements and initial conditions of the system are difficult to realize technically, then, clearly, correction of the inertial system must be based on other sources of information.

We will return to the questions touched on here. Now, however, let us proceed to the integration of the first group of the error equations for those cases in which they reduce to equations with constant coefficients.

5.3.2. Solving the error equations for the case of a stationary object. As was stated in §5.1, the simplest of the cases in which the coefficients of the error equations are constant, is the case in which the object is stationary in the $O_1\xi_*\eta_*\zeta_*$ $(O_1\xi^1\xi^2\xi^3)$ coordinate system. In this case the first group of error equations (5.1) has the form

$$\left.\begin{aligned} &\delta\ddot{x}+\omega_0^2\delta x=\Delta n_x-\Delta\dot{m}_y r,\\ &\delta\ddot{y}+\omega_0^2\delta y=\Delta n_y+\Delta\dot{m}_x r,\\ &\delta\ddot{z}-2\omega_0^2\delta z=\Delta n_z,\end{aligned}\right\} \tag{5.90}$$

where $\omega_0^2 = \mu/r^3$.

Equations (5.90) are written in terms of projections on the axes of trihedron O_1xyz, the z axis of which coincides with the vector $\vec{r}$. Since the object is stationary in the $O_1\xi^1\xi^2\xi^3$ coordinate system, trihedron O_1xyz may be considered to be fixed relative to the ξ^1, ξ^2, ξ^3 axes. We recall that the first group of the error equations of any inertial system, if the object with which it is associated is stationary in the $O_1\xi^1\xi^2\xi^3$ coordinate system, reduce to equations (5.90).

The initial conditions for equations (5.90) may be designated, as previously, in the following manner:

$$\left.\begin{aligned} &\delta x(0) = \delta x^0, \quad \delta y(0) = \delta y^0, \quad \delta z(0) = \delta z^0, \\ &\delta \dot{x}(0) = \delta \dot{x}^0, \quad \delta \dot{y}(0) = \delta \dot{y}^0, \quad \delta \dot{z}(0) = \delta \dot{z}^0. \end{aligned}\right\} \tag{5.91}$$

In accordance with relations (5.2) and the selected orientation of trihedron O_1xyz,

$$\left.\begin{aligned} &\delta \dot{x}^0 = \delta \dot{x}_0^0 + (\delta\omega_y^0 - \Delta m_y^0)\, r, \\ &\delta \dot{y}^0 = \delta \dot{y}_0^0 - (\delta\omega_x^0 - \Delta m_x^0)\, r, \\ &\delta \dot{z}^0 = \delta \dot{z}_0^0. \end{aligned}\right\} \tag{5.92}$$

Equations (5.90) are three independent second-order equations. The general solutions to the homogeneous equations corresponding to equations (5.90) are obvious:

$$\begin{aligned} &\delta x = A_x \sin\omega_0 t + B_x \cos\omega_0 t, \\ &\delta y = A_y \sin\omega_0 t + B_y \cos\omega_0 t, \\ &\delta z = A_z \cosh \omega_0 \sqrt{2}\, t + B_z \sinh \omega_0 \sqrt{2}\, t. \end{aligned} \tag{5.93}$$

The general solutions to the non-homogeneous equations (5.90) may easily be found using the method of variation of the arbitrary parameters A_x, B_x, A_y, B_y, A_z, B_z, for the determination of which the following system of equations is obtained:

$$\begin{aligned} &\dot{A}_x \sin\omega_0 t + \dot{B}_x \cos\omega_0 t = 0, \\ &\dot{A}_x \omega_0 \cos\omega_0 t - \dot{B}_x \omega_0 \sin\omega_0 t = \Delta n_x - \Delta \dot{m}_y r; \\ &\dot{A}_y \sin\omega_0 t + \dot{B}_y \cos\omega_0 t = 0, \\ &\dot{A}_y \omega_0 \cos\omega_0 t - \dot{B}_y \omega_0 \sin\omega_0 t = \Delta n_y + \Delta \dot{m}_x r; \\ &\dot{A}_z \cosh \omega_0 \sqrt{2}\, t + \dot{B}_z \sinh \omega_0 \sqrt{2}\, t = 0, \\ &\dot{A}_z \omega_0 \sqrt{2} \sinh \omega_0 \sqrt{2}\, t + \dot{B}_z \omega_0 \sqrt{2} \cosh \omega_0 \sqrt{2}\, t = \Delta n_z. \end{aligned} \tag{5.94}$$

Solving these systems for $\dot{A}_x$, $\dot{B}_x$, $\dot{A}_y$, $\dot{B}_y$, $\dot{A}_z$, $\dot{B}_z$ and integrating the solutions, we find the functions A_x, B_x, A_y, B_y, A_z, B_z. Substituting them into relations (5.93) and taking into account the initial conditions (5.91), we obtain:

$$\begin{aligned}
\delta x &= \delta x^0 \cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t + \\
&\quad + \frac{1}{\omega_0}\int_0^t (\Delta n_x - \Delta\dot{m}_y r)\sin\omega_0(t-\tau)\,d\tau, \\
\delta y &= \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t + \\
&\quad + \frac{1}{\omega_0}\int_0^t (\Delta n_y + \Delta\dot{m}_x r)\sin\omega_0(t-\tau)\,d\tau, \\
\delta z &= \delta z^0 \cosh\omega_0\sqrt{2}t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\sinh\omega_0\sqrt{2}t + \\
&\quad + \frac{1}{\omega_0\sqrt{2}}\int_0^t \Delta n_z \sinh\omega_0\sqrt{2}(t-\tau)\,d\tau.
\end{aligned} \tag{5.95}$$

Integrating by parts, the first two formulas (5.95) may also be represented in the following form:

$$\left.\begin{aligned}
\delta x &= \delta x^0 \cos\omega_0 t + \frac{\delta\dot{x}^0 + r\Delta m_y^0}{\omega_0}\sin\omega_0 t - \\
&\quad - r\int_0^t \Delta m_y \cos\omega_0(t-\tau)\,d\tau + \\
&\quad + \frac{1}{\omega_0}\int_0^t \Delta n_x \sin\omega_0(t-\tau)\,d\tau, \\
\delta y &= \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0 - r\Delta m_x^0}{\omega_0}\sin\omega_0 t + \\
&\quad + r\int_0^t \Delta m_x \cos\omega_0(t-\tau)\,d\tau + \\
&\quad + \frac{1}{\omega_0}\int_0^t \Delta n_y \sin\omega_0(t-\tau)\,d\tau.
\end{aligned}\right| \tag{5.96}$$

In particular, with Δn_x, Δn_y, Δn_z, Δm_x, Δm_y from equalities (5.95) and (5.96) constant, and taking into account initial conditions (5.92), we find:

$$\delta x = \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta \dot{x}_0^0 + r\left(\Delta\omega_y^0 - \Delta m_y^0\right)}{\omega_0}\sin\omega_0 t.$$

$$\delta y = \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta \dot{y}_0^0 - r\left(\Delta\omega_x^0 - \Delta m_x^0\right)}{\omega_0}\sin\omega_0 t.$$

$$\delta z = -\frac{\Delta n_z}{2\omega_0^2} + \left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\cosh\omega_0\sqrt{2}\,t + \frac{\delta \dot{z}_0^0}{\omega_0\sqrt{2}}\sinh\omega_0\sqrt{2}\,t.$$

(5.97)

5.3.3. Motion of an object at constant velocity along the arc of a fixed great circle. Let us now turn to the case of motion of an object in a plane passing through the point O_1 at a constant distance from the point, i.e., the case represented by equations (5.35). Let us rewrite these equations introducing in place of μ/r^3 the notation ω_0^2, by which this quantity is usually designated:

$$\left.\begin{aligned} &\delta\ddot{x} + (\omega_0^2 - \omega_y^2)\delta x + 2\omega_y\delta\dot{z} = \Delta n_x - \Delta\dot{m}_y r, \\ &\delta\ddot{y} + \omega_0^2\delta y = \Delta n_y + \Delta\dot{m}_x r - \omega_y\Delta m_x r, \\ &\delta\ddot{z} - (2\omega_0^2 + \omega_y^2)\delta z - 2\omega_y\delta\dot{x} = \Delta n_z + 2r\omega_y\Delta m_y. \end{aligned}\right\} \quad (5.98)$$

Since trihedron xyz, in terms of projections on whose axes equations (5.98) are written, is a moving trihedron, the initial conditions for equations (5.98) coincide with those of equations (5.90) and are given by equalities (5.91) and (5.92).

The second equation (5.98), giving the error in the specification of the location of the object in a plane normal to the plane of its motion, is separate from the two others. The solution to this equation is analogous to the solution to the second equation (5.90) and has the form:

$$\delta y = \delta y^0\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t + \frac{1}{\omega_0}\int_0^t\left(\Delta n_y + \Delta\dot{m}_x r - \omega_y\Delta m_x r\right)\sin\omega_0(t-\tau)\,d\tau \quad (5.99)$$

or after integration by parts,

$$\delta y = \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0 - r\Delta m_x^0}{\omega_0}\sin\omega_0 t + \frac{1}{\omega_0}\int_0^t (\Delta n_y - \omega_y \Delta m_x r)\sin\omega_0 (t-\tau)\,d\tau + r\int_0^t \Delta m_x \cos\omega_0 (t-\tau)\,dt. \tag{5.100}$$

Specifically, at constant Δn_y, Δm_x, Δm_z we obtain:

$$\delta y = \frac{\Delta n_y - \omega_y \Delta m_z r}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y - \omega_y \Delta m_z r}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{y}_0^0 - r(\Delta m_x^0 - \Delta m_x)}{\omega_0}\sin\omega_0 t. \tag{5.101}$$

The system of the remaining two second-order equations

$$\left.\begin{aligned} &\delta\ddot{x} + (\omega_0^2 - \omega_y^2)\delta x + 2\omega_y \delta\dot{z} = \Delta n_x - \Delta\dot{m}_y r, \\ &\delta\ddot{z} - (2\omega_0^2 + \omega_y^2)\delta z - 2\omega_y \delta\dot{x} = \Delta r_z + 2r\omega_y \Delta m_y \end{aligned}\right\} \tag{5.102}$$

gives the error in the determination of the coordinates in the plane of motion of the object.

The characteristic equation of system (5.102) has the form:

$$p^4 + p^2(-\omega_0^2 + 2\omega_y^2) - (\omega_0^2 - \omega_y^2)(2\omega_0^2 + \omega_y^2) = 0. \tag{5.103}$$

If

$$\omega_y^2 < \omega_0^2, \tag{5.104}$$

i.e., if the velocity of the object is less than the first cosmic velocity, equation (5.103) has two real and two complex conjugate roots

$$p_{1,2} = \pm\mu, \quad p_{3,4} = \pm j\nu, \tag{5.105}$$

where

$$\begin{aligned} \mu &= \sqrt{\frac{1}{2}\left(\omega_0^2 - 2\omega_y^2 + \omega_0\sqrt{9\omega_0^2 - 8\omega_y^2}\right)}, \\ \nu &= \sqrt{\frac{1}{2}\left(-\omega_0^2 + 2\omega_y^2 + \omega_0\sqrt{9\omega_0^2 - 8\omega_y^2}\right)}. \end{aligned} \tag{5.106}$$

If $\omega_y^2 = 0$, then $\mu = \pm\omega_0 \times \sqrt{2}$ and $\nu = \pm\omega_0$, but if $\omega_y^2 = \omega_0^2$ (the case of the motion of a satellite in a circular orbit)

$$\mu = 0, \quad \nu = \pm\omega_0. \tag{5.107}$$

The dependence of μ and ν on ω_y with continuous variation of ω_y is represented in Figure 5.2.

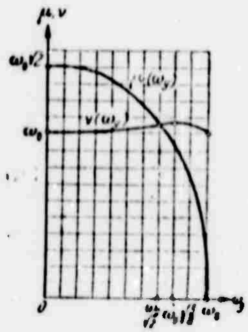

Figure 5.2

The quantity ν, showing small variation in the range $0<\omega_y<\omega_0$ and, as has already been noted, being equal to ω_0 at the points $\omega_y = 0$ and $\omega_y = \omega_0$, reaches a maximum at the point $\omega_y = \omega_0 \times \sqrt{5/8}$, at which $\nu = \omega_0 \times \sqrt{9/8} \approx 1.061\ \omega_0$.

The quantity μ decreases monotonically from the value of $\mu = \omega_0 \times \sqrt{2}$ down to 0 as ω_y varies from 0 to ω_0. At the point $\omega_y = \dfrac{\omega_0}{\sqrt{2}}$ the $\mu(\omega_y)$ and $\nu(\omega_y)$ curves intersect. At this point $\mu = \nu = \omega_0 \times \sqrt{\dfrac{\sqrt{5}}{2}} \approx 1.057\ \omega_0$. It should be noted that in the region from $\omega_y = 0$ to $\omega_y = \omega_0/2$, μ decreases by less than 0.15 from its value at the point $\omega_y = 0$. All of the remaining variation in the value of μ occurs in the segment $\frac{\omega_0}{2}<\omega_y<\omega_0$, where it falls off very rapidly.

Corresponding to the roots of the characteritic equation, the partial solutions to the homogeneous system of equations (5.102) will be the functions

$$\sin \nu t,\ \cos \nu t,\ \sinh \mu t,\ \cosh \mu t. \tag{5.108}$$

The general solution to a homogeneous system of equations in four arbitrary constants may be represented in the following form:

$$\begin{aligned}
\delta x = {} & \frac{A_x}{\mu^2+\nu^2}\left[(\omega_0^2-\omega_y^2+\mu^2)\cos \nu t - \right. \\
& \left. -(\omega_0^2-\omega_y^2-\nu^2)\cosh \mu t\right] + \frac{B_x}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)} \times \\
& \times\left[\nu(\omega_0^2-\omega_y^2+\mu^2)\sin \nu t + \mu(\omega_0^2-\omega_y^2-\nu^2)\sinh \mu t\right] + \\
& + \frac{C_x 2\omega_y \mu\nu}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}(\mu \sin \nu t - \nu \sinh \mu t) + \\
& + \frac{D_x 2\omega_y}{\mu^2+\nu^2}(\cos \nu t - \cosh \mu t), \\
\delta z = {} & \frac{A_x 2\omega_y(\omega_0^2-\omega_y^2)}{\mu\nu(\mu^2+\nu^2)}(\mu \sin \nu t - \nu \sinh \mu t) - \\
& - \frac{B_x 2\omega_y}{\mu^2+\nu^2}(\cos \nu t - \cosh \mu t) + \\
& + \frac{C_x}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\left[\mu^2(\omega_0^2-\omega_y^2-\nu^2)\cos \nu t + \right. \\
& \left. + \nu^2(\omega_0^2-\omega_y^2+\mu^2)\cosh \mu t\right] - \frac{D_x}{\mu\nu(\mu^2+\nu^2)} \times \\
& \times\left[\mu(\omega_0^2-\omega_y^2-\nu^2)\sin \nu t - \nu(\omega_0^2-\omega_y^2+\mu^2)\sinh \mu t\right].
\end{aligned} \tag{5.109}$$

In formulas (5.109) the constants A_x, B_x, C_x, D_x represent the initial values of δx^0, $\delta \dot{x}^0$, δz^0, $\delta \dot{z}^0$.

In order to find the general solution to the non-homogeneous equations (5.102), we may again use the Lagrange method of variation of the arbitrary parameters A_x, B_x, C_x and D_x, which are determined from the following system of four first-order linear differential equations:

$$\begin{aligned}
& \dot{A}_x\left[(\omega_0^2-\omega_y^2+\mu^2)\cos \nu t - (\omega_0^2-\omega_y^2-\nu^2)\cosh \mu t\right] + \\
& + \frac{\dot{B}_x}{\omega_0^2-\omega_y^2}\left[\nu(\omega_0^2-\omega_y^2+\mu^2)\sin \nu t + \right. \\
& \left. + \mu(\omega_0^2-\omega_y^2-\nu^2)\sinh \mu t\right] + \frac{\dot{C}_x 2\omega_y \mu\nu}{\omega_0^2-\omega_y^2} \times \\
& \times (\mu \sin \nu t - \nu \sinh \mu t) + \dot{D}_x 2\omega_y(\cos \nu t - \cosh \mu t) = 0, \\
& \frac{\dot{A}_x 2\omega_y(\omega_0^2-\omega_y^2)}{\mu\nu}(\mu \sin \nu t - \nu \sinh \mu t) -
\end{aligned}$$

$$-\dot{B}_x 2\omega_y(\cos\nu t-\cosh\mu t)+\frac{\dot{C}_x}{\omega_0^2-\omega_y^2}\times$$
$$\times[\mu^2(\omega_0^2-\omega_y^2-\nu^2)\cos\nu t+\nu^2(\omega_0^2-\omega_y^2+\mu^2)\cosh\mu t]-$$
$$-\frac{\dot{D}_x}{\mu\nu}[\mu(\omega_0^2-\omega_y^2-\nu^2)\sin\nu t-$$
$$-\nu(\omega_0^2-\omega_y^2+\mu^2)\sinh\mu t]=0,$$

$$\dot{A}_x[-\nu(\omega_0^2-\omega_y^2+\mu^2)\sin\nu t-\mu(\omega_0^2-\omega_y^2-\nu^2)\sinh\mu t]+$$
$$+\frac{\dot{B}_x}{\omega_0^2-\omega_y^2}[\nu^2(\omega_0^2-\omega_y^2+\mu^2)\cos\nu t+$$
$$+\mu^2(\omega_0^2-\omega_y^2-\nu^2)\cosh\mu t]+$$
$$+\frac{\dot{C}_x 2\omega_y\mu^2\nu^2}{\omega_0^2-\omega_y^2}(\cos\nu t-\cosh\mu t)+$$
$$+\dot{D}_x 2\omega_y(\cos\nu t-\cosh\mu t)=(\mu^2+\nu^2)(\Delta n_x-\Delta\dot{m}_y r).$$

$$\dot{A}_x 2\omega_y(\omega_0^2-\omega_y^2)(\cos\nu t-\cosh\mu t)-$$
$$-\dot{B}_x 2\omega_y(-\nu\sin\nu t-\mu\sinh\mu t)+$$
$$+\frac{\dot{C}_x\mu\nu}{\omega_0^2-\omega_y^2}[-\mu(\omega_0^2-\omega_y^2-\nu^2)\sin\nu t+$$
$$+\nu(\omega_0^2-\omega_y^2+\mu^2)\sinh\mu t]-\dot{D}_x[(\omega_0^2-\omega_y^2-\nu^2)\cos\nu t-$$
$$-(\omega_0^2-\omega_y^2+\mu^2)\cos\nu t]=$$
$$=(\mu^2+\nu^2)(\Delta n_x+2r\omega_y\Delta m_y). \tag{5.110}$$

Solving the system of equations (5.110) for $\dot{A}_x$, $\dot{B}_x$, $\dot{C}_x$, $\dot{D}_x$, integrating the solutions, substituting the resulting expressions into equalities (5.109) and taking account of the initial conditions (5.91), we arrive at the following formulas for the errors δx and δz:

$$\delta x=\frac{1}{(\mu^2+\nu^2)(\omega_0^2-\omega_y^2)}\int_0^t\Delta n_x[\mu(\omega_0^2-\omega_y^2-\nu^2)\times$$
$$\times\sinh\mu(t-\tau)+\nu(\omega_0^2-\omega_y^2+\mu^2)\sin\nu(t-\tau)]d\tau+$$
$$+\frac{r}{\mu^2+\nu^2}\int_0^t\Delta m_y[(\omega_0^2-\omega_y^2-\nu^2)\cos\mu(t-\tau)-$$
$$-(\omega_0^2-\omega_y^2+\mu^2)\cos\nu(t-\tau)]d\tau-$$
$$-\frac{2\omega_y}{\mu^2+\nu^2}\int_0^t\Delta n_x[\cosh\mu(t-\tau)-\cos\nu(t-\tau)]d\tau+ \tag{5.111}$$

$$
\begin{aligned}
&+\frac{\delta x^0}{\mu^2+\nu^2}\left[(\omega_0^2-\omega_y^2+\mu^2)\cos\nu t-(\omega_0^2-\omega_y^2-\nu^2)\cosh\mu t\right]+\\
&+\frac{\delta\dot{x}^0+r\,\Delta m_y^0}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\left[\nu(\omega_0^2-\omega_y^2+\mu^2)\sin\nu t+\right.\\
&\qquad\left.+\mu(\omega_0^2-\omega_y^2-\nu^2)\sinh\mu t\right]+\\
&+\frac{\delta z^0\,2\omega_y\mu\nu}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}(\mu\sin\nu t-\nu\sinh\mu t)+\\
&\qquad+\frac{\delta\dot{z}^0\,2\omega_y}{\mu^2+\nu^2}(\cos\nu t-\cosh\mu t).
\end{aligned}
$$

$$
\begin{aligned}
\delta z=&\frac{2\omega_y}{\mu^2+\nu^2}\int_0^t\Delta n_x\left[\cosh\mu(t-\tau)-\cos\nu(t-\tau)\right]d\tau+\\
&+\frac{1}{\mu\nu(\mu^2+\nu^2)}\int_0^t\Delta n_z\left[\nu(\omega_0^2-\omega_y^2+\mu^2)\sinh\mu(t-\tau)-\right.\\
&\qquad\left.-\mu(\omega_0^2-\omega_y^2-\nu^2)\sin\nu(t-\tau)\right]d\tau-\\
&-\frac{2r\omega_y(\omega_0^2-\omega_y^2)}{\mu\nu(\mu^2+\nu^2)}\int_0^t\Delta m_y\left[\mu\sin\nu(t-\tau)-\right.\\
&\qquad\left.-\nu\sinh\mu(t-\tau)\right]d\tau+\\
&+\frac{\delta x^0\,2\omega_y(\omega_0^2-\omega_y^2)}{\mu\nu(\mu^2+\nu^2)}(\mu\sin\nu t-\nu\sinh\mu t)-\\
&-\frac{(\delta\dot{x}^0+r\Delta m_y^0)\,2\omega_y}{\mu^2+\nu^2}(\cos\nu t-\cosh\mu t)+\\
&+\frac{\delta z^0}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\left[\mu^2(\omega_0^2-\omega_y^2-\nu^2)\cos\nu t+\right.\\
&\qquad\left.+\nu^2(\omega_0^2-\omega_y^2+\mu^2)\cosh\mu t\right]-\\
&-\frac{\delta\dot{z}^0}{\mu\nu(\mu^2+\nu^2)}\left[\mu(\omega_0^2-\omega_y^2-\nu^2)\sin\nu t-\right.\\
&\qquad\left.-\nu(\omega_0^2-\omega_y^2+\mu^2)\sinh\mu t\right]
\end{aligned}
\tag{5.111}
$$

Specifically, at constant Δn_x, Δn_z, Δm_y, we will have:

$$\delta x = \frac{\Delta n_x}{\omega_0^2-\omega_y^2} + \frac{1}{\mu^2+\nu^2}\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2-\omega_y^2}\right)\times$$
$$\times\left[(\omega_0^2-\omega_y^2+\mu^2)\cos\nu t - (\omega_0^2-\omega_y^2-\nu^2)\cosh\mu t\right]+$$
$$+\frac{\delta\dot{x}^0}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\left[\nu(\omega_0^2-\omega_y^2+\mu^2)\sin\nu t+\right.$$
$$\left.+\mu(\omega_0^2-\omega_y^2-\nu^2)\sinh\mu t\right]+$$
$$+\frac{2\omega_y\mu\nu}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\left(\delta z^0+\frac{\Delta n_z+2r\omega_y\Delta m_y}{2\omega_0^2+\omega_y^2}\right)\times$$
$$\times(\mu\sin\nu t-\nu\sinh\mu t)+\frac{2\omega_y\delta\dot{z}^0}{\mu^2+\nu^2}(\cos\nu t-\cosh\mu t).$$
$$\delta z = -\frac{\Delta n_z+2r\omega_y\Delta m_y}{2\omega_0^2+\omega_y^2}+\frac{2\omega_y(\omega_0^2-\omega_y^2)}{\mu\nu(\mu^2+\nu^2)}\times$$
$$\times\left(\delta x^0-\frac{\Delta n_x}{\omega_0^2-\omega_y^2}\right)(\mu\sin\nu t-\nu\sinh\mu t)-$$
$$-\frac{\delta\dot{x}^0 2\omega_y}{\mu^2+\nu^2}(\cos\nu t-\cosh\mu t)+$$
$$+\frac{1}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\left(\delta z^0+\frac{\Delta n_z+2r\omega_y\Delta m_y}{2\omega_0^2+\omega_y^2}\right)\times$$
$$\times\left[\mu^2(\omega_0^2-\omega_y^2-\nu^2)\cos\nu t+\right.$$
$$\left.+\nu^2(\omega_0^2-\omega_y^2+\mu^2)\cosh\mu t\right]-$$
$$-\frac{\delta\dot{z}^0}{\mu\nu(\mu^2+\nu^2)}\left[\mu(\omega_0^2-\omega_y^2-\nu^2)\sin\nu t-\right.$$
$$\left.-\nu(\omega_0^2-\omega_y^2+\mu^2)\sinh\mu t\right].$$

(5.112)

Finally, if the errors in the initial conditions are equal to 0, and the sources of perturbation are only the instrument errors of the newtonometers and gyroscopes, we obtain from equalities (5.101) and (5.102):

$$\delta x = \frac{\Delta n_x}{\omega_0^2-\omega_y^2}\left\{1-\frac{1}{\mu^2+\nu^2}\left[(\omega_0^2-\omega_y^2+\mu^2)\cos\nu t+\right.\right.$$
$$\left.\left.+(\omega_0^2-\omega_y^2-\nu^2)\cosh\mu t\right]\right\}+$$
$$+\frac{2\omega_y\mu\nu(\Delta n_z+2r\omega_y\Delta m_y)}{(\omega_0^2-\omega_y^2)(2\omega_0^2+\omega_y^2)(\mu^2+\nu^2)}(\mu\sin\nu t-\nu\sinh\mu t).$$
$$\delta y = \frac{-\Delta n_y+r\omega_y\Delta m_z}{\omega_0^2}(\cos\omega_0 t-1)+$$
$$+\frac{r\Delta m_z}{\omega_0}\sin\omega_0 t.$$
$$\delta z = \frac{\Delta n_z+2r\omega_y\Delta m_y}{2\omega_0^2+\omega_y^2}-$$
$$-\left\{-1+\frac{1}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\left[\mu^2(\omega_0^2-\right.\right.$$
$$\left.\left.-\omega_y^2-\nu^2)\cos\nu t+\nu^2(\omega_0^2-\omega_y^2+\mu^2)\cosh\mu t\right]\right\}-$$
$$-\frac{2\omega_y\Delta n_x}{\mu\nu(\mu^2+\nu^2)}(\mu\sin\nu t-\nu\sinh\mu t)$$

(5.113)

Relations (5.101), (5.112) and (5.113) enable us to evaluate comparatively easily the errors δx, δy, δz for the case in question as a function of inaccuracies in the initial conditions of the operation of the inertial system, basic instrument errors, and also as a function of the velocity of motion of the object.

Let us turn to the general solution given by formulas (5.111) and (5.100). First of all we note that at $\omega_y = 0$, when

$$\mu = \omega_0 \sqrt{2}, \quad \nu = \omega_0 \tag{5.114}$$

formulas (5.111) and (5.100) reduce to formulas (5.96) and (5.95), respectively.

For formulas (5.111) and (5.100) we can also obtain expressions for δx, δy, δz for the case in which the object is a satellite moving in a circular orbit. In order to obtain these expressions, we must substitute into equalities (5.100) and (5.111) the quantity ω_0 in place of ω_y, and, in addition, substitute in place of μ and ν in equality (5.111), their values (5.107). In this regard, the following should be kept in mind. In formulas (5.111) the denominators of the coefficients contain μ and $\omega_0^2 - \omega_y^2$, which reduce to 0. But it is easy to see that the numerators of the same expressions also reduce to 0. Therefore, in subtituting the values of μ and ν it is necessary at the same time to expand the resulting indeterminacies of the form $\frac{0}{0}$.

We could do this by means of l'Hospital's rule, but it is simpler to proceed as follows. Taking

$$\omega_0^2 - \omega_y^2 = \varepsilon^2, \tag{5.115}$$

where ε^2 is small, we find from formulas (5.106):

$$\mu^2 = 3\varepsilon^2, \quad \nu^2 = \omega_0^2 + \varepsilon^2. \tag{5.116}$$

Substituting (5.115) and (5.116) into formulas (5.111) and expanding where necessary the functions of the arguments $\mu(t - \tau)$ and $\nu(t - \tau)$ in series, we obtain for $\varepsilon^2 \to 0$ the desired expressions. Performing the indicated substitutions and the required transformations, we arrive at the following formulas:

$$\delta x = \frac{1}{\omega_0}\int_0^t \Delta n_x[-3\omega_0(t-\tau)+4\sin\omega_0(t-\tau)]\,d\tau -$$
$$- r\int_0^t \Delta m_y\,d\tau - \frac{2}{\omega_0}\int_0^t \Delta n_z[1-\cos\omega_0(t-\tau)]\,d\tau +$$
$$+\delta x^0 + \frac{\delta\dot{x}^0 + r\Delta m_y^0}{\omega_0}(4\sin\omega_0 t - 3\omega_0 t) +$$
$$+6\delta z^0(\sin\omega_0 t - \omega_0 t) + \frac{2\delta\dot{z}^0}{\omega_0}(\cos\omega_0 t - 1).$$
$$\delta y = \delta y^0\cos\omega_0 t + \frac{\delta\dot{y}^0 - r\Delta m_x^0}{\omega_0}\sin\omega_0 t +$$
$$+\frac{1}{\omega_0}\int_0^t(\Delta n_y - r\omega_0\Delta m_z)\sin\omega_0(t-\tau)\,d\tau +$$
$$+ r\int_0^t \Delta m_z\cos\omega_0(t-\tau)\,d\tau.$$
$$\delta z = \frac{2}{\omega_0}\int_0^t \Delta n_x[1-\cos\omega_0(t-\tau)]\,d\tau +$$
$$+\frac{1}{\omega_0}\int_0^t \Delta n_z\sin\omega_0(t-\tau)\,d\tau + \frac{2(\delta\dot{x}^0 + r\Delta m_y^0)}{\omega_0}\times$$
$$\times(1-\cos\omega_0 t) + \delta z^0(4-3\cos\omega_0 t) + \frac{\delta\dot{z}^0}{\omega_0}\sin\omega_0 t. \qquad (5.117)$$

At constant Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z, these formulas take the form:

$$\delta x = \delta x^0 + \frac{\delta\dot{x}^0}{\omega_0}(4\sin\omega_0 t - 3\omega_0 t) +$$
$$+6\delta z^0(\sin\omega_0 t - \omega_0 t) + \frac{2\delta\dot{z}^0}{\omega_0}(\cos\omega_0 t - 1) +$$
$$+\frac{\Delta n_x}{\omega_0^2}\left[-\frac{3(\omega_0 t)^2}{2} + 4(1-\cos\omega_0 t)\right] +$$
$$+\frac{4r\Delta m_y}{\omega_0}(\sin\omega_0 t - \omega_0 t) +$$
$$+\frac{2\Delta n_z}{\omega_0^2}(\sin\omega_0 t - \omega_0 t).$$
$$\delta y = \delta y^0\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t +$$
$$+\frac{1}{\omega_0^2}(\Delta n_y - r\omega_0\Delta m_z)(1-\cos\omega_0 t).$$
$$\delta z = \frac{2(\delta\dot{x}^0 + r\Delta m_y)}{\omega_0}(1-\cos\omega_0 t) +$$
$$+\delta z^0(4-3\cos\omega_0 t) + \frac{\delta\dot{z}^0}{\omega_0}\sin\omega_0 t +$$
$$+\frac{2\Delta n_x}{\omega_0^2}(\omega_0 t - \sin\omega_0 t) + \frac{\Delta n_z}{\omega_0^2}(1-\cos\omega_0 t)$$

(5.118)

towards the north. The projections ω_y and ω_z entering into the coefficients of these equations are given by equalities (5.36).

We note that at $\varphi = 0$, i.e., for motion along the equator, we arrive at the preceding case. At $v = 0$ and $\varphi = \pi/2$, i.e., at the poles, equations (5.121) reduce to equations (5.90).

For an arbitrary parallel equations (5.121) form a coupled sixth-order system. The characteristic equation of this system reduces to a complete cubic equation in the square of the unknown ($p^2 = q$):

$$q^3 + 2q^2(\omega_y^2 + \omega_z^2) + q[-3\omega_0^4 + 3\omega_0^2(\omega_y^2 - 2\omega_z^2) + (\omega_y^2 + \omega_z^2)^2] - \omega_0^2(\omega_0^2 - \omega_y^2 - \omega_z^2)(2\omega_0^2 + \omega_y^2 - 2\omega_z^2) = 0. \tag{5.122}$$

If

$$\omega_y^2 = \omega_z^2 = 0, \tag{5.123}$$

then equation (5.122) reduces to the simpler equation

$$q^3 - 3\omega_0^4 q - 2\omega_0^6 = 0, \tag{5.124}$$

the roots of which are

$$q_{1,2} = -\omega_0^2, \quad q_3 = 2\omega_0^2, \tag{5.125}$$

which correspond to equations (5.90) to which system (5.121) reduces under the conditions (5.123).

As a result of the continuity of the dependence of the roots of equation (5.122) on ω_y and ω_z, in the vicinity of (5.123) there exists a region in which

$$q_1 = -\mu_1^2, \quad q_2 = -\mu_2^2, \quad q_3 = \mu_3^2, \tag{5.126}$$

such that the roots of the characteritic equation are equal:

$$p_{1,2} = \pm i\mu_1, \quad p_{3,4} = \pm i\mu_2, \quad p_{5,6} = \pm \mu_3, \tag{5.127}$$

where the numbers μ_1, μ_2, μ_3 are real and positive.

It follows from the investigation of the stability of system (5.121) carried out at the beginning of this section, that equalities (5.127) will obtain whenever

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0. \tag{5.128}$$

Specifically, equalities (5.127) will obtain for the case in which the object is stationary relative to the earth when

$$\omega_y = u\cos\varphi, \quad \omega_z = u\sin\varphi, \tag{5.129}$$

where u is the earth rate and φ is the geocentric latitude.

Let us solve the system of equations (5.121) for the case in which the roots of the characteritic equation are defined by equalities (5.127). The system (5.121) is of a rather high order. It is therefore convenient to solve it using the procedures of operational analysis.

We will use $\delta x(p)$, $\delta y(p)$, $\delta z(p)$ to represent the Carson-Heaviside[8] transformations of the functions $\delta x(t)$, $\delta y(t)$, $\delta z(t)$, such that, for example,

$$\frac{\delta x(p)}{p} = \int_0^\infty e^{-pt}\,\delta x(t)\,dt. \tag{5.130}$$

We introduce, further, the following notation for the right sides of system (5.121):

$$\left.\begin{aligned} f_1(t) &= \Delta n_x - \Delta\dot{m}_y r - \omega_z \Delta m_z r, \\ f_2(t) &= \Delta n_y + \Delta\dot{m}_x r - \omega_y \Delta m_z r - \omega_z \Delta m_y r, \\ f_3(t) &= \Delta n_z + 2r\omega_y \Delta m_y. \end{aligned}\right\} \tag{5.131}$$

and we will designate their transformations by $f_1(p)$, $f_2(p)$, $f_3(p)$, respectively.

Subjecting system (5.121) to the Carson-Heaviside transformation, we obtain the following transform equations:

$$
\left.
\begin{aligned}
&(p^2+\omega_0^2-\omega_y^2-\omega_z^2)\delta x(p)-2\omega_z p\,\delta y(p)+\\
&\quad+2\omega_y p\,\delta z(p)=f_1(p)+p^2\delta x^0+p\,\delta\dot{x}^0-\\
&\qquad\qquad-2\omega_z p\,\delta y^0+2\omega_y p\,\delta z^0.\\
&2\omega_z p\,\delta x(p)+(p^2+\omega_0^2-\omega_z^2)\delta y(p)+\\
&\quad+\omega_y\omega_z\,\delta z(p)=f_2(p)+p^2\delta y^0+\\
&\qquad\qquad+p\,\delta\dot{y}^0+2\omega_z p\,\delta x^0.\\
&-2\omega_y p\,\delta x(p)+\omega_z\omega_y\,\delta y(p)+\\
&\quad+(p^2-2\omega_0^2-\omega_y^2)\delta z(p)=\\
&\qquad=f_3(p)+p^2\delta z^0+p\,\delta\dot{z}^0-2\omega_y p\,\delta x^0.
\end{aligned}
\right|
\qquad (5.132)
$$

Here the initial conditions will be quantities given by equalities (5.91) and (5.92).

Solving equations (5.132) for δx(p), δy(p), δz(p) and performing the required transformations, we find:

$$
\begin{aligned}
\delta x(p)=\frac{1}{\Delta(p)}\{&[f_1(p)+p\,\delta\dot{x}^0][p^4-\\
&-p^2(\omega_0^2+\omega_y^2+\omega_z^2)-\omega_0^2(2\omega_0^2+\omega_y^2-2\omega_z^2)]+\\
&+[f_2(p)+p\,\delta\dot{y}^0][2\omega_z p(p^2-2\omega_0^2)]-\\
&-[f_3(p)+p\,\delta\dot{z}^0][2\omega_y p(p^2+\omega_0^2)]+\\
&+\delta x^0 p^2[p^4-p^2(\omega_0^2-3\omega_y^2-3\omega_z^2)-\\
&-\omega_0^2(2\omega_0^2+6\omega_y^2-3\omega_z^2)]+\delta y^0 2\omega_z p[p^2(-\omega_0^2+\\
&+\omega_y^2+\omega_z^2)+\omega_0^2(2\omega_0^2+\omega_y^2-2\omega_z^2)]+\\
&+\delta z^0 2\omega_y p[-p^2(2\omega_0^2+\omega_y^2+\omega_z^2)-\\
&-\omega_0^2(2\omega_0^2+\omega_y^2-2\omega_z^2)]\}.
\end{aligned}
$$

$$
\begin{aligned}
\delta y(p)=\frac{1}{\Delta(p)}\{&-[f_1(p)+p\,\delta\dot{x}^0][2\omega_z p(p^2-\\
&-2\omega_0^2)]+[f_2(p)+p\,\delta\dot{y}^0][p^4-p^2(\omega_0^2-2\omega_y^2+\\
&+\omega_z^2)-(\omega_0^2-\omega_y^2-\omega_z^2)(2\omega_0^2+\omega_y^2)]+[f_3(p)+\\
&+p\,\delta\dot{z}^0][\omega_y\omega_z(3p^2-\omega_0^2+\omega_y^2+\omega_z^2)]+\\
&+\delta x^0 2\omega_z(\omega_0^2-\omega_y^2-\omega_z^2)p(p^2-2\omega_0^2)+\\
&+\delta y^0 p^2[p^4-p^2(\omega_0^2-2\omega_y^2-3\omega_z^2)-\\
&-\omega_0^2(2\omega_0^2-\omega_y^2+6\omega_z^2)+\omega_y^2\omega_z^2+\omega_y^4]+\\
&+\delta z^0\omega_y\omega_z p^2(-p^2+7\omega_0^2+\omega_y^2+\omega_z^2)].
\end{aligned}
$$

(5.133)

$$\begin{aligned}\delta z(p)=\frac{1}{\Delta(p)}\{[f_1(p)+p\,\delta\dot{x}^0][2\omega_y p(p^2+\omega_0^2)]+\\+[f_2(p)+p\,\delta\dot{y}^0][\omega_y\omega_z(3p^2-\omega_0^2+\omega_y^2+\omega_z^2)]+\\+[f_3(p)+p\,\delta\dot{z}^0][p^4+p^2(2\omega_0^2+2\omega_z^2-\omega_y^2)+\\+(\omega_0^2-\omega_z^2)(\omega_0^2-\omega_y^2-\omega_z^2)]-\\-\delta x^0 2\omega_y p(\omega_0^2-\omega_y^2-\omega_z^2)(p^2+\omega_0^2)-\\-\delta y^0\omega_y\omega_z p^2(p^2+5\omega_0^2-\omega_y^2-\omega_z^2)+\\+\delta z^0 p^2[p^4+p^2(2\omega_0^2+2\omega_z^2+3\omega_y^2)+\\+\omega_0^2(\omega_0^2+3\omega_y^2-2\omega_z^2)+\omega_y^2\omega_z^2+\omega_z^4]\}.\end{aligned} \qquad (5.133)$$

where in accordance with equalities (5.127)

$$\Delta(p)=(p^2+\mu_1^2)(p^2+\mu_2^2)(p^2-\mu_3^2). \qquad (5.134)$$

Let us convert from the transforms (5.133) to the original forms $\delta x(t)$, $\delta y(t)$, $\delta z(t)$.

Examining the right sides of equalities (5.133), we note that they include expressions of the form

$$P(p)=\frac{F(p)Q(p)}{\Delta(p)}, \qquad (5.135)$$

where $F(p)$ is one of the transforms $f_1(p)$, $f_2(p)$, $f_3(p)$ and $Q(p)$ is a polynomial of order not greater than the fourth, and expressions of the form

$$R(p)=\frac{paS(p)}{\Delta(p)}, \qquad (5.136)$$

where a is one of the initial conditions δx^0, δy^0, δz^0, $\delta\dot{x}^0$, $\delta\dot{y}^0$, $\delta\dot{z}^0$, and $S(p)$ is a polynomial of order not greater than the fifth.

But if

$$\frac{pQ(p)}{\Delta(p)}\doteqdot q(t), \quad F(p)\doteqdot f(t), \qquad (5.137)$$

where the symbol ":" denotes correspondence between the pre-image and the transformation, then, as is well known,

$$\frac{F(p)Q(p)}{\Delta(p)}\doteqdot\int_0^t f(\tau)q(t-\tau)\,d\tau. \qquad (5.138)$$

Thus, the problem reduces to that of finding the pre-image of the transformation

$$R(p)=\frac{p(b_0p^5+b_1p^4+b_2p^3+b_3p^2+b_4p+b_5)}{\Delta(p)}. \tag{5.139}$$

in which the coefficients b_1 may include coefficients equal to 0.

In order to find the pre-image corresponding to transformation R(p), we expand the right side of equality (5.139) into a sum of the form

$$R(p)=\frac{p(A_1p+B_1)}{p^2+\mu_1^2}+\frac{p(A_2p+B_2)}{p^2+\mu_2^2}+\frac{p(A_3p+B_3)}{p^2-\mu_3^2}. \tag{5.140}$$

Equating the right sides of equalities (5.139) and (5.140), we find A_i and B_i. Expressions for A_1 and B_1 are obtained in this manner as follows:

$$\left.\begin{aligned} A_1&=\frac{b_4-\mu_1^2b_2+\mu_1^4b_0}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)},\\ B_1&=\frac{b_5-\mu_1^2b_3+\mu_1^4b_1}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\end{aligned}\right\} \tag{5.141}$$

Expressions for A_2 and B_2 may be obtained from formulas (5.141) if μ_2 is substituted for μ_1 and vice versa. In order to obtain A_3 and B_3, it is necessary to substitute into the numerators of formulas (5.141) the quantity μ_3^2 in place of μ_1^2, and to replace the denominators by $(\mu_1^2 + \mu_3^2)\times(\mu_2^2 + \mu_3^2)$.

For the terms in the right side of (5.140) the conversion to the pre-images is obvious, since

$$\frac{p^2}{p^2-\mu_3^2} \doteq \cosh\mu_3 t, \qquad \frac{\mu_3 p}{p^2-\mu_3^2} \doteq \sinh\mu_3 t.$$

$$\frac{p^2}{p^2+\mu_{1,2}^2} \doteq \cos\mu_{1,2}t, \quad \frac{\mu_{1,2}p}{p^2+\mu_{1,2}^2} \doteq \sin\mu_{1,2}t. \tag{5.142}$$

Performing the indicated sequence of calculations, we obtain an exact solution to system (5.121) in the following form:

$$
\begin{aligned}
\delta x = \int_0^t \{ & f_1(\tau)[a_{11}\sin\mu_1(t-\tau) + a_{12}\sin\mu_2(t-\tau) + \\
& + a_{13}\ \sinh\mu_3(t-\tau)] + f_2(\tau)[a_{21}\cos\mu_1(t-\tau) + \\
& + a_{22}\cos\mu_2(t-\tau) + a_{23}\ \sinh\mu_3(t-\tau)] + \\
& + f_3(\tau)[a_{31}\cos\mu_1(t-\tau) + a_{32}\cos\mu_2(t-\tau) + \\
& + a_{33}\ \cosh\mu_3(t-\tau)]\}\, d\tau + \\
& + \delta\dot{x}^0(a_{11}\sin\mu_1 t + a_{12}\sin\mu_2 t + a_{13}\ \sinh\mu_3 t) + \\
& + \delta\dot{y}^0(a_{21}\cos\mu_1 t + a_{22}\cos\mu_2 t + a_{23}\cosh\ \mu_3 t) + \\
& + \delta\dot{z}^0(a_{31}\cos\mu_1 t + a_{32}\cos\mu_2 t + a_{33}\cosh\ \mu_3 t) + \\
& + \delta x^0(a^0_{11}\cos\mu_1 t + a^0_{12}\cos\mu_2 t + a^0_{13}\cosh\ \mu_3 t) + \\
& + \delta y^0(a^0_{21}\sin\mu_1 t + a^0_{22}\sin\mu_2 t + a^0_{23}\ \sinh\mu_3 t) + \\
& \quad + \delta z^0(a^0_{31}\sin\mu_1 t + a^0_{32}\sin\mu_2 t + a^0_{33}\ \sinh\mu_3 t).
\end{aligned}
$$

$$
\begin{aligned}
\delta y = \int_0^t \{ & f_1(\tau)[b_{11}\cos\mu_1(t-\tau) + b_{12}\cos\mu_2(t-\tau) + \\
& + b_{13}\ \cosh\mu_3(t-\tau)] + f_2(\tau)[b_{21}\sin\mu_1(t-\tau) + \\
& + b_{22}\sin\mu_2(t-\tau) + b_{23}\ \sinh\ \mu_3(t-\tau)] + \\
& + f_3(\tau)[b_{31}\sin\mu_1(t-\tau) + b_{32}\sin\mu_2(t-\tau) + \\
& + b_{33}\ \sinh\mu_3(t-\tau)]\}\, d\tau + \\
& + \delta\dot{x}^0(b_{11}\cos\mu_1 t + b_{12}\cos\mu_2 t + b_{13}\cosh\ \mu_3 t) + \\
& + \delta\dot{y}^0(b_{21}\sin\mu_1 t + b_{22}\sin\mu_2 t + b_{23}\sinh\ \mu_3 t) + \\
& + \delta\dot{z}^0(b_{31}\sin\mu_1 t + b_{32}\sin\mu_2 t + b_{33}\sinh\ \mu_3 t) + \\
& + \delta x^0(b^0_{11}\sin\mu_1 t + b^0_{12}\sin\mu_2 t + b^0_{13}\sinh\ \mu_3 t) + \\
& + \delta y^0(b^0_{21}\cos\mu_1 t + b^0_{22}\cos\mu_2 t + b^0_{23}\sinh\ \mu_3 t) + \\
& \quad + \delta z^0(b^0_{31}\cos\mu_1 t + b^0_{32}\cos\mu_2 t + b^0_{33}\cosh\ \mu_3 t).
\end{aligned}
$$

$$
\begin{aligned}
\delta z = \int_0^t \{ & f_1(\tau)[c_{11}\cos\mu_1(t-\tau) + c_{12}\cos\mu_2(t-\tau) + \\
& + c_{13}\,\mathrm{ch}\,\mu_3(t-\tau)] + f_2(\tau)[c_{21}\sin\mu_1(t-\tau) + \\
& + c_{22}\sin\mu_2(t-\tau) + c_{23}\ \sinh\ \mu_3(t-\tau)] + \\
& + f_3(\tau)[c_{31}\sin\mu_1(t-\tau) + c_{32}\sin\mu_2(t-\tau) + \\
& + c_{33}\ \sinh\ \mu_3(t-\tau)]\}\, d\tau + \\
& + \delta\dot{x}^0(c_{11}\cos\mu_1 t + c_{12}\cos\mu_2 t + c_{13}\cosh\ \mu_3 t) + \\
& + \delta\dot{y}^0(c_{21}\sin\mu_1 t + c_{22}\sin\mu_2 t + c_{23}\ \sinh\mu_3 t) + \\
& + \delta\dot{z}^0(c_{31}\sin\mu_1 t + c_{32}\sin\mu_2 t + c_{33}\ \sinh\mu_3 t) + \\
& + \delta x^0(c^0_{11}\sin\mu_1 t + c^0_{12}\sin\mu_2 t + c^0_{13}\ \cosh\mu_3 t) + \\
& + \delta y^0(c^0_{21}\cos\mu_1 t + c^0_{22}\cos\mu_2 t + c^0_{23}\ \cosh\ \mu_3 t) + \\
& \quad + \delta z^0(c^0_{31}\cos\mu_1 t + c^0_{32}\cos\mu_2 t + c^0_{33}\ \cosh\ \mu_3 t).
\end{aligned}
\tag{5.143}
$$

The quantities a_{ij}, a^0_{ij}, b_{ij}, b^0_{ij}, c_{ij}, c^0_{ij} are expressed by means of the coefficients of system (5.121) and the roots of the characteristic equation (5.127) by the following equalities:

$$
\begin{aligned}
a_{11} &= \frac{-\omega_0^2(2\omega_0^2+\omega_y^2-2\omega_z^2)+\mu_1^2(\omega_0^2+\omega_y^2+\omega_z^2)+\mu_1^4}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
a_{21} &= \frac{-2\omega_z(2\omega_0^2+\mu_1^2)}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
a_{31} &= \frac{-2\omega_y(\omega_0^2-\mu_1^2)}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
a_{11}^0 &= \frac{-\omega_0^2(2\omega_0^2+6\omega_z^2-3\omega_y^2)+\mu_1^2(\omega_0^2-3\omega_y^2-3\omega_z^2)+\mu_1^4}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
a_{21}^0 &= \frac{2\omega_z\left[\omega_0^2(2\omega_0^2+\omega_y^2-2\omega_z^2)+\mu_1^2(\omega_0^2-\omega_y^2-\omega_z^2)\right]}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
a_{31}^0 &= \frac{-2\omega_y\left[\omega_0^2(2\omega_0^2+\omega_y^2-2\omega_z^2)-\mu_1^2(2\omega_0^2+\omega_y^2+\omega_z^2)\right]}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
b_{11} &= \frac{2\omega_z(2\omega_0^2+\mu_1^2)}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
b_{21} &= \frac{-(\omega_0^2-\omega_y^2-\omega_z^2)(2\omega_0^2+\omega_y^2)+\mu_1^2(\omega_0^2-2\omega_y^2+\omega_z^2)+\mu_1^4}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
b_{31} &= \frac{-\omega_y\omega_z(\omega_0^2-\omega_y^2-\omega_z^2+3\mu_1^2)}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
b_{11}^0 &= \frac{-2\omega_z(\omega_0^2-\omega_y^2-\omega_z^2)(2\omega_0^2+\mu_1^2)}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
b_{21}^0 &= \frac{-\omega_0^2(2\omega_0^2-\omega_y^2+6\omega_z^2)+\omega_y^2(\omega_y^2+\omega_z^2)+\mu_1^2(\omega_0^2-2\omega_y^2-3\omega_z^2)+\mu_1^4}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
b_{31}^0 &= \frac{\omega_y\omega_z(7\omega_0^2+\omega_y^2+\omega_z^2+\mu_1^2)}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
c_{11} &= \frac{2\omega_y(\omega_0^2-\mu_1^2)}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
c_{21} &= \frac{-\omega_y\omega_z(\omega_0^2-\omega_y^2-\omega_z^2+3\mu_1^2)}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
c_{31} &= \frac{(\omega_0^2-\omega_z^2)(\omega_0^2-\omega_y^2-\omega_z^2)-\mu_1^2(2\omega_0^2+2\omega_z^2-\omega_y^2)+\mu_1^4}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
c_{11}^0 &= \frac{-2\omega_y(\omega_0^2-\omega_y^2-\omega_z^2)(\omega_0^2-\mu_1^2)}{\mu_1(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
c_{21}^0 &= \frac{-\omega_y\omega_z(5\omega_0^2-\omega_y^2-\omega_z^2-\mu_1^2)}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.\\
c_{31}^0 &= \frac{\omega_0^2(\omega_0^2+3\omega_y^2-2\omega_z^2)+\omega_z^2(\omega_y^2+\omega_z^2)-\mu_1^2(2\omega_0^2+2\omega_z^2+3\omega_y^2)+\mu_1^4}{(\mu_1^2-\mu_2^2)(\mu_1^2+\mu_3^2)}.
\end{aligned}
\qquad (5.144)
$$

Only eighteen coefficients of the 54 entering into formulas (5.143), namely the coefficients a_{i1}, a^0_{i1}, b_{i1}, b^0_{i1}, c_{i1}, c^0_{i1}, (= 1, 2, 3) are written out here. The remaining 36 coefficients are tained from relations (5.144) in the following manner. In order to tain coefficients a_{i2}, a^0_{i2}, b_{i2}, b^0_{i2}, c_{i2}, c^0_{i2}, μ_1 and μ_2 must change aces everywhere they occur in the relations (5.144). In order to nd the coefficients a_{i3}, a^0_{i3}, b_{i3}, b^0_{i3}, c_{i3}, c^0_{i3}, μ_3^2 must be substituted r μ_1^2, and μ_3^4 for μ_1^4 in the numerators of relations (5.144), and in t e denominators $\mu_1(\mu_1^2 - \mu_2^2)x(\mu_1^2 + \mu_3^2)$ and $\mu(\mu_1^2 - \mu_2^2)x(\mu_1^2 + \mu_3^2)$ must be r placed by $\mu_3(\mu_1^2 + \mu_3^2)x(\mu_2^2 + \mu_3^2)$ and $(\mu_1^2 + \mu_3^2)x(\mu_2^2 + \mu_3^2)$, respectively.

The solution (5.143) to equations (5.121) is unwieldy and in general difficult to inspect. In addition, we do not have explicit expressions in terms of the coefficients of the initial system for the roots μ_1, μ_2, μ_3 of the characteristic equation (5.122) of system (5.121), which appear in solution (5.143). It is, of course, possible to determine them by using the Cardan solutions for the roots of a cubic equation, but this leads to further complication of the form of solution (5.143).

5.3.5. <u>Motion of an object along a parallel at low velocity</u>. The solution to system (5.121) may be simplified to a significant degree y assuming that

$$\omega_0^2 \gg \omega_y^2 = \left(u\cos\varphi + \frac{v}{r}\right)^2. \tag{5.145}$$

dition (5.145) permits coverage of a fairly wide class of motions.

Let us take the error equations in the form (5.28). Instead of 5 121) we will then have:

$$\begin{aligned}
&\delta\ddot{x}_1 + (\omega_0^2 - \omega_{y_1}^2)\delta x_1 + \omega_{x_1}\omega_{y_1}\delta y_1 + \dot{\omega}_{y_1}\delta z + \\
&\qquad + 2\omega_{y_1}\delta\dot{z} = \Delta n_{x_1} - \Delta\dot{m}_{y_1} r - \omega_{z_1}\Delta m_z r, \\
&\delta\ddot{y}_1 + (\omega_0^2 - \omega_{x_1}^2)\delta y_1 + \omega_{x_1}\omega_{y_1}\delta x_1 - \dot{\omega}_{x_1}\delta z - \\
&\qquad - 2\omega_{x_1}\delta\dot{z} = \Delta n_{y_1} + \Delta\dot{m}_{x_1} r - \omega_{y_1}\Delta m_z r, \\
&\delta\ddot{z} - (2\omega_0^2 + \omega_{x_1}^2 + \omega_{y_1}^2)\delta z - \dot{\omega}_{y_1}\delta x_1 - 2\omega_{y_1}\delta\dot{x}_1 + \\
&+ \dot{\omega}_{x_1}\delta y_1 + 2\omega_{x_1}\delta\dot{y}_1 = \Delta n_z + 2r(\omega_{x_1}\Delta m_{x_1} + \omega_{y_1}\Delta m_{y_1})
\end{aligned} \tag{5.146}$$

Equations (5.121), we recall, are written in terms of projections on the axes of a moving trihedron oriented to the points of the compass and on a sphere surrounding the earth. Equations (5.146) are also error equations in terms of projections on the axes of a moving trihedron, but such that the projection of the absolute rate of its rotation around the z axis is equal to 0. This trihedron was termed above a free-azimuth trihedron. In order to distinguish it from a trihedron oriented to the points of the compass, we will designate it here by $x_1 y_1 z$.

If trihedra xyz and $x_1 y_1 z$ coincide at the initial moment, their relative position will afterwards be determined by the following table of direction cosines:

$$\begin{array}{c|ccc|} & x & y & z \\ x_1 & \cos\psi & \sin\psi & 0 \\ y_1 & -\sin\psi & \cos\psi & 0 \\ z & 0 & 0 & 1. \end{array} \tag{5.147}$$

The angle ψ is found from the condition

$$\omega_z' = \omega_z + \dot{\psi} = 0 \tag{5.148}$$

and therefore

$$\psi = -\int_0^t \omega_z \, dt. \tag{5.149}$$

In accordance with table (5.147) we have:

$$\left.\begin{aligned} \delta x_1 &= \delta x \cos\psi + \delta y \sin\psi, & \delta y_1 &= -\delta x \sin\psi + \delta y \cos\psi, \\ \omega_{x_1} &= \omega_x \cos\psi + \omega_y \sin\psi, & \omega_{y_1} &= -\omega_x \sin\psi + \omega_y \cos\psi. \end{aligned}\right\} \tag{5.150}$$

Analogous formulas relate the errors Δn_x, Δn_y, Δm_x, Δm_y, to the errors Δn_{x_1}, Δn_{y_1}, Δm_{x_1}, Δm_{y_1}. In addition, it is obvious that δz, Δn_z, Δm_z, in equations (5.146) have the same significance and value as in equations (5.121).

Therefore, if in equations (5.121) the projection ω_x is equal to 0, and ω_y and ω_z are constant, then ω_{x_1} and ω_{y_1} in equations (5.146) are functions of time. Since ω_z is constant, it follows from formulas (5.149) and (5.150) that these will be periodic functions of time. Therefore, at first glance we have only complicated the problem by moving from equations (5.121) with constant coefficients to equations (5.146) with variable (periodic) coefficients.

However, closer examination shows that this is not the case. The structure of equations (5.146) differs somewhat from that of equations (5.121). The first two equations (5.121) contain the derivatives $\delta\dot{x}$ and $\delta\dot{y}$, but the first two equations (5.146) do not contain the derivatives $\delta\dot{x}_1$ and $\delta\dot{y}_1$. The coefficients of δx and δy in the first two equations (5.121) contain ω_z^2. The quantity ω_z may be quite large in the case of an object moving along a parallel close to a pole even at low velocity, as follows from expression (5.36) for δ_z. In addition to ω_0^2, the quantities $\omega_{x_1}^2$, $\omega_{y_1}^2$, $\omega_{x_1}\omega_{y_1}$ are the coefficients of δx_1 and δy_1 in the first two equations (5.146). They are always bounded in absolute value. Specifically, in the case of motion of an object along a parallel, if condition (5.145) is observed, the following equalities obtain:

$$\omega_0^2 \gg \omega_{x_1}^2, \quad \omega_0^2 \gg \omega_{y_1}^2, \quad \omega_0^2 \gg |\omega_{x_1}\omega_{y_1}|. \tag{5.151}$$

The coefficients of the system (5.146), of course, implicity contain ω_z. It enters as a multiplier into $\dot{\omega}_{x_1}$ and $\dot{\omega}_{y_1}$, as can easily be seen by differentiating. But $\dot{\omega}_{x_1}$ and $\dot{\omega}_{y_1}$ enter into the first two equations (5.146) only as coefficients of δz, and into the last equation only as coefficients of δx_1 and δy_1.

These characteristics of the structure of equations (5.146) and conditions (5.145) permit us to separate the last equation of (5.146) from the first two, by representing it in a first approximation as follows:

$$\delta\ddot{z} - 2\omega_0^2\delta z = \Delta n_z + 2r\omega_y\Delta m_y. \tag{5.152}$$

Then, ignoring in the first two equations (5.146) the terms containing quadratic functions of the projections ω_{x_1} and ω_{y_1}, and moving to the right sides terms containing δz and $\delta\dot{z}$, we obtain equations for the determination of δx_1 and δy_1 in the following form:

$$\left.\begin{aligned}\delta\ddot{x}_1 + \omega_0^2\delta x_1 &= \Delta n_{x_1} - \Delta\dot{m}_{y_1}r - \omega_{x_1}\Delta m_z r - \\ &\quad - \dot{\omega}_{y_1}\delta z - 2\omega_{y_1}\delta\dot{z}, \\ \delta\ddot{y}_1 + \omega_0^2\delta y_1 &= \Delta n_{y_1} + \Delta\dot{m}_{x_1}r - \omega_{y_1}\Delta m_z r + \\ &\quad + \dot{\omega}_{x_1}\delta z + 2\omega_{x_1}\delta\dot{z}.\end{aligned}\right\} \tag{5.153}$$

The quantities δz and $\delta\dot{z}$ occurring in the right sides of these equations may be found from the solution to equation (5.152).

Equations (5.152) and (5.153) correspond to the following simplifications of the system (5.121). In the left side of the last equation (5.121) all terms containing the multiplier ω_y are discarded, and it reduces, thereby, to equation (5.152). In the first two equations (5.121) terms containing δz and $\delta\dot{z}$ are moved to the right side, and, in addition, in the coefficient of δx in the first equation ω_y^2 is considered to be small relative to $\omega_0^2 - \omega_z^2$ and therefore ignored. The first two equations (5.121) therefore take the form:

$$\left.\begin{aligned}\delta\ddot{x} + (\omega_0^2 - \omega_z^2)\delta x - 2\omega_z\delta\dot{y} &= f_1(t) - 2\omega_y\delta\dot{z}, \\ \delta\ddot{y} + (\omega_0^2 - \omega_z^2)\delta y + 2\omega_z\delta\dot{x} &= f_2(t) - \omega_y\omega_z\delta z,\end{aligned}\right\} \tag{5.154}$$

where $f_1(t)$ and $f_2(t)$ in accordance with definitions (5.131) are the right sides of the first two equations (5.121).

Change of variables in (5.149) and (5.150) reduces this system to equations (5.153). Therefore, at $\omega_y = 0$ equations (5.152) and (5.153) correspond exactly to equations (5.121).

It should be noted that the transition to the simplified equations (5.152) and (5.153) is also possible for arbitrary motion of an object at a constant distance from the center of the earth, and not only for motion along a parallel at constant velocity. This follows from the fact that inequalities (5.145) and (5.151) remain valid for variable ω_x, ω_y, ω_z.

Comparing equations (5.152) and (5.153) with equations (5.90), corresponding to the case of a stationary object, we note that they differ only in their right sides. Therefore the solution to these equations may be obtained by analogy to the solution (5.95) to equations (5.90).

Introducing into the right sides of equations (5.153) the notations

$$\left.\begin{aligned} \Delta n_{x_1} - \Delta \dot{m}_{y_1} r - \omega_{x_1} \Delta m_z r = f_1'(t), \\ \Delta n_{y_1} + \Delta \dot{m}_{x_1} r - \omega_{y_1} \Delta m_z r = f_2'(t), \end{aligned}\right\} \tag{5.155}$$

we obtain the following formulas for the solution to equations (5.152) and (5.153):

$$\begin{aligned} \delta x_1 &= \delta x_1^0 \cos \omega_0 t + \frac{\delta \dot{x}_1^0}{\omega_0} \sin \omega_0 t + \\ &+ \frac{1}{\omega_0} \int_0^t [f_1'(\tau) - \dot{\omega}_{y_1} \delta z(\tau) - 2\omega_{y_1} \delta \dot{z}(\tau)] \sin \omega_0 (t-\tau)\, d\tau, \\ \delta y_1 &= \delta y_1^0 \cos \omega_0 t + \frac{\delta \dot{y}_1^0}{\omega_0} \sin \omega_0 t + \\ &+ \frac{1}{\omega_0} \int_0^t [f_2'(\tau) + \dot{\omega}_{x_1} \delta z(\tau) + 2\omega_{x_1} \delta \dot{z}(\tau)] \sin \omega_0 (t-\tau)\, d\tau, \\ \delta z &= \delta z^0 \cosh \omega_0 \sqrt{2}\, t + \frac{\delta \dot{z}^0}{\omega_0 \sqrt{2}} \sinh \omega_0 \sqrt{2}\, t + \\ &+ \frac{1}{\omega_0 \sqrt{2}} \int_0^t f_3(\tau) \sinh \omega_0 \sqrt{2} (t-\tau)\, d\tau, \end{aligned} \tag{5.156}$$

where f_3 designates, in accordance with the last equality (5.131), the right side of equation (5.152).

In order to convert from the errors δx_1 and δy_1 to δx and δy, formulas inverse to formulas (5.150) must be used:

$$\left.\begin{aligned}\delta x &= \delta x_1 \cos\omega_z t + \delta y_1 \sin\omega_z t,\\ \delta y &= -\delta x_1 \sin\omega_z t + \delta y_1 \cos\omega_z t.\end{aligned}\right\} \tag{5.157}$$

Multiplying, in accordance with these formulas, the first equality (5.156) by $\cos\omega_z t$ and the second by $\sin\omega_z t$ and adding, we obtain expressions for δx. Multiplying the second equality (5.156) by $\cos\omega_z t$ and subtracting from it the first equality multiplied by $\sin\omega_z t$, we find δy.

Performing the indicated operations, we arrive at the approximation formulas:

$$\begin{aligned}\delta x = {} & (\delta x^0 \cos\omega_z t + \delta y^0 \sin\omega_z t)\cos\omega_0 t +\\ & + \frac{1}{\omega_0}[(\delta\dot{x}^0 - \omega_z \delta y^0)\cos\omega_z t + (\delta\dot{y}^0 + \omega_z \delta x^0)\sin\omega_z t]\sin\omega_0 t +\\ & + \frac{1}{\omega_0}\int_0^t [(f_1 - 2\omega_y \delta\dot{z})\cos\omega_z(t-\tau) +\\ & \qquad + (f_2 - \omega_y\omega_z \delta z)\sin\omega_z(t-\tau)]\cos\omega_0(t-\tau)\,d\tau,\\ \delta y = {} & (\delta y^0 \cos\omega_z t - \delta x^0 \sin\omega_z t)\cos\omega_0 t +\\ & + \frac{1}{\omega_0}[(\delta\dot{y}^0 + \omega_z \delta x^0)\cos\omega_z t - (\delta\dot{x}^0 - \omega_z \delta y^0)\sin\omega_z t]\sin\omega_0 t +\\ & + \frac{1}{\omega_0}\int_0^t [-(f_1 - 2\omega_y \delta\dot{z})\sin\omega_z(t-\tau) +\\ & \qquad + (f_2 - \omega_y\omega_z \delta z)\cos\omega_z(t-\tau)]\sin\omega_0(t-\tau)\,d\tau.\end{aligned} \tag{5.158}$$

where the quantities δz and $\delta\dot{z}$ entering into the integrands are determined from the third formula (5.156).

For constant instrument errors, recalling the value of f_3, we obtain:

$$\begin{aligned}\delta z = {} & \delta z^0 \cosh\omega_0\sqrt{2}\,t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\sinh\omega_0\sqrt{2}\,t +\\ & + \frac{\Delta n_z + 2\omega_y \Delta m_y}{2\omega_0^2}(\cosh\omega_0\sqrt{2}\,t - 1).\end{aligned} \tag{5.159}$$

Differentiation of this formula yields:

$$\delta\dot{z} = \delta\dot{z}^0 \cos\omega_0\sqrt{2}t + \left(\omega_0\sqrt{2}\,\delta z^0 + \frac{\Delta n_z + 2r\omega_y\Delta m_y}{\omega_0\sqrt{2}}\right)\sinh\omega_0\sqrt{2}t. \tag{5.160}$$

These values of δz and $\delta\dot{z}$, as well as the values of f_1 and f_2 defined by expressions (5.131), are now substituted into equalities (5.158) and the resulting expression integrated. The following integrals are required here:

$$I_1 = \int_0^t \cos\omega_z(t-\tau)\sin\omega_0(t-\tau)\,d\tau = \frac{\omega_0}{\omega_0^2-\omega_z^2}\left(1-\cos\omega_0 t\cos\omega_z t - \frac{\omega_z}{\omega_0}\sin\omega_0 t\sin\omega_z t\right).$$

$$I_2 = \int_0^t \sin\omega_z(t-\tau)\sin\omega_0(t-\tau)\,d\tau = \frac{\omega_0}{\omega_0^2-\omega_z^2}\left(\frac{\omega_z}{\omega_0}\sin\omega_0 t\cos\omega_z t - \cos\omega_0 t\sin\omega_z t\right).$$

$$I_3 = \int_0^t \sinh\omega_0\sqrt{2}\tau\cos\omega_z(t-\tau)\sin\omega_0(t-\tau)\,d\tau = \frac{\omega_0}{9\omega_0^4+2\omega_0^2\omega_z^2+\omega_z^4}\Big[(3\omega_0^2-\omega_z^2)\sinh\omega_0\sqrt{2}t - \sqrt{2}(3\omega_0^2+\omega_z^2)\sin\omega_0 t\cos\omega_z t + 2\omega_0\sqrt{2}\omega_z\cos\omega_0 t\sin\omega_z t\Big].$$

$$I_4 = \int_0^t \sinh\omega_0\sqrt{2}\tau\sin\omega_z(t-\tau)\sin\omega_0(t-\tau)\,d\tau = \frac{\omega_0\sqrt{2}}{9\omega_0^4+2\omega_0^2\omega_z^2+\omega_z^4}\Big[2\omega_0\omega_z \quad \omega_0\sqrt{2}t - 2\omega_0\omega_z\cos\omega_0 t\cos\omega_z t - (3\omega_0^2+\omega_z^2)\sin\omega_0 t\sin\omega_z t\Big]$$

$$I_5 = \int_0^t \cosh\omega_0\sqrt{2}\tau\sin\omega_z(t-\tau)\sin\omega_0(t-\tau)\,d\tau = \frac{\omega_0}{9\omega_0^4+2\omega_0^2\omega_z^2+\omega_z^4}\Big[2\sqrt{2}\omega_0\omega_z\sinh\omega_0\sqrt{2}t + (3\omega_0^2-\omega_z^2)\cos\omega_0 t\sin\omega_z t + (\omega_0^2+\omega_z^2)\frac{\omega_z}{\omega_0}\sin\omega_0 t\cos\omega_z t\Big].$$

$$I_6 = \int_0^t \cosh\omega_0\sqrt{2}\tau\cos\omega_z(t-\tau)\sin\omega_0(t-\tau)\,d\tau = \frac{\omega_0}{9\omega_0^4+2\omega_0^2\omega_z^2+\omega_z^4}\Big[(3\omega_0^2-\omega_z^2)\cosh\omega_0\sqrt{2}t - (3\omega_0^2-\omega_z^2)\cos\omega_0 t\cos\omega_z t + (\omega_0^2+\omega_z^2)\frac{\omega_z}{\omega_0}\sin\omega_0 t\sin\omega_z t\Big] \tag{5.161}$$

For the sake of further simplification of the form of solutions (5.158), we will confine ourselves to the case in which, in addition to condition (5.145), the following condition obtains:

$$\omega_z^2 \ll \omega_0^2. \tag{5.162}$$

We may then take:

$$\left.\begin{aligned}
I_1 &= \frac{1}{\omega_0}(1 - \cos\omega_0 t \cos\omega_z t),\\
I_2 &= -\frac{1}{\omega_0}\cos\omega_0 t \sin\omega_z t,\\
I_3 &= \frac{1}{3\omega_0}(\sinh\omega_0\sqrt{2}t - \sqrt{2}\sin\omega_0 t\cos\omega_z t),\\
I_4 &= \frac{\sqrt{2}}{3\omega_0}\left(\frac{2\omega_z}{3\omega_0}\cosh\omega_0\sqrt{2}t - \sin\omega_0 t\sin\omega_z t\right),\\
I_5 &= \frac{1}{3\omega_0}\left(\frac{2\sqrt{2}\omega_z}{3\omega_0}\sinh\omega_0\sqrt{2}t + \cos\omega_0 t\sin\omega_z t\right),\\
I_6 &= \frac{1}{3\omega_0}(\cosh\omega_0\sqrt{2}t - \cos\omega_0 t\cos\omega_z t).
\end{aligned}\right| \tag{5.163}$$

In the fourth and fifth equalities (5.163) the terms $\frac{2\omega_z}{3\omega_0}\cosh\omega_0\sqrt{2}t$ and $\frac{2\sqrt{2}\omega_z}{3\omega_0}\sinh\omega_0\sqrt{2}t$ appear in the parentheses. They contain the small multiplier ω_z, but as a result of the rapid increase of the hyperbolic sine and cosine, they are significant.

Substituting expressions (5.159), (5.160), and (5.163) in formulas (5.158) and once again taking into account inequality (5.162), we obtain the following approximate formulas for δx and δy:

$$\left.\begin{aligned}
\delta x &= (\delta x^0\cos\omega_z t + \delta y^0\sin\omega_z t)\cos\omega_0 t +\\
&+ \left(\frac{\delta\dot{x}^0}{\omega_0}\cos\omega_z t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_z t\right)\sin\omega_0 t -\\
&- \frac{2\sqrt{2}\omega_y}{3\omega_0}\left(\delta z^0\sinh\omega_0\sqrt{2}t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\cosh\omega_0\sqrt{2}t\right) +\\
&+ \frac{\Delta n_x - r\omega_z\Delta m_y}{\omega_0^2}(1 - \cos\omega_0 t\cos\omega_z t) -\\
&- \frac{\Delta n_y - r\omega_y\Delta m_z - r\omega_z\Delta m_y}{\omega_0^2}\cos\omega_0 t\sin\omega_z t -\\
&- \frac{\sqrt{2}\omega_y(\Delta n_z + 2r\omega_y\Delta m_y)}{3\omega_0^3}\sinh\omega_0\sqrt{2}t.
\end{aligned}\right| \tag{5.164}$$

$$
\begin{aligned}
\delta y = {} & (\delta y^0 \cos \omega_z t - \delta x^0 \sin \omega_z t) \cos \omega_0 t + \\
& + \left(\frac{\delta \dot{y}^0}{\omega_0} \cos \omega_z t - \frac{\delta \dot{x}^0}{\omega_0} \sin \omega_z t \right) \sin \omega_0 t + \\
& + \frac{5\omega_y \omega_z}{9\omega_0^2} \left(\delta z^0 \cosh \omega_0 \sqrt{2} t + \frac{\delta \dot{z}^0}{\omega_0 \sqrt{2}} \sinh \omega_0 \sqrt{2} t \right) + \\
& + \frac{\Delta n_x - r\omega_z \Delta m_z}{\omega_0^2} \cos \omega_0 t \sin \omega_z t + \\
& + \frac{\Delta n_y - r\omega_y \Delta m_z - r\omega_z \Delta m_y}{\omega_0^2} (1 - \cos \omega_0 t \cos \omega_z t) + \\
& + \frac{5\omega_y \omega_z (\Delta n_z + 2r\omega_y \Delta m_y)}{18\omega_0^4} \cosh \omega_0 \sqrt{2} t.
\end{aligned}
\tag{5.164}
$$

5.3.6. <u>The case of an object which is stationary relative to the earth</u>. In order to obtain expressions for the errors δx, δy, δz for the case of an object which is stationary relative to the earth, the values of ω_y and ω_z give by (5.129) must be substituted in formulas (5.164) and (5.159).

The approximate solution to equations (5.121) consisting of formulas (5.158) and the third formula (5.156) may also, of course, be obtained by direct simplification of the exact solution (5.143). We will show this, limiting ourselves to the case of an object which is stationary relative to the earth.

In this case, in the characteristic equation (5.122) the quantities ω_y and ω_z must be replaced by their values (5.129). Due to the fact that u^2 is small relative to ω_0^2, the value of the root q_3 of equation (5.122) is close to that of $2\omega_0^2$, and the roots q_1 and q_2 approach ω_0^2. Therefore, taking $q_3 = 2\omega_0^2$ as a first approximation and applying twice Newton's method of approximation to a root, we obtain accurate to within terms of the order u^4/ω_0^4 inclusively

$$q_3 = 2\omega_0^2 - \frac{5}{3} u^2 \cos^2 \varphi - \frac{u^4 \cos^2 \varphi}{27\omega_0^2} (1 - 5 \sin^2 \varphi). \tag{5.165}$$

Equation (5.122) now reduces, when divided by $q - q_3$, to a quadratic equation with the roots

$$q_{1,2} = -\omega_0^2 - u^2\left(1 - \frac{5}{6}\cos^2\varphi\right) + \frac{u^4\cos^2\varphi}{54\omega_0^2}(4 - 5\sin^2\varphi) \pm$$
$$\pm\sqrt{4\omega_0^2u^2\sin^2\varphi + \frac{u^4\cos^2\varphi}{36}(1 + 55\sin^2\varphi)}.$$

Now, according to equalities (5.126)

$$\left.\begin{aligned}
&\mu_3 = \omega_0\sqrt{2}\left[1 - \frac{5u^2\cos^2\varphi}{12\omega_0^2} - \frac{u^4\cos^2\varphi}{32\cdot 27\omega_0^4}(107 - 115\sin^2\varphi)\right],\\
&\mu_{1,2} = \omega_0\left[1 + \frac{u^2\cos^2\varphi}{12\omega_0^2} - \frac{u^4\cos^2\varphi}{16\cdot 27\omega_0^4}(19 - 140\sin^2\varphi) \mp\right.\\
&\left.\mp\left(\frac{1}{2} - \frac{u^2\cos^2\varphi}{24\omega_0^2}\right)\sqrt{\frac{4u^2\sin^2\varphi}{\omega_0^2} + \frac{u^4\cos^2\varphi}{36\omega_0^4}(1 - 55\sin^2\varphi)}\right].
\end{aligned}\right\} \tag{5.166}$$

If in equalities (5.166) only terms of the order u/ω_0 are retained, i.e., if we assume that

$$\mu_3 = \omega_0\sqrt{2}, \quad \mu_{1,2} = \omega_0 \mp u\sin\varphi. \tag{5.167}$$

then the coefficients a_{ij}, b_{ij}, c_{ij} of solution (5.143) take the following approximate values:

$$\left.\begin{aligned}
&a_{13} = 0,\ a_{23} = 0,\ a^0_{13} = 0,\ a^0_{23} = 0,\ b_{13} = 0,\ b_{23} = 0,\ b_{33} = 0,\\
&b^0_{13} = 0,\ b^0_{23} = 0,\ b^0_{33} = 0,\ c_{13} = 0,\ c^0_{13} = 0,\ c_{31} = 0,\ c^0_{31} = 0,\\
&c_{32} = 0,\ c^0_{32} = 0;\\
&a_{21} = \frac{1}{2\omega_0},\ a_{22} = -\frac{1}{2\omega_0},\ a_{11} = a_{12} = \frac{1}{2\omega_0},\ a_{31} = a_{32} = \frac{u\cos\varphi}{3\omega_0^2},\\
&a^0_{21} = \frac{1}{2},\ a^0_{22} = -\frac{1}{2},\ a^0_{11} = a^0_{12} = \frac{1}{2},\ a^0_{31} = a^0_{32} = \frac{2u\cos\varphi}{3\omega_0},\\
&a_{33} = -\frac{2u\cos\varphi}{3\omega_0^2},\ a^0_{33} = -\frac{4u\cos\varphi}{3\omega_0\sqrt{2}};\\
&b_{21} = b_{22} = \frac{1}{2\omega_0},\ b_{11} = \frac{1}{2\omega_0},\ b_{12} = -\frac{1}{2\omega_0},\\
&\qquad\qquad b_{31} = \frac{u\cos\varphi}{3\omega_0^2},\ b_{32} = -\frac{u\cos\varphi}{3\omega_0^2},\\
&b^0_{21} = b^0_{22} = -\frac{1}{2},\ b^0_{11} = \frac{1}{2},\ b^0_{12} = -\frac{1}{2},\\
&\qquad\qquad b^0_{31} = \frac{2u\cos\varphi}{3\omega_0},\ b^0_{32} = -\frac{2u\cos\varphi}{3\omega_0};\\
&c_{11} = \frac{u\cos\varphi}{3\omega_0^2},\ c_{12} = -\frac{u\cos\varphi}{3\omega_0^2},\ c_{21} = c_{22} = -\frac{u\cos\varphi}{3\omega_0^2},\\
&c_{23} = \frac{2u\cos\varphi}{3\omega_0^2},\ c_{33} = \frac{1}{\omega_0\sqrt{2}},\ c^0_{11} = \frac{u\cos\varphi}{3\omega_0},\ c^0_{12} = -\frac{u\cos\varphi}{3\omega_0},\\
&c^0_{21} = c^0_{22} = -\frac{u\cos\varphi}{3\omega_0^2},\ c^0_{23} = -\frac{2u\cos\varphi}{3\omega_0\sqrt{2}},\ c^0_{33} = 1.
\end{aligned}\right\} \tag{5.168}$$

It is evident from equalities (5.168) that a large proportion of the coefficients of $\sinh \mu_3 t$ and $\cosh \mu_3 t$ in formulas (5.143) are equal to 0 in the first approximation. But since these functions rapidly increase in time, it is necessary to determine the magnitudes of their coefficients.

If in expressions (5.166) for the roots of the characteristic equation we retain quantities of the order u^2/ω_0^2, then we may determine the first terms of the expansions in u/ω_0 of the coefficients b_{13}, b_{13}^0, b_{33}, b_{33}^0, a_{13}, a_{13}^0, a_{23}, a_{23}^0, c_{13}, c_{13}^0. They are:

$$\left.\begin{aligned}
&b_{13} = -\frac{10u^3 \sin\varphi \cos^2\varphi}{27\omega_0^4}, && b_{13}^0 = \frac{10u^3 \sin\varphi \cos^2\varphi}{27\omega_0^3},\\
&b_{33} = \frac{5u^3 \sin\varphi \cos\varphi}{9\omega_0^3}, && b_{33}^0 = \frac{5u^3 \sin\varphi \cos\varphi}{9\omega_0^2},\\
&a_{13} = -\frac{8u^3 \cos^2\varphi}{9\sqrt{2}\,\omega_0^3}, && a_{13}^0 = \frac{4u^3 \cos^2\varphi}{9\omega_0^2},\\
&a_{23} = -\frac{10u^3 \sin\varphi \cos^2\varphi}{27\omega_0^4}, && a_{23}^0 = -\frac{8u^3 \sin\varphi \cos^2\varphi}{27\omega_0^3},\\
&c_{13} = -\frac{5u^3 \sin\varphi \cos\varphi}{9\sqrt{2}\,\omega_0^3}, && c_{13}^0 = \frac{7u^3 \sin\varphi \cos\varphi}{9\omega_0^2}.
\end{aligned}\right\} \qquad (5.169)$$

In order to determine the remaining coefficients it is necessary also to retain in equalities (5.166) quantities of the order u^4/ω_0^4, i.e., to take expressions (5.166) in their entirety. Substituting the latter into relations (5.144) gives:

$$\left.\begin{aligned}
&b_{23} = \frac{-25u^4 \sin^2\varphi \cos^2\varphi}{81\sqrt{2}\,\omega_0^5}, && b_{23}^0 = -\frac{35u^4 \sin^2\varphi \cos^2\varphi}{81\omega_0^4},\\
&c_{31} = c_{32} = \frac{2u^3 \cos^2\varphi}{9\omega_0^3}, && c_{31}^0 = c_{32}^0 = -\frac{4u^3 \cos^2\varphi}{9\omega_0^2}.
\end{aligned}\right\} \qquad (5.170)$$

Further simplification of solution (5.143) may be based on the following considerations. We will consider that errors in the introduction of the initial conditions, i.e., the errors δx^0, δy^0, δz^0, $\delta\dot{x}^0/\omega_0$, $\delta\dot{y}^0/\omega_0$, $\delta\dot{z}^0/\omega_0\sqrt{2}$ are quantities of the same order. We will also consider f_1/ω_0^2, f_2/ω_0^2, $f_3/2\omega_0^2$ as being of the same order relative to one another and relative to errors in the initial conditions.

The assumptions made with regard to the initial conditions and the perturbing influences enable us to retain the properties of solution (5.143) which are required for the evaluation of errors in the functioning of the system, leaving in it only a portion of the coefficients of the first and second approximation, namely:

$$a_{11},\ a_{12},\ a_{21},\ a_{22},\ a_{33},\ a_{11}^0,\ a_{12}^0,\ a_{21}^0,\ a_{22}^0,\ a_{33}^0,\ b_{33},\ b_{33}^0,$$
$$b_{11},\ b_{12},\ b_{11}^0,\ b_{12}^0,\ c_{33},\ c_{33}^0.$$

In this case the solution take the form:

$$\left.\begin{aligned}
\delta x &= \frac{1}{2\omega_0}\int_0^t \{f_1(\tau)[\sin\mu_1(t-\tau)+\sin\mu_2(t-\tau)] - \\
&\quad - f_2(\tau)[\cos\mu_1(t-\tau)-\cos\mu_2(t-\tau)]\}\,d\tau - \\
&\quad - \frac{2u\cos\varphi}{3\omega_0^2}\int_0^t f_3(\tau)\,\mathrm{ch}\,\mu_3(t-\tau)\,d\tau + \\
&\quad + \frac{\delta x^0}{2}(\cos\mu_1 t+\cos\mu_2 t) + \frac{\delta\dot{x}^0}{2\omega_0}(\sin\mu_1 t+\sin\mu_2 t) + \\
&\quad + \frac{\delta y^0}{2}(\sin\mu_1 t-\sin\mu_2 t) + \frac{\delta\dot{y}^0}{2\omega_0}(-\cos\mu_1 t+\cos\mu_2 t) - \\
&\quad - \delta z^0\frac{4u\cos\varphi}{3\omega_0\sqrt{2}}\sinh\mu_3 t - \delta\dot{z}^0\frac{2u\cos\varphi}{3\omega_0^2}\cosh\mu_3 t, \\
\delta y &= \frac{1}{2\omega_0}\int_0^t \{f_2(\tau)[\sin\mu_1(t-\tau)+\sin\mu_2(t-\tau)] + \\
&\quad + f_1(\tau)[\cos\mu_2(t-\tau)-\cos\mu_1(t-\tau)]\}\,d\tau + \\
&\quad + \frac{5u^2\sin\varphi\cos\varphi}{9\sqrt{2}\,\omega_0^3}\int_0^t f_3(\tau)\sinh\mu_3(t-\tau)\,d\tau + \\
&\quad + \frac{\delta y^0}{2}(\cos\mu_1 t+\cos\mu_2 t) + \frac{\delta\dot{y}^0}{2\omega_0}(\sin\mu_1 t+\sin\mu_2 t) + \\
&\quad + \frac{\delta x^0}{2}(-\sin\mu_1 t+\sin\mu_2 t) + \frac{\delta\dot{x}^0}{2\omega_0}(\cos\mu_1 t-\cos\mu_2 t) + \\
&\quad + \frac{5u^2\sin\varphi\cos\varphi}{9\omega_0^2}\left(\delta z^0\cosh\mu_3 t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\sinh\mu_3 t\right), \\
\delta z &= \frac{1}{\omega_0\sqrt{2}}\int_0^t f_3(\tau)\sinh\mu_3(t-\tau)\,d\tau + \\
&\quad + \delta z^0\cosh\mu_3 t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\sinh\mu_3 t,
\end{aligned}\right|$$

(5.171)

where according to equalities (5.167):

$$\mu_3=\omega_0\sqrt{2},\quad \mu_1=\omega_0-u\sin\varphi,\quad \mu_2=\omega_0+u\sin\varphi. \tag{5.172}$$

The approximate formulas (5.171) are equivalent to the approximate formulas (5.158) and (5.156). With constant instrument errors this is easily shown to be the case if in place of f_1, f_2, f_3 in equalities (5.171) their values (5.131) are substituted, and in place of μ_1, μ_2, μ_3 their values (5.172), and also if expressions (5.129) ω_y and ω_z and the condition of the smallness of u^2 relative to e taken into account.

§5.4. Integrating the First Group of the Error Equations for the Case of Keplerian Motion.

5.4.1. The possibility of integration. As has already been noted, in the case of the first group of the error equations (5.1) may be integrated in quadratic forms. This possibility derives from the following circumstances.

According to vector equations (5.17), the following vector equation corresponds to error equations (5.1):

$$\frac{d^2\delta r}{dt^2}+\frac{\mu}{r^3}\delta r-\frac{\mu r}{r^3}\frac{3(r\cdot\delta r)}{r^2}=\\=\Delta n-2\Delta m\times\frac{dr}{dt}+r\times\frac{d\Delta m}{dt}. \qquad (5.173)$$

The last two terms on the left side of this equation are simply $\delta\left(\frac{\mu \vec{r}}{r^3}\right)$, i.e., equation (5.173) is equivalent to equation

$$\frac{d^2\delta r}{dt^2}+\delta\left(\frac{\mu r}{r^3}\right)=\Delta n-2\Delta m\times\frac{dr}{dt}+r\times\frac{d\Delta m}{dt}. \qquad (5.174)$$

Let us consider the homogeneous equation (5.174). It may be represented in the following form:

$$\delta\left(\frac{d^2r}{dt^2}+\frac{\mu r}{r^3}\right)=0. \qquad (5.175)$$

But equation

$$\frac{d^2r}{dt^2}+\frac{\mu}{r^3}r=0 \qquad (5.176)$$

is the equation of motion of a point of unit mass moving in a spherical Newtonian gravitational field, i.e., an equation of Keplerian motion. Therefore, the homogeneous error equation (5.175) is a variation of the Keplerian equation (5.176). Here, of course, it must be kept in mind that the sensitive masses of the newtonometers must be considered to be located in the center of mass of the object.

The general integral of equation (5.176), containing six arbitrary constants, is known. According to the Poincare theorem[9], the independent partial solutions to equation (5.175) are found in the form of the derivatives of the general solution to equation (5.176) with arbitrary constants, which permits us to find a general solution to the homogeneous equation (5.173), which is also in turn a funtion of six arbitrary constants.

But equation (5.173) is a linear differential equation with variable coefficients. And if the general solution to a homogeneous linear differential equation is known, then the general solution to a non-homogeneous equation in quadratic forms may be immediately obtained. The Lagrange method of variation of the parameters of the general solution to a homogeneous equation may be used for this purpose.

Let us consider the following. If the sensitive mass of a newtonometer is located in the center of mass of an object in Keplerian motion, then the newtonometer reading is clearly zero. However, analysis of the error equations for this case is not without significance. This is because the solution to the error equations for Keplerian motion in the first approximation will simultaneously be the solution to the error equations for motions differing slightly from Keplerian motion: deceleration in the upper layers of the atmosphere, maneuvering of an orbital space vehicle (with small thrust), etc.

5.4.2. <u>The basic characteristics of Keplerian motion</u>. Before moving directly to solution of the error equations for the case of Keplerian motion, let us recall certain of the properties[10] of this motion which we will require below.

As has already been noted, Keplerian motion satisfies the vector equation (5.176), which corresponds in its projections on the ξ_*, η_*, ζ_* axes to the three scaler equations:

$$\ddot{\xi}_* + \frac{\mu\xi_*}{r^3} = 0, \quad \ddot{\eta}_* + \frac{\mu\eta_*}{r^3} = 0, \quad \ddot{\zeta}_* + \frac{\mu\zeta_*}{r^3} = 0. \tag{5.177}$$

Equations (5.177) have six independent first integrals, which define the general solution to these equations.

Three of the first integrals are called <u>space integrals</u>. In order to obtain one of them, we multiply the second equation (5.177) by ζ_*, and the third by η_*, and then subtract the second result from the first. We then obtain:

$$\zeta_*\ddot{\eta}_* - \ddot{\zeta}_*\eta_* = \frac{d}{dt}(\dot{\eta}_*\zeta_* - \dot{\zeta}_*\eta_*) = 0, \tag{5.178}$$

whence,

$$\dot{\eta}_*\zeta_* - \dot{\zeta}_*\eta_* = c_1 = \text{const.} \tag{5.179}$$

The other two integrals of this form are obtained in a completely analogous manner. Combining them with integral (5.179), we have:

$$\left.\begin{array}{c} \dot{\eta}_*\zeta_* - \dot{\zeta}_*\eta_* = c_1, \quad \dot{\zeta}_*\xi_* - \dot{\xi}_*\zeta_* = c_2, \\ \dot{\xi}_*\eta_* - \dot{\eta}_*\xi_* = c_3. \end{array}\right\} \tag{5.180}$$

Specifically, it follows from equalities (5.180) that the trajectory which is the solution to equation (5.177) is in a plane passing through the center of the earth. Indeed, multiplying the first of these equalities by ζ_*, the second by η_*, and the third by ζ_* and adding the results, we arrive at the equation of a plane

$$c_1\xi_* + c_2\eta_* + c_3\zeta_* = 0, \tag{5.181}$$

passing through the center of the earth.

Integrals (5.180) may also be obtained in a somewhat different manner. Multiplying equation (5.176) by the vector $\vec{r}$, we find:

$$r\times\frac{d^2r}{dt^2}=\frac{d}{dt}\left(r\times\frac{dr}{dt}\right)=0, \tag{5.182}$$

i.e.,

$$r\times\frac{dr}{dt}=c. \tag{5.183}$$

The projections of the vector $\vec{c}$ defined by equation (5.183) on the ξ_*, η_*, ζ_* axes are equalities (5.180). Thus, the first integrals (5.180) derive from the moment of momentum theorem.

The quantity

$$dS=r\times dr \tag{5.184}$$

is, as is well known, the oriented doubled space of the trihedron formed by the vectors $\vec{r}$, $d\vec{r}$ and $\vec{r}+d\vec{r}$, or, equivalently, the oriented doubled space of the sector marked out by the radius vector $\vec{r}$ during the time dt. From equalities (5.183) and (5.184) we see that

$$\dot{S}=\text{const.} \tag{5.185}$$

The next integral of equation (5.176) will be the energy integral. In order to obtain it, we perform the scaler multiplication of equation (5.176) by $d\vec{r}/dt$:

$$\frac{d^2r}{dt^2}\cdot\frac{dr}{dt}+\frac{\mu r}{r^3}\cdot\frac{dr}{dt}=0. \tag{5.186}$$

Introducing $\vec{v}=d\vec{r}/dt$, we find:

$$d\left(\frac{v^2}{2}\right)+\frac{\mu r}{r^3}\cdot dr=0. \tag{5.187}$$

The second term on the left side of expression (5.187) is the influence of gravitational forces on the displacement $d\vec{r}$. Since the gravitational forces are potential, this influence is equal to the differential of the force function V of the gravitational field:

$$\frac{\mu r}{r^3}\cdot dr=dV=d\left(\frac{\mu}{r}\right). \tag{5.188}$$

Therefore, instead of relation (5.187) we may write:

$$d\left(\frac{v^2}{2} + \frac{\mu}{r}\right) = 0.$$

The integration of this relation is the energy integral:

$$\frac{v^2}{2} + \frac{\mu}{r} = h = \text{const.} \qquad (5.189)$$

For the further integration of equation (5.176) we will make use of the fact that the solution to this equation is a plane curve.

Let us place a right orthogonal coordinate system $O_1\xi'\eta'\zeta'$ in the plane of motion, with the ξ' and η' axes in the plane. Let us further introduce the right orthogonal system O_1xyz. Let us superpose its xz plane on the $\xi'\eta'$ plane, and let us direct the z axis toward the moving point O (along r), and the y axis along the ζ' axis. The position of the coordinate system O_1xyz relative to the $O_1\xi'\eta'\zeta'$ system is then defined by a single angle σ in accordance with the table of direction cosines below:

	x	y	z
ξ'	$-\sin\sigma$	0	$\cos\sigma$
η'	$\cos\sigma$	0	$\sin\sigma$
ζ'	0	1	$0.$

(5.190)

According to the definition of the orbital trihedron xyz,

$$\mathbf{r} = r\mathbf{z}, \quad \mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{z} + \dot{\sigma}r\mathbf{x}. \qquad (5.191)$$

In order to obtain the solution to equation (5.176) we must, clearly, express r and σ as a function of time and the initial conditions. However, it is simpler to first express r as a function of the angle σ, i.e., to find the orbit equation.

Taking the second equality (5.191) into account, the space integral (5.185) is expressed by the equality

$$r^2\dot{\sigma} = c = \text{const.} \qquad (5.192)$$

On the other hand, from relation (5.189) we obtain:

$$\frac{1}{2}\frac{d(v^2)}{dt} = -\frac{\mu}{r^2}\frac{dr}{dt}. \tag{5.193}$$

Let us convert from the independent variable t to the dependent variable σ by rewriting equation (5.193) as follows:

$$\frac{1}{2}\frac{d(v^2)}{d\sigma}\frac{d\sigma}{dt} = -\frac{\mu}{r^2}\frac{dr}{d\sigma}\frac{d\sigma}{dt}. \tag{5.194}$$

Eliminating the case in which

$$\frac{d\sigma}{dt} = 0, \tag{5.195}$$

i.e., excluding the case of motion along the radius vector $\vec{r}$, we arrive at the equality

$$\frac{1}{2}\frac{d(v^2)}{d\sigma} = -\frac{\mu}{r^2}\frac{dr}{d\sigma}. \tag{5.196}$$

Let us further express v^2 in terms of r and σ. In order to do this we will use the second equality (5.191) and the space integral in the form (5.192), from which it follows that

$$v^2 = \frac{c^2}{r^4}\left[\left(\frac{dr}{d\sigma}\right)^2 + r^2\right]. \tag{5.197}$$

If we now substitute expression (5.197) into equality (5.196), we obtain the differential orbit equation in terms of the coordinates r and σ; however, in order to obtain it in its simplest form, it is useful to convert in formulas (5.196) and (5.197) from the variable r to a new variable

$$u = \frac{1}{r}. \tag{5.198}$$

Performing this conversion and substituting expression (5.197) into (5.196), we arrive at the equality

$$\left(\frac{d^2u}{d\sigma^2} + u\right)\frac{du}{d\sigma} = \frac{\mu}{c^2}\frac{du}{d\sigma}. \tag{5.199}$$

Assuming that

$$\frac{du}{d\sigma} \neq 0, \tag{5.200}$$

i.e., excluding a circular orbit for which

$$r = r^0 = \text{const},$$

for the general case of a Keplerian orbit we arrive at the equation

$$\frac{d^2u}{d\sigma^2} + \sigma = \frac{\mu}{c^2}, \tag{5.201}$$

from which the variable u is easily found:

$$u = \frac{\mu}{c^2} - A_1 \cos\sigma - A_2 \sin\sigma. \tag{5.202}$$

Introducing now the new constants

$$p = \frac{c^2}{\mu}, \quad e^2 = \frac{(A_1^2 + A_2^2)c^4}{\mu^2}, \quad \sigma_1 = \operatorname{arctg}\frac{A_2}{A_1} \tag{5.203}$$

and converting from the variable u again to r, we obtain

$$r = \frac{p}{1 - e\cos(\sigma - \sigma_1)}. \tag{5.204}$$

It is known from analytic geometry" that equation (5.204) is the equation of a conic section in polar coordinates, with the focus of the section as the origin and the focal axis of the section as the angle between σ_1 and the ξ' axis.

The quantity p -- the focal parameter of the conic section -- is equal to the length of the radius vector of the orbit directed from the focus perpendicularly to the focal axis. The parameter p defines the linear dimensions of the orbit.

The quantity e is called the eccentricity of the orbit and determines its form. For $e<1$ the orbit is an ellipse, one of whose foci is located at the center O_1 of the earth; for $e>1$ we have a hyperbolic orbit, and for $e = 1$ a parabolic orbit. If we use v^0 and r^0 to designate the initial velocity of the object and the initial value of r, then the form of the orbit will be entirely determined by the quantity

$$k = \frac{v^{0\,2} r^0}{2\mu}. \tag{5.205}$$

characterizing the relation between the initial kinetic energy of the object and the work which must be done for it to recede to infinity.

when $0 < k < 1$, $e < 1$ always

when $k = 1$, $e = 1$ always

when $k > 1$, $e > 1$ always

We note that if the initial point of the motion (the starting point) is located on the surface of the earth, then the speed of the object corresponding to the value $k = 1$ is usually termed the second cosmic velocity; the term first cosmic velocity denotes the velocity required for the object to move in a circular orbit; this velocity corresponds to the value $k = 1/2$.

We will confine ourselves below to the most practically interesting case of an elliptic orbit, i.e., for $e<1$. We will designate the large semi-axis by a and introduce the notation.

$$v = \sigma - \sigma_1 - \pi \tag{5.206}$$

(in celestial mechanics the variable v is termed the "true anomaly"). Formula (5.204) now takes the form:

$$r = \frac{a(1-e^2)}{1+e\cos v}. \tag{5.207}$$

The angle $\omega = \pi + \sigma_1$ defines the direction to the perigee of the orbit.

In order to finish the integration of equation (5.176), it is sufficient to find σ or, equivalently, v as a function of time t. From formula (5.207) the space integral, (5.192) and equalities (5.1206) and (5.203) we have

$$\dot{v} = \frac{\sqrt{\mu}}{[a^3(1-e^2)^3]^{1/2}}(1+e\cos v)^2. \tag{5.208}$$

Separating variables and integrating, we obtain:

$$\frac{\sqrt{\mu}}{a^{3/2}(1-e^2)^{3/2}}(t-t_0) = \int_0^v \frac{dv}{(1+e\cos v)^2}. \tag{5.209}$$

where t_0 is the time at which the object passes through the perigee of the orbit.

We transform the integral in the right side of equality (5.209) by converting from the true anomaly v to the eccentric anomaly E according to the formula

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \tag{5.210}$$

Since it follows from formula (5.210) that

$$dv = \frac{\sqrt{1-e^2}\,dE}{1-e\cos E}, \quad 1+e\cos v = \frac{1-e^2}{1-e\cos E}, \tag{5.211}$$

equality (5.209) takes the form:

$$\frac{\sqrt{\mu}}{a^{3/2}}(t-t_0) = \int_0^E (1-e\cos E)\,dE. \tag{5.212}$$

Performing the integration on the right side of the resulting relation, we arrive at the Kepler equation

$$E - e\sin E = \frac{\sqrt{\mu}}{a^{3/2}}(t-t_0) = M, \tag{5.213}$$

relating the eccentric anomaly E to the mean anomaly M.

Noting that, taking into account the second equality (5.211), formula (5.207) may be written in the form

$$r = a(1-e\cos E), \tag{5.214}$$

and introducing the following notation for the peridicity of the motion of the object in its orbit

$$\nu = \frac{\sqrt{\mu}}{a^{3/2}}, \tag{5.215}$$

we arrive at the following formulas:

$$\left.\begin{aligned} M &= \nu(t-t_0)+M_0, \\ \nu &= \frac{\sqrt{\mu}}{a^{3/2}}, \\ E - e\sin E &= M, \\ r &= a(1-e\cos E), \\ \tan\frac{v}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \\ \sigma &= v+\omega. \end{aligned}\right\} \tag{5.216}$$

In order to solve the third of these equalities for t, i.e., in order to solve Kepler's equation, the usual procedure is to expand $E - M = e\sin E$ in a trigonometric series in sines whose arguments are multiples of M:

$$E - M = \sum_{k=1}^{\infty} a_k \sin kM. \tag{5.217}$$

The possibility of this expansion derives from the fact that $E - M$ is an odd periodic function.

The coefficients of the expansion (5.217) are calculated from the formulas

$$a_k = \frac{2}{\pi} \int_0^{\pi} (E - M) \sin kM \, dM. \tag{5.218}$$

Integrating expression (5.218) by parts, we obtain:

$$\begin{aligned} a_k &= \frac{2}{\pi k} \left[-(E - M) \cos kM \Big|_0^{\pi} - \int_0^{\pi} \cos kM \, d(E - M) \right] = \\ &= \frac{2}{\pi k} \int_0^{\pi} \cos kM \, dE = \frac{2}{\pi k} \int_0^{\pi} \cos k(E - e \sin E) \, dE. \end{aligned} \tag{5.219}$$

The functions

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(ny - x \sin y) \, dy$$

were used for the solution of the Kepler equation by Bessel and bear his name. According to the definition of these functions,

$$a_k = \frac{2}{k} J_k(ke), \quad E = M + 2\sum_{k=1}^{\infty} \frac{J_k(ke)}{k} \sin kM. \tag{5.220}$$

Returning to the Kepler equation, we obtain:

$$\sin E = \frac{2}{e} \sum_{k=1}^{\infty} \frac{J_k(ke)}{k} \sin kM. \tag{5.221}$$

Formulas (5.216) define the motion of the object in a plane orbit. They contain four arbitrary constants: t_0, e, a, ω. The general solution of equation (5.176) should contain six arbitrary constants. The two missing constants should be supplied by the definition of the orbital plane relative to the coordinate system fixed in inertial space.

In celestial mechanics the plane of an orbit is usually specified by the angles Ω and i in such a way that these angles, together with the angle $v + \omega$, form a system of Euler angles. However, in the analysis of equations in variations Euler angles are not always convenient. We therefore define the position of the orbital plane, i.e., the plane $\xi'\eta'$, somewhat differently with regard to the coordinate system $O_1\xi_*\eta_*\zeta_*$. For this purpose we introduce the angles α and β (Figure 5.3). The orientation of the coordinate system $O_1\xi'\eta'\zeta'$ relative to the coordinate system $O_1\xi_*\eta_*\zeta_*$ is specified by the following direction cosines:

$$\left.\begin{array}{cccc} & \xi' & \eta' & \zeta' \\ \xi_* & \cos\beta & 0 & \sin\beta \\ \eta_* & \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta \\ \zeta_* & -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta. \end{array}\right\} \tag{5.222}$$

According to table (5.222) and the definition of the angle σ, we obtain the following formulas for the coordinates ξ_*, η_*, ζ_*:

$$\left.\begin{aligned} \xi_* &= r\cos\sigma\cos\beta, \\ \eta_* &= r(\cos\sigma\sin\alpha\sin\beta + \sin\sigma\cos\alpha), \\ \zeta_* &= r(-\cos\sigma\cos\alpha\sin\beta + \sin\sigma\sin\alpha). \end{aligned}\right\} \tag{5.223}$$

Thus, the solution to equation (5.176) will be the vector

$$r = \xi_*\xi_* + \eta_*\eta_* + \zeta_*\zeta_*. \tag{5.224}$$

defined by formulas (5.223) and (5.216) and a function of six arbitrary constants: α, β, ω, e, a, t_0.

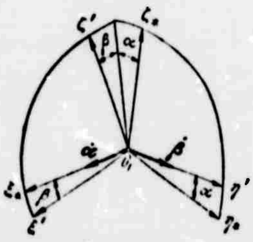

Figure 5.3.

Henceforth we will consider the object to be moving in the plane $\xi_* \eta_*$ such that

$$\alpha = \beta = 0. \tag{5.225}$$

We now proceed to the solution of the equation in variations (5.173).

5.4.3. <u>The integration of homogeneous error equations</u>. The vector equation (5.173), projected on the x, y, z axes of the orbital trihedron, results in the following system of equations:

$$\left.\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x + \dot{\omega}_y\,\delta z + 2\omega_y\,\delta\dot{z} = \\
&\qquad = \Delta n_x - 2\,\Delta m_y\dot{r} - \Delta\dot{m}_y r, \\
&\delta\ddot{y} + \frac{\mu}{r^3}\delta y = \Delta n_y + 2\,\Delta m_x\dot{r} + \Delta\dot{m}_x r - \omega_y\,\Delta m_z r, \\
&\delta\ddot{z} - \left(\frac{2\mu}{r^3} + \omega_y^2\right)\delta z - \dot{\omega}_y\,\delta x - 2\omega_y\,\delta\dot{x} = \\
&\qquad = \Delta n_z + 2r\omega_y\,\Delta m_y.
\end{aligned}\right\} \tag{5.226}$$

The coefficients of equations (5.226) contain, in addition to r, the quantities $\dot{r}$ and ω_y. These may be found from formulas (5.216):

$$\dot{r} = \frac{\nu e a}{\sqrt{1-e^2}}\sin v, \quad \omega_y = \dot{\vartheta} = \dot{v} = \nu\sqrt{1-e^2}\,\frac{a^2}{r^2}, \tag{5.227}$$

where ν is defined by equality (5.215).

The system (5.226) breaks down into a second order equation in δy and a system of fourth order equations in δx and δz.

It will be more convenient below to represent equation (5.173) in terms of projections on the x, y, z axes somewhat differently from (5.226). In order to obtain this representation, we introduce the following change of variables in equations (5.226):

$$\left.\begin{aligned}
&x_1 = \delta x, \quad x_2 = \delta z, \\
&x_3 = \dot{x}_1 + \omega_y x_2, \quad x_4 = \dot{x}_2 - \omega_y x_1, \\
&x_5 = \delta y, \quad x_6 = \dot{x}_5
\end{aligned}\right\} \tag{5.228}$$

The introduction of the variables x_i into the first and third equations (5.226) gives a system of fourth-order equations written in the Cauchy form:

$$\left.\begin{aligned} \dot{x}_1 &= -\omega_y x_2 + x_3, \\ \dot{x}_2 &= \omega_y x_1 + x_4, \\ \dot{x}_3 &= -\omega_y x_4 - \frac{\mu}{r^3} x_1 + \Delta n_x - 2\Delta m_y \dot{r} - \Delta \dot{m}_y r, \\ \dot{x}_4 &= \omega_y x_3 + \frac{2\mu}{r^3} x_2 + \Delta n_z + 2r\omega_y \Delta m_y. \end{aligned}\right| \tag{5.229}$$

and from the second equation (5.226) a system of second-order equations is obtained:

$$\left.\begin{aligned} \dot{x}_5 &= x_6, \\ \dot{x}_6 &= -\frac{\mu}{r^3} x_5 + \Delta n_y + 2\Delta m_x \dot{r} + \Delta \dot{m}_x r - \omega_y \Delta m_z r. \end{aligned}\right\} \tag{5.230}$$

In equations (5.229) and (5.230), in accordance with (5.228), the variables x_1, x_2, x_5 are the projections of the error vector $\delta\vec{r}$ on the x, y, z axes. The variables x_3, x_4, x_6 are the projections on the same axes of the total time derivative $d\delta r/dt$ of the vector $\delta\vec{r}$ This is easily seen by noting that

$$\frac{d\,\delta r}{dt} = \delta\dot{r} + \omega \times \delta r. \tag{5.231}$$

and recalling that $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_5$, are the components of $\delta\dot{\vec{r}}$ along the x, y, z axes, and that only the projection ω_v of the projections of the absolute rate of the rotation of the trihedron xyz about its axis is different from 0.

As was noted at the beginning of this section, according to the Poincare theorem the partial solutions to the homogeneous equation in variations (5.173) or, equivalently, equations (5.229) and (5.230), will be the partial derivatives of the solution to equation (5.176) in arbitrary constants, i.e., the partial derivatives of the vector $\vec{r}$, defined by equalities (5.224), (5.223) and (5.216) in terms of the constants α, β, ω, e, a, t_0.

Differentiating the vector $\vec{r}$ with respect to these constants and taking into account equalities (5.225) we obtain:

$$
\left.
\begin{aligned}
&\frac{\partial r}{\partial \alpha} = r(\sin v\cos\omega + \cos v\sin\omega)y,\\
&\frac{\partial r}{\partial \beta} = -r(\cos v\cos\omega - \sin v\sin\omega)y,\\
&\frac{\partial r}{\partial \omega} = r\frac{\partial u}{\partial \omega}x + \frac{\partial r}{\partial \omega}z,\\
&\frac{\partial r}{\partial e} = r\frac{\partial u}{\partial e}x + \frac{\partial r}{\partial e}z,\\
&\frac{\partial r}{\partial a} = r\frac{\partial u}{\partial a}x + \frac{\partial r}{\partial a}z,\\
&\frac{\partial r}{\partial t_0} = r\frac{\partial u}{\partial t_0}x + \frac{\partial r}{\partial t_0}z,
\end{aligned}
\right| \qquad (5.232)
$$

where $\vec{x}$, $\vec{y}$, $\vec{z}$ are the unit vectors of the corresponding axes.

Performing the differentiation and simplifying as required [using relations (5.216)], we find expressions for $r\frac{\partial u}{\partial t_0}$, $r\frac{\partial u}{\partial a}$, $r\frac{\partial u}{\partial e}$, $r\frac{\partial u}{\partial \omega}$, $\frac{\partial r}{\partial t_0}$, $\frac{\partial r}{\partial a}$, $\frac{\partial r}{\partial e}$, $\frac{\partial r}{\partial \omega}$ in the following form:

$$
\left.
\begin{aligned}
&r\frac{\partial u}{\partial t_0} = -\frac{\nu a(1+e\cos v)}{\sqrt{1-e^2}},\\
&r\frac{\partial u}{\partial a} = -\frac{3\nu(t-t_0)}{2\sqrt{1-e^2}}(1+e\cos v),\\
&r\frac{\partial u}{\partial e} = \frac{a\sin v(2+e\cos v)}{1+e\cos v},\\
&r\frac{\partial u}{\partial \omega} = r,\\
&\frac{\partial r}{\partial t_0} = -\frac{ae\nu\sin v}{\sqrt{1-e^2}},\\
&\frac{\partial r}{\partial a} = \frac{r}{a} - \frac{3\nu(t-t_0)e\sin v}{2\sqrt{1-e^2}},\\
&\frac{\partial r}{\partial e} = -a\cos v,\\
&\frac{\partial r}{\partial \omega} = 0.
\end{aligned}
\right| \qquad (5.233)
$$

It is convenient to take as independent solutions to the homogeneous equation (5.173) specially selected linear combinations q_i of the derivatives (5.232) of the vector $\vec{r}$:

$$
\left.
\begin{aligned}
&q_1 = \frac{\partial r}{\partial a}, \quad q_2 = \frac{1}{a}\frac{\partial r}{\partial e},\\
&q_3 = -\frac{1}{ae(1-e)^2}\frac{\partial r}{\partial \omega} - \frac{\sqrt{1-e^2}}{ae\nu}\frac{\partial r}{\partial t_0},\\
&q_4 = \frac{1}{a}\frac{\partial r}{\partial \omega},\\
&q_5 = \frac{1}{a}\left(\frac{\partial r}{\partial \alpha}\sin\omega - \frac{\partial r}{\partial \beta}\cos\omega\right),\\
&q_6 = \frac{1}{a}\left(\frac{\partial r}{\partial \alpha}\cos\omega + \frac{\partial r}{\partial \beta}\sin\omega\right).
\end{aligned}
\right| \qquad (5.234)
$$

Taking into account relations (5.233), the expressions for q_i take the form:

$$\left.\begin{aligned}
q_1 &= -\frac{3\nu(t-t_0)(1+e\cos v)}{2\sqrt{1-e^2}}x + \\
&\qquad + \left[\frac{r}{a} - \frac{3\nu(t-t_0)e\sin v}{2\sqrt{1-e^2}}\right]z, \\
q_2 &= \frac{2+e\cos v}{1+e\cos v}\sin vx - \cos vz, \\
q_3 &= \frac{2+e\cos v}{1+e\cos v}\cos vx + \sin vz, \\
q_4 &= \frac{r}{a}x, \\
q_5 &= \frac{r}{a}\cos vy, \\
q_6 &= \frac{r}{a}\sin vy.
\end{aligned}\right| \quad (5.235)$$

(5.235)

We will use $\vec{p}_i$ to denote the total time derivatives of the vector $\vec{q}_i$. Then, taking into account relations (5.227) and the obvious equalities:

$$\frac{dy}{dt} = 0, \quad \frac{dz}{dt} = \omega_y x, \quad \frac{dx}{dt} = -\omega_y z, \tag{5.236}$$

We obtain from relations (5.235) the following formulas for $\vec{p}_i$:

$$\left.\begin{aligned}
p_1 &= -\nu\left\{\frac{1+e\cos v}{2\sqrt{1-e^2}}x + \right. \\
&\qquad \left. + \left[\frac{e\sin v}{2\sqrt{1-e^2}} - \frac{3}{2}\nu(t-t_0)\frac{a^2}{r^2}\right]z\right\}, \\
p_2 &= \frac{\nu}{\sqrt{1-e^2}}\frac{a}{r}\left(\frac{e+\cos v}{1+e\cos v}x - \sin vz\right), \\
p_3 &= -\frac{\nu}{\sqrt{1-e^2}}\frac{a}{r}\left(\frac{\sin v}{1+e\cos v}x + \cos vz\right), \\
p_4 &= \frac{\nu}{\sqrt{1-e^2}}[e\sin vx - (1+e\cos v)z], \\
p_5 &= -\frac{\nu\sin v}{\sqrt{1-e^2}}y, \\
p_6 &= \frac{\nu}{\sqrt{1-e^2}}(\cos v + e)y.
\end{aligned}\right| \quad (5.237)$$

(5.237)

The vectors $\vec{q}_i$ and $\vec{p}_i$ form the system of integrals of the homogeneous equation (5.173). The projections of the vectors $\vec{q}_1$, $\vec{q}_2$, $\vec{q}_3$, $\vec{q}_4$, $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$ and $\vec{p}_4$ on the y axis of the orbital trihedron are equal to 0. The projections of these vectors on the x and z axes therefore constitute the system of integrals of the homogeneous equations (5.229). Analogously, the projections of the vectors $\vec{q}_5$, $\vec{q}_6$, $\vec{p}_5$, $\vec{p}_6$ on the y axis constitute the system of integrals of equations (5.230).

Let us form two matrices from these projections:

$$A=\begin{vmatrix} q_1\cdot x & q_2\cdot x & q_3\cdot x & q_4\cdot x \\ q_1\cdot z & q_2\cdot z & q_3\cdot z & q_4\cdot z \\ p_1\cdot x & p_2\cdot x & p_3\cdot x & p_4\cdot x \\ p_1\cdot z & p_2\cdot z & p_3\cdot z & p_4\cdot z \end{vmatrix}, \quad B=\begin{vmatrix} q_5\cdot y & q_6\cdot y \\ p_5\cdot y & p_6\cdot y \end{vmatrix}. \tag{5.238}$$

In accordance with formulas (5.235) and (5.237), the elements of matrix A are.

$$\left.\begin{aligned}
&A_{11}=-\frac{3\nu(t-t_0)(1+e\cos v)}{2\sqrt{1-e^2}},\\
&A_{12}=\frac{2+e\cos v}{1+e\cos v}\sin v,\\
&A_{13}=\frac{2+e\cos v}{1+e\cos v}\cos v, \quad A_{14}=\frac{r}{a},\\
&A_{21}=\frac{r}{a}-\frac{3\nu(t-t_0)e\sin v}{2\sqrt{1-e^2}}, \quad A_{22}=-\cos v,\\
&A_{23}=\sin v, \quad A_{24}=0,\\
&A_{31}=\frac{-\nu(1+e\cos v)}{2\sqrt{1-e^2}}, \quad A_{32}=\frac{\nu}{\sqrt{1-e^2}}\frac{a}{r}\frac{e+\cos v}{1+e\cos v},\\
&A_{33}=\frac{-\nu a\sin v}{r\sqrt{1-e^2}(1+e\cos v)}, \quad A_{34}=\frac{\nu e\sin v}{\sqrt{1-e^2}},\\
&A_{41}=-\nu\left[\frac{e\sin v}{2\sqrt{1-e^2}}-\frac{3}{2}\nu(t-t_0)\frac{a^2}{r^2}\right],\\
&A_{42}=-\frac{\nu\sin v}{\sqrt{1-e^2}}\frac{a}{r},\\
&A_{43}=\frac{-\nu\cos v}{\sqrt{1-e^2}}\frac{a}{r}, \quad A_{44}=\frac{-\nu(1+e\cos v)}{\sqrt{1-e^2}},
\end{aligned}\right| \tag{5.239}$$

where ν is determined by equation (5.215).

For the elements of matrix B we have the following expressions:

$$\left.\begin{aligned}
&B_{11}=\frac{r}{a}\cos v, \quad B_{12}=\frac{r}{a}\sin v,\\
&B_{21}=\frac{-\nu\sin v}{\sqrt{1-e^2}}, \quad B_{22}=\frac{\nu(\cos v+e)}{\sqrt{1-e^2}}.
\end{aligned}\right| \tag{5.240}$$

The determinants $|A|$ and $|B|$ of matrices A and B are the Wronskians of the systems of partial solutions to the homogeneous equations (5.229) and (5.230), respectively.

Since the matrices of the coefficients of the rights sides of the homogeneous equations (5.229) and (5.230) contain no diagonal elements, according to the well known Ostrogradsky-Liouville Theorem the Wronskians of these systems are constant[11] for all values of t. They are therefore easily calculated by setting $t = t_0$. Since for this value of t the angle v is also equal to 0, it follows from relations (5.239), (5.240) and (5.207) that:

$$\left.\begin{array}{l}
A_{11}=0,\quad A_{12}=0,\quad A_{23}=0,\quad A_{24}=0,\\
A_{33}=0,\quad A_{34}=0,\\
A_{41}=0,\quad A_{42}=0,\\
A_{13}=\dfrac{2+e}{1+e},\ A_{14}=1-e,\ A_{21}=1-e,\ A_{22}=-1,\\
A_{31}=\dfrac{-v(1+e)}{2\sqrt{1-e^2}},\quad A_{32}=\dfrac{v}{\sqrt{1-e^2}\,(1-e)},\\
A_{43}=\dfrac{-v}{(1-e)\sqrt{1-e^2}},\quad A_{44}=\dfrac{-v(1+e)}{\sqrt{1-e^2}},\\
B_{11}=1-e,\ B_{12}=0,\ B_{21}=0,\ B_{22}=\dfrac{v(1+e)}{\sqrt{1-e^2}}.
\end{array}\right| \qquad (5.241)$$

We now find:

$$|A|=-\frac{v^2}{2},\quad |B|=v\sqrt{1-e^2}. \qquad (5.242)$$

The Wronskians of these systems of partial solutions are non-zero. Therefore, the partial solutions in questions are linearly independent. They remain independent for e = 0, i.e., for a circular orbit.

We note that if the derivatives (5.232) of the radius vector $\vec{r}$ in arbitrary constants are taken as $\vec{q}_i$ in place of (5.235), the Wronskian of the corresponding system of solutions reduces to 0 for e = 0. This fact, which is easily demonstrated by composing and expanding the determinant in question, results from the fact that in solving equation (5.176) the circular orbit was excluded by condition (5.200). The linear independence of the solutions defining matrix A was achieved for e = 0 as a result of the fact that the linear combinations (5.235) of the derivatives (5.232), rather than the derivatives (5.232) themselves, were taken as the partial solutions, as proposed by A. I. Lur'ye[12].

Since the system of solutions defined by the vectors $\vec{q}_i$ and $\vec{p}_i$ is linearly independent, i.e., a fundamental system, the general solution to the homogeneous equation (5.173) may be represented in the form:

$$\delta r = \sum_{i=1}^{6} C_i q_i, \quad \frac{d\,\delta r}{dt} = \sum_{i=1}^{6} C_i p_i.$$

(5.243)

Then for the general solution to the system of equations (5.229) we obtain the formulas

$$x_i = \sum_{j=1}^{4} A_{ij} C_j \quad (i = 1, 2, 3, 4).$$

(5.244)

and for the general solution to the system (5.230), the formulas

$$x_i = \sum_{j=1}^{2} B_{ij} C_{4+j} \quad (i = 5, 6) \tag{5.245}$$

5.4.4. Integration of non-homogeneous equations. The solution to the non-homogeneous equation (5.173) may now be found from solution (5.243) by the method of variation of arbitrary constants. Taking C_i as functions of time, we obtain:

$$\delta r = \sum_{i=1}^{6} C_i(t) q_i, \quad \frac{d\,\delta r}{dt} = \sum_{i=1}^{6} C_i(t) p_i. \tag{5.246}$$

We obtain the following system of linear equations for $C_i(t)$:

$$\left.\begin{aligned} &\sum_{i=1}^{6} \dot{C}_i(t) q_i = 0, \\ &\sum_{i=1}^{6} \dot{C}_i(t) p_i = \Delta n - 2\Delta m \times \frac{dr}{dt} + r \times \frac{d\Delta m}{dt}. \end{aligned}\right| \tag{5.247}$$

In terms of projections on the x, y, z axes the system of equations (5.247) breaks down, in accordance with formulas (5.244) and (5.245) into two systems: one fourth-order system and one second-order system. The first, serving for the determination of $\dot{C}_1(t)$, $\dot{C}_2(t)$, $\dot{C}_3(t)$, $\dot{C}_4(t)$ has the form:

$$\left.\begin{aligned} &\sum_{i=1}^{4} \dot{C}_i(t) A_{1i} = 0, \quad \sum_{i=1}^{4} \dot{C}_i(t) A_{2i} = 0, \\ &\sum_{i=1}^{4} \dot{C}_i(t) A_{3i} = \Delta n_x - 2\Delta m_y \dot{r} - \Delta \dot{m}_y r, \\ &\sum_{i=1}^{4} \dot{C}_i(t) A_{4i} = \Delta n_z + 2r\omega_y \Delta m_y. \end{aligned}\right| \tag{5.248}$$

The functions $\dot{C}_5(t)$ and $\dot{C}_6(t)$ are found from the second system:

$$\left.\begin{aligned}
&\sum_{i=1}^{2} \dot{C}_{i+4}(t) B_{1i} = 0, \\
&\sum_{i=1}^{2} \dot{C}_{i+4}(t) B_{2i} = \Delta n_y + 2\Delta m_x \dot{r} + \Delta \dot{m}_x r - \omega_y \Delta m_z r.
\end{aligned}\right\} \tag{5.249}$$

The determinants of the linear systems (5.248) and (5.249) are the Wronskians $|A|$ and $|B|$. They are non-zero and the systems (5.248) and (5.249) are uniquely solvable in $\dot{C}_i(t)$

In order to solve the systems of equations (5.248) and (5.249) for $\dot{C}_i$, we construct matrices D and E, the reciprocals of matrices A and B:

$$D = A^{-1}, \quad E = B^{-1}. \tag{5.250}$$

After the necessary transformations we obtain the following values for the elements of matrix D:

$$\left.\begin{aligned}
&D_{11} = 0, \quad D_{12} = 2\frac{a^2}{r^2}, \quad D_{13} = \frac{2(1 + e\cos v)}{\nu\sqrt{1-e^2}}, \\
&D_{14} = \frac{2e\sin v}{\nu\sqrt{1-e^2}}; \\
&D_{21} = \sin v, \quad D_{22} = \frac{(1 + e\cos v)(e + \cos v)}{1-e^2}, \\
&D_{23} = \frac{\sqrt{1-e^2}}{\nu}\,\frac{e + 2\cos v + e\cos^2 v}{1 + e\cos v}, \\
&D_{21} = \frac{\sqrt{1-e^2}}{\nu}\sin v; \\
&D_{31} = \cos v; \\
&D_{32} = \frac{3\nu(t-t_0)}{\sqrt{1-e^2}}\,\frac{a^2 e}{r^2} - \frac{1 + e\cos v + e^2}{1-e^2}\sin v, \\
&D_{33} = \frac{1}{\nu}\left[-\frac{\sqrt{1-e^2}}{1 + e\cos v}(2 + e\cos v)\sin v + \right. \\
&\qquad\qquad \left. + \frac{3\nu(t-t_0)}{1-e^2}(1 + e\cos v)e\right], \\
&D_{31} = \frac{1}{\nu}\left[\frac{3\nu(t-t_0)}{1-e^2}e^2\sin v - \right. \\
&\qquad\qquad \left. - \frac{\sqrt{1-e^2}}{1 + e\cos v}(2e - \cos v - e\cos v)\right];
\end{aligned}\right| \tag{5.251}$$

$$
\left.\begin{aligned}
D_{41} &= -\frac{1}{\sqrt{1-e^2}},\\
D_{42} &= \frac{3\nu(t-t_0)}{(1-e^2)^{5/2}}\frac{a^2}{r^2} - e\sin v\,\frac{2+e\cos v}{(1-e^2)^2},\\
D_{43} &= \frac{1}{\nu}\Big[-e\sin v\,\frac{2+e\cos v}{\sqrt{1-e^2}\,(1+e\cos v)} +\\
&\qquad + \frac{3\nu(t-t_0)}{(1-e^2)^2}(1+e\cos v)\Big],\\
D_{44} &= \frac{1}{\nu}\left[\frac{3\nu(t-t_0)}{(1-e^2)^2}e\sin v + \frac{e\cos v + e^2\cos^2 v - 2}{\sqrt{1-e^2}\,(1+e\cos v)}\right].
\end{aligned}\right| \qquad (5.251)
$$

The elements of matrix E are:

$$
\left.\begin{aligned}
E_{11} &= \frac{\cos v + e}{1-e^2}, & E_{12} &= -\frac{\sqrt{1-e^2}\sin v}{\nu(1+e\cos v)},\\
E_{21} &= \frac{\sin v}{1-e^2}, & E_{22} &= \frac{\sqrt{1-e^2}\cos v}{\nu(1+e\cos v)}.
\end{aligned}\right\} \qquad (5.252)
$$

Using matrices D and E, we find:

$$
\left.\begin{aligned}
\dot{C}_i(t) &= (\Delta n_x - 2\Delta m_y\dot{r} - \Delta\dot{m}_y r)D_{i3} +\\
&\qquad + (\Delta n_z + 2r\omega_y\Delta m_y)D_{i4},\\
\dot{C}_{i+4}(t) &= (\Delta n_y + 2\Delta m_x\dot{r} + \Delta\dot{m}_x r - \omega_y\Delta m_z r)E_{i2}.
\end{aligned}\right\} \qquad (5.253)
$$

Integrating the latter equalities, we obtain the following expressions for $C_i(t)$:

$$
\left.\begin{aligned}
C_i(t) &= \int_0^t [(\Delta n_x - 2\Delta m_y\dot{r} - \Delta\dot{m}_y r)D_{i3} +\\
&\qquad + (\Delta n_z + 2r\omega_y\Delta m_y)D_{i4}]\,dt + C_i^0,\\
C_{i+4}(t) &= \int_0^t (\Delta n_y + 2\Delta m_x\dot{r} +\\
&\qquad + \Delta\dot{m}_x r - \omega_y\Delta m_z r)E_{i2}\,dt + C_{i+4}^0,
\end{aligned}\right| \qquad (5.254)
$$

in which C_i^0 are determined by the initial conditions of equations (5.229) and (5.230).

For equations (5.229) the initial conditions will be:

$$
\left.\begin{aligned}
x_1^0 &= \delta x^0, \quad x_2^0 = \delta z^0,\\
x_3^0 &= \delta\dot{x}^0 + \omega_y^0\delta z^0, \quad x_4^0 = \delta\dot{z}^0 - \omega_y^0\delta x^0.
\end{aligned}\right\} \qquad (5.255)
$$

For equations (5.230):

$$x_5^0 = \delta y^0, \quad x_6^0 = \delta \dot{y}^0. \tag{5.256}$$

Using matrices D and E and expressions (5.255) and (5.256), we find that

$$C_i^0 = \sum_{j=1}^{4} D_{ij}^0 x_j^0, \quad C_{i+4}^0 = \sum_{j=1}^{2} E_{ij}^0 x_{j+4}^0. \tag{5.257}$$

Projecting the first equality (5.246) on the x, y, z axes and taking into account relations (5.254) and (5.257), we obtain finally the following expressions[13] for $\delta x(t)$, $\delta y(t)$, $\delta z(t)$:

$$\left.\begin{aligned}
\delta x &= \sum_{i=1}^{4} A_{1i} \left\{ \int_0^t [(\Delta n_x - 2\Delta m_y \dot{r} - \Delta \dot{m}_y r) D_{i3} + \right. \\
&\qquad \left. + (\Delta n_z + 2r\omega_y \Delta m_y) D_{i4}]\, dt + \sum_{j=1}^{4} D_{ij}^0 x_j^0 \right\}, \\
\delta y &= \sum_{i=1}^{2} B_{1i} \left[\int_0^t (\Delta n_y + 2\Delta m_x \dot{r} + \Delta \dot{m}_x r - \right. \\
&\qquad \left. - \omega_y \Delta m_z r) E_{i2}\, dt + \sum_{j=1}^{2} E_{ij}^0 x_{j+4}^0 \right], \\
\delta z &= \sum_{i=1}^{4} A_{2i} \left\{ \int_0^t [(\Delta n_x - 2\Delta m_y \dot{r} - \Delta \dot{m}_y r) D_{i3} + \right. \\
&\qquad \left. + (\Delta n_z + 2r\omega_y \Delta m_y) D_{i4}]\, dt + \sum_{j=1}^{4} D_{ij}^0 x_j^0 \right\}.
\end{aligned}\right| \tag{5.258}$$

It is easily demonstrated that, for e = 0, i.e., in the case of a circular orbit, formulas (5.258) reduce to the earlier formulas (5.117).

In fact, for a circular orbit

$$\left.\begin{aligned}
e = 0, \quad r = a, \quad \dot{r} = 0, \quad v = \omega_0, \quad t_0 = 0, \quad v = \omega_0 t, \\
\omega_y = \omega_0, \quad \dot{\omega}_y = 0.
\end{aligned}\right\} \tag{5.259}$$

Substituting these values into formulas (5.239), (5.240), (5.251) and (5.252), we find:

$$
\left.
\begin{array}{l}
A_{11} = -\frac{3\omega_0 t}{2}, \quad A_{12} = 2\sin\omega_0 t, \quad A_{13} = 2\cos\omega_0 t, \\
A_{14} = 1, \\
A_{21} = 1, \quad A_{22} = -\cos\omega_0 t, \quad A_{23} = \sin\omega_0 t, \quad A_{24} = 0, \\
B_{11} = \cos\omega_0 t, \quad B_{12} = \sin\omega_0 t, \\
D_{11} = \frac{2}{\omega_0}, \quad D_{21} = \frac{2}{\omega_0}\cos\omega_0 t, \quad D_{31} = -\frac{2}{\omega_0}\sin\omega_0 t, \\
D_{41} = 3t, \quad D_{14} = 0, \quad D_{24} = \frac{1}{\omega_0}\sin\omega_0 t,
\end{array}
\right|
$$

$$
\left.
\begin{array}{l}
D_{34} = \frac{1}{\omega_0}\cos\omega_0 t, \quad D_{44} = -\frac{2}{\omega_0}, \\
E_{12} = 0, \quad E_{22} = 1, \\
D^0_{11} = 0, \quad D^0_{12} = 2, \quad D^0_{13} = \frac{2}{\omega_0}, \quad D^0_{14} = 0, \\
D^0_{21} = 0, \quad D^0_{22} = 1, \quad D^0_{23} = \frac{2}{\omega_0}, \quad D^0_{24} = 0, \\
D^0_{31} = 1, \quad D^0_{32} = 0, \quad D^0_{33} = 0, \quad D^0_{34} = \frac{1}{\omega_0}, \\
D^0_{41} = -1, \quad D^0_{42} = 0, \quad D^0_{43} = 0, \quad D^0_{44} = -\frac{2}{\omega_0}, \\
E^0_{11} = \omega_0, \quad E^0_{12} = 0, \quad E^0_{21} = 0, \quad E^0_{22} = 1.
\end{array}
\right| \qquad (5.260)
$$

Substituting now these values of the matrix elements in relations (5.258), integrating by parts the terms in the integrands containing $\Delta\dot{m}_x$, and $\Delta\dot{m}_y$, and taking into account the relations between x^0_j and δx^0, δy^0, δz^0, $\delta\dot{x}^0$, $\delta\dot{y}^0$, $\delta\dot{z}^0$, i.e., equalities (5.255) and (5.256), we arrive at formulas (5.117).

In conclusion we make one comment regarding the second group of error equations (5.3) for the case of Keplerian motion.

The equations of the second group were examined in §5.2, where the general solution to these equations in quadratic forms was found and expressed in formulas (5.48).

In the case of Keplerian motion, when the $\xi_*\eta_*(\xi^1\xi^2)$ plane is taken as the orbital plane, the general solution (5.48) to the error equations of the second group is expressed by formulas (5.71), (5.72), and (5.73) or the equivalent formulas (5.77).

Formulas (5.77) contain ω_y, and equalities (5.71) and (5.72) contain the angle σ. In order to obtain explicit expressions for θ_{1x} and θ_{1z}, it is necessary to substitute in formula (5.77) for

ω_y the value of this projection of the angular velocity deriving from relations (5.216) and (5.227), or, in formulas (5.71) and (5.72), to substitute the value of σ obtained from the solution to system (5.216).

Formulas (5.258) give the solution to the error equations (5.226). In §5.4.1 it was stated that the homogeneous equation (5.173), and therefore also the homogeneous equation (5.258), are equations in variations of Keplerian motion. Therefore, formulas (5.258), if it is asssumed that $\Delta n_x = 0$, $\Delta n_y = 0$, $\Delta n_z = 0$, $\Delta m_x = 0$, $\Delta m_y = 0$, $\Delta m_z = 0$, also give the solution to the problem of the deviation of the motion of an artificial satellite of the earth from its calculated (nominal) trajectory (in a spherical field) for incorrect initial conditions. Moreover, formulas (5.258) enable us to calculate the change in the trajectory of a satellite (or an orbital aircraft) under the influence of small perturbing forces (resistance of the upper layers of the atmosphere, maneuvering with small thrust, etc.). In order to do this we have only to substitute into formulas (5.258) the values χ_x, χ_y, χ_z of the projections of these forces on the axes of the orbital (nominal) trihedron for the functions $\Delta n_x - 2\Delta m_y \dot{r} - \Delta \dot{m}_y r$, $\Delta n_y + 2\Delta m_x \dot{r} + \Delta \dot{m}_x r - \omega_y \Delta m_z r$, $\Delta n_z + 2r\omega_y \Delta m_y$. It is evident, in particular, that it is possible to calculate in this manner the change in the trajectory of a satellite under the influence of a non-spherical component of the gravitational field. The corresponding values of the functions χ_x, χ_y, χ_z are found from the formulas for the projections of the strength of the regularized gravitational field of the earth derived in §2.2.

It follows that formulas (5.258) also give error expressions for the case of an object in near-Keplerian motion. Indeed, in this case the coefficients of the error equations (5.226) will differ from the coefficients for initial Keplerian motion by amounts of the first order of smallness. Multiplication of these small quantities by the small quantities δx, δy, δz, $\delta \dot{x}$, $\delta \dot{y}$, $\delta \dot{z}$, Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z gives magnitudes of the second order of smallness. Therefore, equations (5.226) and their solution (5.258) remain valid to a first approximation for near-Keplerian motion.

§5.5. Errors in the Determination of the Coordinates of an Object and Its Orientation. Errors Deriving from Inaccuracies in Instrument Readings and Initial Conditions.

5.5.1. <u>The case of an object which is stationary in inertial space</u>. In the preceding sections of this chapter we analyzed the first (5.1) and second (5.3) groups of differential error equations of an inertial system. In §5.2 a solution in quadratic forms for the case of arbitrary motion of an object was found for the second group of error equations. In §5.3 and §5.4 exact solutions for the equations of the first group were found only for certain special cases.

However, as was stated above, for example in §5.1, the solution to equations (5.1) and (5.3) in itself does not give expressions for errors in the determination by an inertial system of the coordinates of the object and the parameters of its orientation. It is these expressions, however, which are the goal of the analysis of the error equations of an inertial system.

In order to obtain these errors and to establish their relation to inaccuracies in instrument readings and initial conditions, it is necessary to consider in addition to equations (5.1) and (5.3) relations (5.5), (5.6), (5.8) and (5.9).

However, relations (5.8) and (5.9) do not bear on the solution to equations (5.1) and (5.3). These relations give the errors θ_{3x}, θ_{3y}, θ_{3z} in the determination of the orientation of the object deriving from the instrument errors $\Delta\alpha$, $\Delta\beta$, $\Delta\lambda$ in the measurement of the angles of rotation of the gimbal rings of the gyroscopic platform of the inertial system. The errors θ_{3x}, θ_{3y}, θ_{3z} are independent of the solutions to equations (5.1) and (5.3) and, according to equalities (5.9), are simply added to the errors θ_x, θ_y, θ_z obtained from the solution to equations (5.6). At the same time θ_x, θ_y, θ_z are completely independent of $\Delta\alpha$, $\Delta\beta$, $\Delta\lambda$.

Therefore, it is important here to make use of relations (5.5) and (5.6), combining the solutions to equations (5.1) and (5.3), the right sides of which are functions in part of the same quantities Δm_x, Δm_y, Δm_z.

Analysis of the errors in the determination by an inertial system of the coordinates of the object and its orientation, i.e., analysis of the functional relation between the errors δx_3, δy_3, δz_3 and θ_x, θ_y, θ_z, entering into formulas (5.5) and (5.6), on inaccuracies in the instrument readings of the inertial system,and in the initial conditions of its functioning, becomes now our immediate task. It is, of course, evident that we will be able to achieve an exact solution to this problem only for those cases of motion of the object in which equations (5.1) allow exact integration.

The simplest case that in which the object is stationary in the $O_1\xi_*\eta_*\zeta_*$ coordinate system. In this case the solution to equations (5.1) is given by formulas (5.95) and (5.96), and the solution to equations (5.3) by formulas (5.58).

Formulas (5.95) and (5.96) were obtained under the assumption that the z axis of the xyz trihedron was directed along the vector $\vec{r}$. In this case, in the first three formulas (5.5) it follows that:

$$x = y = 0, \quad z = r \tag{5.261}$$

Then

$$\delta x_1 = \theta_{1y} r, \quad \delta y_1 = -\theta_{1x} r, \quad \delta z_1 = 0. \tag{5.262}$$

If θ_{1x} and θ_{1y} from equalities (5.58) are substituted into these expressions, and the resulting expressions together with relations (5.95) and (5.96) are substituted into the last three formulas (5.5), we obtain:

$$
\left.
\begin{aligned}
\delta x_3 &= \delta x^0 \cos\omega_0 t + \frac{\delta\dot{x}^0 + r\Delta m_y^0}{\omega_0}\sin\omega_0 t + r\theta_{1y}^0 + \\
&\quad + \frac{1}{\omega_0}\int_0^t \Delta n_x \sin\omega_0 (t-\tau)\,d\tau + \\
&\quad + r\int_0^t \Delta m_y\,[1 - \cos\omega_0 (t-\tau)]\,d\tau, \\
\delta y_3 &= \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0 - r\Delta m_x^0}{\omega_0}\sin\omega_0 t - \\
&\quad - r\theta_{1x}^0 + \frac{1}{\omega_0}\int_0^t \Delta n_y \sin\omega_0 (t-\tau)\,d\tau + \\
&\quad + r\int_0^t \Delta m_x\,[\cos\omega_0 (t-\tau) - 1]\,d\tau, \\
\delta z_3 &= \delta z^0 \cosh\omega_0 \sqrt{2}t + \frac{\delta\dot{z}^0}{\sqrt{2}\omega_0}\sinh\omega_0 \sqrt{2}t + \\
&\quad + \frac{1}{\omega_0\sqrt{2}}\int_0^t \Delta n_z \sinh\omega_0 \sqrt{2}(t-\tau)\,d\tau.
\end{aligned}
\right| \tag{5.263}
$$

Relations (5.263) give for the case under consideration the total errors in the determination by the inertial system of coordinates of the object as a function of instrument errors and errors in the initial conditions.

In order to obtain the errors in the determination of the orientation of the object, we refer to formulas (5.6) or, taking into account expression (5.261), to formulas (5.22).

Here, as was explained in §3.5 and §5.1, two cases must be distinguished. The first corresponds to the structure of an inertial system in which the orientation of the xyz trihedron is not set as a function of the coordinates determined by the inertial system. In this case, in formulas (5.6)

$$
\delta\alpha_{ij} = 0, \quad \frac{\partial\alpha_{ij}}{\partial\vartheta_k} = 0. \tag{5.264}
$$

Therefore,

$$
\theta_x = -\theta_{1x}, \quad \theta_y = -\theta_{1y}, \quad \theta_z = -\theta_{1z}. \tag{5.265}
$$

and in accordance with equalities (5.58) we have:

$$\left.\begin{aligned} \theta_x &= -\int_0^t \Delta m_x\,dt - \theta_{1x}^0, \quad \theta_y = -\int_0^t \Delta m_y\,dt - \theta_{1y}^0, \\ \theta_z &= -\int_0^t \Delta m_z\,dt - \theta_{1z}^0. \end{aligned}\right| \tag{5.266}$$

In the second case the orientation of the xyz trihedron is a function of the coordinates determined by the inertial system.

If the xyz trihedron in the unperturbed position is a moving free-azimuth ($\omega_z = 0$) trihedron, then according to expressions (5.31)

$$\theta_x = -\frac{\delta y}{r}, \quad \theta_y = \frac{\delta x}{r}, \quad \theta_z = -\theta_{1z}. \tag{5.267}$$

Substituting into the first two equalities (5.267) the values δy and δx from relations (5.95) and (5.96) and into the third the value θ_{1z} from formulas (5.58), we find:

$$\left.\begin{aligned} \theta_x = &-\frac{\delta y^0}{r}\cos\omega_0 t - \frac{\delta\dot{y}^0 - r\,\Delta m_x^0}{r\omega_0}\sin\omega_0 t - \\ &-\frac{1}{r\omega_0}\int_0^t \Delta n_y \sin\omega_0(t-\tau)\,d\tau - \\ &\qquad -\int_0^t \Delta m_x \cos\omega_0(t-\tau)\,d\tau, \\ \theta_y = &\frac{\delta x^0}{r}\cos\omega_0 t + \frac{\delta\dot{x}^0 + r\,\Delta m_y^0}{r\omega_0}\sin\omega_0 t + \\ &+\frac{1}{r\omega_0}\int_0^t \Delta n_x \sin\omega_0(t-\tau)\,d\tau - \int_0^t \Delta m_y \cos\omega_0(t-\tau)\,d\tau, \\ \theta_z = &-\int_0^t \Delta m_z\,dt - \theta_{1z}^0. \end{aligned}\right| \tag{5.268}$$

If the xyz trihedron, being in the unperturbed position a moving trihedron, is oriented to the points of the compass, then the third formula (5.32) applies instead of the third equality (5.268):

$$\theta_z = -\theta_{1z} + \operatorname{tg}\varphi\,\frac{\delta x_3}{r}, \tag{5.269}$$

which after substitution of θ_{1z} from (5.56) and δx_3 from (5.263), takes the form:

$$\theta_z = -\theta_{1z}^0 - \int_0^t \Delta m_z \, dt + \operatorname{tg}\varphi \left\{ \frac{\delta x^0}{r} \cos \omega_0 t + \frac{\delta \dot{x}^0 + r \Delta m_y^0}{r \omega_0} \sin \omega_0 t + \frac{1}{r\omega_0} \int_0^t \Delta n_x \sin \omega_0 (t - \tau) \, d\tau + \int_0^t \Delta m_y [1 - \cos \omega_0 (t - \tau)] \, d\tau + \theta_{1y}^0 \right\}. \tag{5.270}$$

Let us consider expressions (5.263), (5.266), (5.268) and (5.270) for the total errors in the determination of the coordinates and orientation of the object. Let us first consider formulas (5.263). Since the first two of these formulas are analogous (which is evident from symmetry considerations), it is sufficient to consider the formula for δx_3, for example, i.e., the first formula (5.263).

It is evident from this formula that the portion of the error δx_3 deriving from the initial conditions δx^0 and $\delta \dot{x}^0$ and the initial value of Δm_y^0, is a harmonic oscillation with a frequency ω_0. Since we denoted the quantity μ / r^3 by ω_0^2, the period corresponding to the frequency ω_0 is calculated according to the formula

$$T = 2\pi \sqrt{\frac{r^3}{\mu}} = 2\pi \sqrt{\frac{r}{g}}. \tag{5.271}$$

If the distance r from the center of the earth to the moving object is approximately equal to the radius of the earth, then T $\approx$ 84 min. This period is usually termed the Schuler period (after the German physicist who first noted the remarkable properties possessed by pendulums with this period)[14].

The initial value θ_{1y}^0 gives rise to a constant error in the determination of the coordinate x. The instrument error Δn_x is integrated with a weight $\frac{1}{\omega_0} \sin \omega_0 (t - \tau)$, and the instrument error Δm_y - c with a weight $r[1 - \cos \omega_0 (t - \tau)]$.

For constant Δn_x, Δm_y, Δn_y, Δm_x the first two formulas (5.263) take the form:

$$\left.\begin{aligned}\delta x_3 &= \frac{\Delta n_x}{\omega_0^2} + r\theta_{1y}^0 + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \\ &\quad + \frac{\delta\dot{x}_0}{\omega_0}\sin\omega_0 t + r\Delta m_y t.\\ \delta y_3 &= \frac{\Delta n_y}{\omega_0^2} - r\theta_{1x}^0 + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \\ &\quad + \frac{\delta\dot{y}_0}{\omega_0}\sin\omega_0 t - r\Delta m_x t.\end{aligned}\right\} \tag{5.272}$$

The error δx_3 in the determination of the coordinate x is composed of the error $\frac{\Delta n_x}{\omega^2_0} + r\theta^0_{1y}$, the error oscillating with a frequency ω_0, and the error $r\Delta m_y t$, increasing proportionally to time.

For the quantitative evaluation of the dependence of the errors δx_3, δy_3 and δz_3 in the determination of the coordinates on the instrument errors of the sensing elements of the inertial system and the errors in the initial condition, we introduce some numbers.

If the object in which the inertial system is placed is located near the surface of the earth, we may assume

$$r = 6.4 \cdot 10^6 \text{ m}, \quad g = 9.8 \text{ m/sec}^2. \tag{5.273}$$

Then

$$\omega_0 = 1.25 \cdot 10^{-3} \text{ 1/sec}, \quad \omega_0^2 = 1.56 \cdot 10^{-6} \text{ 1/sec}^2. \tag{5.274}$$

From the first formula (5.272) it follows that the partial error δx_3 = 1 km causes a newtonometer error of $\Delta n_x = 7.8 \cdot 10^{-4}$ m/sec^2 ($\simeq 8 \cdot 10^{-5}$ g), the error $\theta^0_{1y} = 1.6 \cdot 10^{-4}$ rad ($\simeq$ 0.55 angular min), the error δx^0 = 1 km, and the error $\delta\dot{x}^0$ = 1.25 m/sec. For the partial error δx_3, giving rise to the error Δm_y, not to exceed 1 km in the course of one hour of the operation of the system, Δm_y should not exceed

$4.4 \cdot 10^{-8}$ 1/sec ($\approx 0.009°$/hour). It is evident that for numerical values of the error δy_3 evaluations may be obtained by the same procedure as was used above to determine δx_3.

The third formula (5.263) shows that the error δz_3 in the determination by the inertial system of the distance r to the center of the earth is a function of the errors δz^0 and $\delta \dot{z}^0$ in the initial conditions and the error Δn_z deriving from the newtonometer oriented along the z axis, but is not a function of the errors Δn_x or Δn_y of the newtonometers oriented oriented along the x and y axes, or the errors Δm_x, Δm_y, Δm_z in the measurement of the absolute angular velocity. However, as is evident from the first two formulas (5.263), the error Δm_x does not enter into the errors δx_3 or δy_3.

It follows from the last formula (5.263) that for long-term operation of the system δz_3 increases exponentially. For a constant error Δn_z we will have:

$$\delta z_3 = -\frac{\Delta n_z}{2\omega_0^2} + \left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\cosh \omega_0 \sqrt{2}\, t + \frac{\delta \dot{z}^0}{\omega_0 \sqrt{2}} \sinh \omega_0 \sqrt{2}\, t. \tag{5.275}$$

The error δz_3 increases very rapidly with time, which is explained by the rapid growth of the hyperbolic functions $\cosh \omega_0 \sqrt{2}\, t$ and $\sinh \omega_0 \sqrt{2}$ Approximate values of these functions for various moments of time are presented below.

	t, min			
	15	30	45	60
$\mathrm{ch}\, \omega_0 \sqrt{2}\, t$	2,13	10	45	200
$\mathrm{sh}\, \omega_0 \sqrt{2}\, t$	2,35	10	45	200

Thus, as a result of the errors Δn_z, δz^0, and $\delta \dot{z}^0$, the time during which the inertial system is able to autonomously determine the distance r to the center of the earth to an acceptable degree of accuracy, may be small. For example, for the values of the newtonometer errors and errors in the initial conditions used above in computing the error δx_3, the time in question is of the order of 10 - 15 min. In fact, let the errors Δn_z and $\delta \dot{z}^0$ have the same values as those which in the preceding calculation gave an error of 1 km for δx_3, i.e., let $\Delta n_z = 7.8 \cdot 10^{-4}$ m/sec^2 and $\delta \dot{z}^0 = 1.25$ m/sec. Then the error δz_3 reaches 1 km in $\approx$ 18 min due to Δn_z alone, and in $\approx$ 10 min due to $\delta \dot{z}^0$ alone. In order for δz_3 not to exceed 1 km in 15 min due to the error δz^0, we must take $\delta z^0 \approx 0.43$ km.

It should be noted that in the case under consideration, namely that of a stationary object, the increase in δz_3 has no effect on the magnitudes of the errors δx_3 and δy_3. If this property of the inertial system were preserved for a moving object an interesting possibility would arise. This would be that the errors in the determination of the coordinates of the object on the surface of the earth (for example, geographical latitudes and longitudes), i.e., the errors in the determination of the direction of r from the center of the earth to the object, could be small even for a significant duration of operation of the system, in spite of the large error δz_3 in the determination of the distance r to the center of the earth.

However, in the case of a moving object the errors δx_3 and δy_3 also have components which increase exponentially with time, although these components contain the velocity of motion as a factor, and consequently may be small for small velocities. We will consider this question in greater detail when we consider errors in the functioning of an inertial system for various cases of motion of the object, but let us now return to the case of a stationary object.

We have discussed relations (5.263), (5.272) and (5.275), characterizing errors in the determination of coordinates. Let us now turn to orientation errors, i.e., to formulas (5.266), (5.268) and (5.270).

Formulas (5.266) were obtained for the case in which the orientation of the object is defined relative to a trihedron, the position of whose axes relative to the $O_1\xi_*\eta_*\zeta_*$ coordinate system is not a function of the coordinates determined by the inertial system. This could be, in particular, the case in which the system is based on a free gyrostabilized platform, and the orientation of the object is defined relative to its axes. This case obtains when the basis of the system is a spatial gauge of absolute angular velocity or a time-maneuverable gyroplatform, if the orientation of the object is determined relative to the axes of the platform of the element measuring absolute angular velocity or the gyroplatform. Finally, included here also is the more general case in which the orientation of the axes of the sensing elements is a function of the coordinates determined by the initial system, but the orientation of the object is determined relative to directions which change their orientation as a function only of time.

For all of these cases, as is evident from formulas (5.266), the errors in the determination of the orientation of the object are composed of their initial values and the integrals over time of the instrument errors Δm_x, Δm_y, Δm_z. For constant Δm_x, Δm_y, Δm_z, the orientation errors increase only as a function of time:

$$\left.\begin{array}{c}\theta_x = -\Delta m_x t - \theta^0_{1x}, \quad \theta_y = -\Delta m_y t - \theta^0_{1y}, \\ \theta_z = -\Delta m_z t - \theta^0_{1z}.\end{array}\right\} \qquad (5.276)$$

Formulas (5.268) and (5.270) are valid for cases in which the position of the trihedron relative to which the orientation of the object is determined (or the trihedron associated with the platform, if its orientation is in question), relative to the coordinate system $O_1\xi_*\eta_*\zeta_*$, is a function of the coordinates, formulas (5.268) having been derived for the case in which this trihedron is a moving free-azimuth trihedron.

It follows from the first two formulas (5.268) that the errors θ_x and θ_y are composed of harmonic oscillations with a frequency ω_0 and amplitudes determined by the initial values δx^0, δy^0, $\delta \dot{x}^0$, $\delta \dot{y}^0$,

Δm_x^0, Δm_y^0, and of the instrument errors Δn_x, Δn_y, Δm_x, Δm_y. The portions of θ_x and θ_y corresponding to these instrument errors are obtained by integrating them with the weights $\frac{1}{r\omega_0}\sin\omega_0(t-\tau)$ and $\frac{1}{r\omega_0}\cos\omega_0(t-\tau)$. For constant Δn_x, Δn_y, Δm_x, Δm_y formulas (5.268) give:

$$\left.\begin{aligned} \theta_x &= -\frac{\Delta n_y}{r\omega_0^2} - \frac{1}{r}\left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t - \frac{\delta \dot{y}^0}{r\omega_0}\sin\omega_0 t, \\ \theta_y &= \frac{\Delta n_x}{r\omega_0^2} + \frac{1}{r}\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta \dot{x}^0}{r\omega_0}\sin\omega_0 t, \\ \theta_z &= -\Delta m_z t - \theta_{z}^0. \end{aligned}\right\} \tag{5.277}$$

Thus, for const int instrument errors, θ_x and θ_y fluctuate about the values

$$\theta_x' = -\frac{\Delta n_y}{r\omega_0^2}, \quad \theta_y' = \frac{\Delta n_x}{r\omega_0^2} \tag{5.278}$$

with a frequency ω_0 and amplitudes

$$\left.\begin{aligned} \theta_x^* &= \frac{1}{r}\sqrt{\left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)^2 + \frac{1}{\omega_0^2}(\delta \dot{y}^0)^2}, \\ \theta_y^* &= \frac{1}{r}\sqrt{\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)^2 + \frac{1}{\omega_0^2}(\delta \dot{x}^0)^2}. \end{aligned}\right\} \tag{5.279}$$

The fundamental difference between the first two formulas (5.277) and the corresponding formulas (5.276) consist, therefore, in the fact that the latter do not entail an error in the orientation of the platform relative to the angles θ_x and θ_y which increases with time, although in both cases the errors in the determination of the coordinates have components which increase with time.

The third formula (5.277) for a constant value of Δm_z coincides with the third formula (5.276). The deviation θ_z in the azimuth is not a function of the errors in the determination of the coordinates of the object and for constant Δm_z it increases proportionally with time. Thus, errors in orientation are determined primarily by the instrument errors Δm_z. This fact gives rise in the case in question to more stringent requirements on the accuracy of the azimuth gyroscopes (the heading gyroscopes) of the inertial system.

Let us now consider formula (5.270). This formula and the two first formulas (5.268) describe the orientation errors for the case in which the trihedron xyz in the unperturbed position is oriented according to the points of the compass. Although in the preceding cases, i.e., the third formulas (5.266) and (5.268), θ_z was not a function of errors in the determination of the coordinates, this is now no longer the case.

The third term on the right side of formula (5.270) contains as a factor (the expression in brackets) the component δx_3 of the total coordinate error. For $\phi = 0$, this formula, of course, reduces to the third formula (5.268). For $\phi = \frac{\pi}{2}$, when tan ϕ tends to infinity equality (5.270), of course, becomes meaningless. In this case, the only conclusion that may be drawn from it is that in the immediate vicinity of a pole small δx_3 lead to a finite error θ_z, i.e., to a finite error in the determination of the bearing to the pole. This result is obvious from purely geometrical considerations. It demonstrates once again that if the trihedron bound to the platform of the inertial system is oriented to the points of the compass, it is necessary to exclude the vicinity of the pole from the possible areas in which the object may move.

5.5.2. <u>Motion of an object on a fixed great circle. Motion of a satellite in a circular orbit</u>. We have considered the errors in the determination of coordinates and orientation parameters for an object which is stationary in the $O_1\xi_*\eta_*\zeta_*$ coordinate system, for which case the solution to equations (5.1) and (5.3) was given by formulas (5.95), (5.96) and (5.58).

Let us now consider the motion of an object in a plane passing through a point O_1 at a constant distance from the center of the earth, for which case the solution to equations (5.1) is given by formulas (5.100), (5.111), and (5.117), and the solution to equations (5.3) by formulas (5.60)

In this case the xyz trihedron will be a moving trihedron. Its xz plane coincides with the $\xi_*\eta_*$ plane, i.e., with the plane of motion of the object, the z axis is directed along the vector $\vec{r}$, and y axis is normal to the plane of motion. The projections δx, δy, δz of the error vector $\delta\vec{r}$ on the axes of this trihedron are the solutions (5.100), (5.111) and (5.117), and the projections θ_{1x}, θ_{1y}, θ_{1z} of the error vector θ_1 are the solutions (5.60).

From relations (5.262) and the last formulas (5.5) we have:

$$\delta x_3 = \delta x + \theta_{1y} r, \quad \delta y_3 = \delta y - \theta_{1x} r, \quad \delta z_3 = \delta z. \tag{5.280}$$

These formulas characterize the total error in the determination of the coordinates. For the case in which $\omega_y < \omega_0$, the values of δx, δy, δz from expressions (5.100) and (5.111) and the values of θ_{1x} and θ_{1y} from equalities (5.60) must be substituted into them. When $\omega_y = \omega_0$, i.e., for the case of motion of a satellite in a circular orbit, expressions (5.117) must be used instead of formulas (5.100) and (5.111), and ω_y in equalities (5.60) must be replaced by ω_0.

Our task is to analyze the relation between the errors δx_3, δy_3, δz_3 and instrument errors and errors in initial conditions. It is also understandably of interest to compare the values of δx_3, δy_3, δz_3 for the type of motion in question with their expressions (5.263) for the case of a stationary object, and also to examine the special case of $\omega_y = \omega_0$, i.e., the case of motion of a satellite in a circular orbit.

It follows from the third formula (5.280) that the total error δz_3 in the determination of the distance r to the center of the earth is not a function of the solution (5.60) to equations (5.3) and, for $\omega_y < \omega_0$, is given by the second equality (5.111), which for constant instrument errors reduces to the second equality (5.112).

Let us investigate the relation between δz_3 and the initial conditions and instrument errors, taking the latter to be constant, i.e., using the second equality (5.112) to determine δz_3. Let us rewrite this equality, separating terms which are functions of the initial conditions from terms containing instrument errors:

$$
\begin{aligned}
\delta z_3 = \frac{\Delta n_z + 2r\omega_y \Delta m_y}{2\omega_0^2 + \omega_y^2} \Big\{ -1 + \frac{1}{(\omega_0^2 - \omega_y^2)(\mu^2 + \nu^2)} \times \\
\times \left[\mu^2 (\omega_0^2 - \omega_y^2 - \nu^2) \cos \nu t + \nu^2 (\omega_0^2 - \omega_y^2 + \mu^2) \cosh \mu t \right] \Big\} - \\
- \frac{2\omega_y \Delta n_x}{(\mu^2 + \nu^2)\mu\nu} (\mu \sin \nu t - \nu \sinh \mu t) + \\
+ \frac{\delta z^0}{(\omega_0^2 - \omega_y^2)(\mu^2 + \nu^2)} \left[\mu^2 (\omega_0^2 - \omega_y^2 - \nu^2) \cos \nu t + \right. \\
\left. + \nu^2 (\omega_0^2 - \omega_y^2 + \mu^2) \cosh \mu t \right] - \\
- \frac{\delta \dot{z}^0}{(\mu^2 + \nu^2)\mu\nu} \left[\mu (\omega_0^2 - \omega_y^2 - \nu^2) \sin \nu t - \right. \\
\left. - \nu (\omega_0^2 - \omega_y^2 + \mu^2) \sinh \mu t \right] + \\
+ \frac{2\omega_y (\omega_0^2 - \omega_y^2) \delta x^0}{(\mu^2 + \nu^2)\mu\nu} (\mu \sin \nu t - \nu \sinh \mu t) - \\
- \frac{2\omega_y \delta \dot{x}^0}{\mu^2 + \nu^2} (\cos \nu t - \cosh \mu t).
\end{aligned}
\tag{5.281}
$$

Comparing this expression with formula (5.275) for δz_3 for the case of a stationary object, we note first of all that expression (5.281) is a function not only of Δn_z, δz^0, and $\delta \dot{z}^0$, which appear in formula (5.275), but also of δx^0, $\delta \dot{x}^0$, Δn_x, Δm_y, these latter terms being multiplied by ω_y. For $\omega_y = 0$ expression (5.281) reduces to formula (5.275).

It is interesting to examine the difference between expression (5.281) and formula (5.275) for small values of ω_y. Let

$$\omega_y^2 \ll \omega_0^2. \tag{5.282}$$

Expanding expressions (5.106) in powers of ω_y/ω_0, we obtain the following approximate values:

$$\mu = \omega_0 \sqrt{2} \left(1 - \frac{5}{12} \frac{\omega_y^2}{\omega_0^2} \right), \quad \nu = \omega_0 \left(1 + \frac{1}{6} \frac{\omega_y^2}{\omega_0^2} \right). \tag{5.283}$$

Substituting these values into equality (5.281) and retaining only terms in the first power of ω_y/ω_0, we find:

$$\delta z_3 = \frac{\Delta n_z}{2\omega_0^2}\left(\cosh \omega_0 \sqrt{2}\, t - 1\right) + \delta z^0 \cosh \omega_0 \sqrt{2}\, t +$$
$$+ \frac{\delta \dot{z}^0}{\omega_0 \sqrt{2}} \sinh \omega_0 \sqrt{2}\, t + \frac{\omega_y}{\omega_0}\left[\frac{r\Delta m_y}{\omega_0}\left(\cosh \omega_0 \sqrt{2}\, t - 1\right) + \right.$$
$$+ \frac{2}{3}\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\left(\sin \omega_0 t - \frac{1}{\sqrt{2}} \sinh \omega_0 \sqrt{2}\, t\right) -$$
$$\left. - \frac{2}{3}\frac{\delta \dot{x}^0}{\omega_0}\left(\cos \omega_0 t - \cosh \omega_0 \sqrt{2}\, t\right)\right]. \tag{5.284}$$

The first three terms of this formula are identical to the right side of equality (5.275), while the remaining terms contain the factor ω_y/ω_0. Thus, for small values of ω_y formula (5.275) is a good approximation to formula (5.281).

If ω_y is close to ω_0, then, defining

$$\omega_y^2 = \omega_0^2 - \varepsilon^2, \tag{5.285}$$

where ε^2 is small in relation to ω_0^2, we obtain from relations (5.106):

$$\mu = \varepsilon\sqrt{3}, \quad \nu = \omega_0\left(1 + \frac{\varepsilon^2}{2\omega_0^2}\right) \cdots \tag{5.286}$$

Substituting these values into equality (5.281), we are able to find an approximate formula for δz_3, which for $\varepsilon = 0$ reduces to an exact formula for the case $\omega_y = \omega_0$.

For the latter case, from the third formulas (5.280) and (5.118) we have:

$$\delta z_3 = \frac{2(\delta \dot{x}^0 + r\Delta m_y)}{\omega_0}(1 - \cos \omega_0 t) +$$
$$+ \delta z^0 (4 - 3\cos \omega_0 t) + \frac{\delta \dot{z}^0}{\omega_0} \sin \omega_0 t +$$
$$+ \frac{2\Delta n_x}{\omega_0^2}(\omega_0 t - \sin \omega_0 t) + \frac{\Delta n_z}{\omega_0^2}(1 - \cos \omega_0 t). \tag{5.287}$$

Expression (5.287), unlike expression (5.284), is quite different from formula (5.275), which specifies δz_3 for the case of a stationary object. If in formula (5.275) the error δz_3 increases expedientially, then in expression (5.287) there is only one term which increases without limit with time, and its increase is only proportional to time. This term is $\frac{2\Delta n_x}{\omega_0} t$.

Let us now turn to the second formula (5.280), which specifies the error δy_3 in a plane normal to the plane of motion. We obtain from relations (5.101) and (5.60) for constant instrument errors:

$$\delta y_3 = \frac{\Delta n_y - \omega_y \Delta m_z r}{\omega_y^2} + \frac{r \Delta m_x}{\omega_y} + \left(\delta y^0 - \frac{\Delta n_y - \omega_y \Delta m_z r}{\omega_y^2}\right)\cos\omega_y t + \frac{\delta\dot{y}^0}{\omega_y}\sin\omega_y t - r\left(\theta^0_{1x} + \frac{\Delta m_x}{\omega_y}\right)\cos\omega_y t + r\left(-\frac{\Delta m_z}{\omega_y} + \theta^0_{1x}\right)\sin\omega_y t. \quad (5.288)$$

Let us compare equality (5.288) with the expression for δy_3 for the case of a stationary object, i.e., with the second formula (5.272). The first point to be noticed is that equality (5.288) differs from formula (5.272) in that in the numerator of equality (5.288) the first term contains the quantity $\Delta n_y - \omega_y \Delta m_z r$ instead of Δn_y, as in formula (5.272). More interesting, however, is the fact that formula (5.272) contains the term $-r\Delta m_x t$, which is proportional to time, while in equality (5.288) only harmonic oscillations of δy_3 at a frequency of ω_y correspond to the errors Δm_x and Δm_z. For small ω_y we obtain the approximate equality

$$\delta y_3 \approx \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t - r\theta^0_{1x} - r\Delta m_x t. \quad (5.289)$$

corresponding to formula (5.272).

For $\omega_y = \omega_0$, i.e., for the case of motion of a satellite in a circular orbit, we have:

$$\delta y_3 = \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2} - r\theta^0_{1x}\right)\cos\omega_0 t + \left(\frac{\delta\dot{y}^0}{\omega_0} + r\theta^0_{1x}\right)\sin\omega_0 t \quad (5.290)$$

The first formula (5.280) remains to be considered. From this formula, the first formula (5.112) and the second formula (5.60) we find for constant instrument errors:

$$\begin{aligned}\delta x_3 = {} & r\theta^0_{1y} + \frac{\Delta n_x}{\omega_0^2 - \omega_y^2} + \Delta m_y t + \\ & + \frac{1}{\mu^2 + \nu^2}\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2 - \omega_y^2}\right)\left[(\omega_0^2 - \omega_y^2 + \mu^2)\cos \nu t - \right. \\ & \left. - (\omega_0^2 - \omega_y^2 - \nu^2)\cosh \mu t\right] + \\ & + \frac{\delta \dot{x}^0}{(\omega_0^2 - \omega_y^2)(\mu^2 + \nu^2)}\left[\nu(\omega_0^2 - \omega_y^2 + \mu^2)\sin \nu t + \right. \\ & \left. + \mu(\omega_0^2 - \omega_y^2 - \nu^2)\sinh \mu t\right] + \\ & + \frac{2\omega_y \mu\nu}{(\omega_0^2 - \omega_y^2)(\mu^2 + \nu^2)}\left(\delta z^0 + \frac{\Delta n_z + 2r\omega_y \Delta m_y}{2\omega_0^2 + \omega_y^2}\right)(\mu \sin \nu t - \nu \sinh \mu t) + \\ & + \frac{2\omega_y \delta \dot{z}^0}{\mu^2 + \nu^2}(\cos \nu t - \cosh \mu t).\end{aligned} \tag{5.291}$$

For small values of ω_y we arrive at the approximate formula [taking into account equalities (5.283)]:

$$\begin{aligned}\delta x_3 = {} & \frac{\Delta n_x}{\omega_0^2} + r\theta^0_{1y} + r\Delta m_y t + \\ & + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos \omega_0 t + \frac{\delta \dot{x}^0}{\omega_0}\sin \omega_0 t + \\ & + \frac{2\omega_y \sqrt{2}}{3\omega_0}\left[\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)(\sqrt{2}\sin \omega_0 t - \sinh \omega_0 \sqrt{2}\, t) + \right. \\ & \left. + \frac{\delta \dot{z}^0}{\omega_0 \sqrt{2}}(\cos \omega_0 t - \cosh \omega_0 \sqrt{2}\, t)\right]\end{aligned} \tag{5.292}$$

The first five terms of this formula form expression (5.272) for the error δx_3 for the case of a stationary object. The latter terms, containing the factor ω_y, distinguish formula (5.292) from the first formula (5.272).

The difference between the expressions for δx_3 for the case under consideration and for the case of a stationary object is much more important than the difference between δy_3 and δz_3, since expression (5.292) contains hyperbolic functions which increase rapidly with time; for the case of a stationary object only the expression for δz_3 contains such functions. As a result of this, the error δn_z in the reading of the newtonometer directed along the z axis and the errors in the initial conditions δz^0 and $\delta \dot{z}^0$ begin to play a significant role in the formation of the error δx_3.

Since ω_y is small, we may let

$$\left.\begin{aligned}\frac{2\omega_y}{3\omega_0}\left(\delta z^0+\frac{\Delta n_z}{2\omega_0^2}\right)\ll\frac{\delta\dot{x}^0}{\omega_0},\\ \frac{2\omega_y\delta\dot{z}^0}{3\omega_0^2}\ll\left(\delta x^0-\frac{\Delta n_x}{\omega_0^2}\right).\end{aligned}\right| \tag{5.293}$$

Therefore, equality (5.292) may be simplified and written in the form

$$\begin{aligned}\delta x_3=\frac{\Delta n_x}{\omega_0^2}+r\theta_{1y}^0+r\Delta m_y t+\\ +\left(\delta x^0-\frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t+\frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t-\\ -\frac{2\omega_y\sqrt{2}}{3\omega_0}\left[\left(\delta z^0+\frac{\Delta n_z}{2\omega_0^2}\right)\sinh\omega_0\sqrt{2}t+\frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\cosh\omega_0\sqrt{2}t\right].\end{aligned} \tag{5.294}$$

For the quantitative evaluation of the influence of the last term on δx_3, it is useful to consider the following example. Let $\delta z^0 + \dfrac{\Delta n_z}{2\omega_0^2}$ and $\delta\dot{z}^0/\omega_0$ be of the same order of magnitude as $\delta x^0 - \dfrac{\Delta n_x}{\omega_0^2}$ and $\delta\dot{x}^0/\omega_0$. Under this condition the errors $\delta z^0 + \dfrac{\Delta n_z}{2\omega_0^2}$ and $\delta\dot{z}^0/\omega_0$ will begin, obviously, to exert a significant influence on δx_3 only when, in proportion to the increase of the function $\sinh \omega_0\sqrt{2}t$ and $\cosh \omega_0\sqrt{2}t$, the quantities $(2\omega_y\sqrt{2}/3\omega_0)\sinh \omega_0\sqrt{2}t$ and $(2\omega_y/3\omega_0)\cosh \omega_0\sqrt{2}t$ take on values close to unity. The time required for this to occur is found from equalities

$$\sinh\omega_0\sqrt{2}t_1=\frac{3\omega_0}{2\omega_y\sqrt{2}},\quad \cosh\omega_0\sqrt{2}t_2=\frac{3\omega_0}{2\omega_y}. \tag{5.295}$$

Since the argument $\omega_0\sqrt{2}t$ takes on a rather large value, we assume that

$$\cosh\omega_0\sqrt{2}t\approx\sinh\omega_0\sqrt{2}t\approx\frac{1}{2}e^{\omega_0\sqrt{2}t}. \tag{5.296}$$

Substituting these values into equalities (5.295), we obtain:

$$t_1 = \frac{1}{\omega_0 \sqrt{2}} \ln \frac{3\omega_0}{\omega_y \sqrt{2}}, \quad t_2 = \frac{1}{\omega_0 \sqrt{2}} \ln \frac{3\omega_0}{\omega_y}. \tag{5.297}$$

If we assume that ω_y is of the same order of magnitude as the earth rate u, then

$$\frac{\omega_0}{\omega_y} = \frac{\omega_0}{u} \approx 17 \text{ and } \omega_0 = 1.25 \cdot 10^{-3} \text{ 1/sec.}$$

For the first equality (5.297) we obtain in this case

$$t_1 \sim 35 \text{ min}$$

It is obvious that t_2 is somewhat larger than t_1.

If ω_y is of the order of magnitude of 3u, i.e., if the object moves relative to the surface of the earth with a velocity of the order of 1000 m/sec, then $\omega_0/\omega_y \approx 6$ and $t_1 \sim 25$ min, and at a velocity of the order 2000 m/sec, $\omega_0/\omega_y \approx 3.2$ and $t_1 \sim 20$ min. The validity of the approximate formula (5.294) for $\omega_y/\omega_0 \sim 1/3$ should not be in doubt. If the character of the variation of the roots μ and ν of the characteristic equation (5.103) as a function of ω_y (Figure 5.2) is taken into account, it is easily shown that the accuracy of formula (5.294) is satisfactory for $\omega_y/\omega_0 = 1/2$.

The resulting values of t_1 characterize, clearly, the time during which autonomous operation of the inertial system is possible under the condition that error δx_3 does not exceed the allowable limits. Of course, these evaluations characterize the time t_1 only for velocities corresponding to the condition $\omega_y{}^2 \ll \omega_0^2$.

For a satellite moving in a circular orbit, i.e., for the case in which $\omega_y = \omega_0$, we obtain, taking into account the first formula (5.118), the following expression in place of formula (5.291):

$$\begin{aligned}\delta x_3 = {} & r\theta_{1y}^0 + \delta x^0 + \frac{\delta\dot{x}^0}{\omega_0}(4\sin\omega_0 t - 3\omega_0 t) + \\ & + 6\,\delta z^0(\sin\omega_0 t - \omega_0 t) + \frac{2\delta\dot{z}_0}{\omega_0}(\cos\omega_0 t - 1) + \\ & + \frac{\Delta n_x}{\omega_0^2}\left[4(1-\cos\omega_0 t) - \frac{3\omega_0^2 t^2}{2}\right] + \\ & + \frac{r\Delta m_y}{\omega_0}(4\sin\omega_0 t - 3\omega_0 t) + \frac{2\Delta n_z}{\omega_0^2}(\sin\omega_0 t - \omega_0 t)\end{aligned} \qquad (5.298)$$

It follows from comparison of expressions (5.298), (5.290), and (5.287) with expressions (5.292), (5.289) and (5.284) that, if in the case of slow motion of the object, when $\omega_y^2 << \omega_0^2$, the major influence on the formation of the errors δx_3 and δz_3 for extended operation of the inertial system is Δn_z, δz^0, and $\delta\dot{z}^0$entering into the coefficients of the hyperbolic functions, then for a satellite moving in a circular orbit, i.e., for $\omega_y = \omega_0$, the greatest influence is exerted by the error Δn_x.

This concludes our discussion of the relation between errors in the determination of coordinates and instrument errors for the cases in question.

Let us now turn to errors in the orientation of the platform.

For the case in which the position of the platform (the trihedron xyz) is not a function of the coordinates, we have

$$\theta_x = -\theta_{1x}, \quad \theta_y = -\theta_{1y}, \quad \theta_z = -\theta_{1z}. \qquad (5.299)$$

If this trihedron is a free-azimuth moving trihedron, then

$$\theta_x = -\frac{\Delta y}{r}, \quad \theta_y = \frac{\Delta x}{r}, \quad \theta_z = -\theta_{1z}. \qquad (5.300)$$

The case described by formula (5.269) may be excluded from the discussion, since, for motion in the $\xi_*\eta_*$ plane and with the ζ_* axis as the polar axis, we see that in this formula $\varphi = 0$, which reduces it to the third equality (5.299).

From equalities (5.299) and (5.60) for constant instrument errors Δm_x, Δm_y, Δm_z we find,

$$
\left.
\begin{aligned}
\theta_x &= \frac{\Delta m_z}{\omega_y} - \left(\theta_{1x}^0 + \frac{\Delta m_x}{\omega_y}\right)\cos\omega_y t + \\
&\qquad + \left(\theta_{1z}^0 - \frac{\Delta m_z}{\omega_y}\right)\sin\omega_y t, \\
\theta_y &= -\Delta m_y t - \theta_{1y}^0, \\
\theta_z &= -\frac{\Delta m_x}{\omega_y} - \left(\theta_{1z}^0 - \frac{\Delta m_z}{\omega_y}\right)\cos\omega_y t - \\
&\qquad - \left(\theta_{1x}^0 + \frac{\Delta m_x}{\omega}\right)\sin\omega_y t.
\end{aligned}
\right\} \tag{5.301}
$$

As may be seen from the second formula (5.301), the expression for the orientation error θ_y in the plane of motion has the same form as in the case of a stationary basis. This error increase proportionally with time. The expressions for the orientation errors θ_x and θ_z in a plane normal to the plane of motion consist of constant components and harmonic oscillations at a frequency ω_y. The oscillations occur relative to the displaced equilibrium positions

$$\theta_x' = \frac{\Delta m_z}{\omega_y}, \quad \theta_z' = -\frac{\Delta m_x}{\omega_y} \tag{5.302}$$

and have identical amplitudes

$$\theta_x^* = \theta_z^* = \sqrt{\left(\theta_{1x}^0 + \frac{\Delta m_x}{\omega_y}\right)^2 + \left(\theta_{1z}^0 - \frac{\Delta m_z}{\omega_y}\right)^2}. \tag{5.303}$$

It follows from (5.301) and (5.302) that the total orientation errors about angles θ_x and θ_z for $\theta_{1x}^0 = \theta_{1z}^0 = 0$ do not exceed the magnitudes

$$\frac{1}{\omega_y}\left(|\Delta m_z| + \sqrt{\Delta m_x^2 + \Delta m_z^2}\right), \quad \frac{1}{\omega_y}\left(|\Delta m_x| + \sqrt{\Delta m_x^2 + \Delta m_z^2}\right) \tag{5.304}$$

correspondingly.

For $|\Delta m_x| = |\Delta m_z| = \Delta m$ the total errors are equal. If the requirement is imposed that the total orientation errors should not in this case exceed 1 angular min ($\approx 2.9 \cdot 10^{-4}$ rad), we obtain the inequality

$$\frac{\Delta m}{\omega_y}\left(1 + \sqrt{2}\right) < 2.9 \cdot 10^{-4}. \tag{5.305}$$

relating Δm and ω_y. If we now assume that ω_y is of the same order of magnitude as the earth rate
($u = 7.3 \cdot 10^{-4}$ rad/sec),

$$\Delta m < 0.002^{\circ}/\text{hr} \; . \tag{5.306}$$

For $\omega_y = 5u$ the magnitude of the allowable error Δm increases to approximately 0.01°/hour, and for $\omega_y = \omega_0$ up to 0.03°/hour.

If $\omega_y t$ is sufficiently small such that it may be assumed that $\cos \omega_y t = 1$ and $\sin \omega_y t = \omega_y t$, expressions (5.301) for θ_x and θ_z take the form:

$$\theta_x = -\theta_{1x}^0 - \Delta m_x t, \quad \theta_z = -\theta_{1z}^0 - \Delta m_z t, \tag{5.307}$$

which coincides with the corresponding formulas (5.276) for the case of a stationary object.

Let us now consider the case described by formulas (5.300).

The third formula (5.300) coincides with the third formula (5.299), the corresponding values of θ_z being those given by equalities (5.301) and (5.307).

In order to analyze the first two formulas (5.300) for the cases under consideration (for constant instrument errors), the values of δx and δy from expressions (5.101) and (5.112) for the case in which $\omega_y < \omega_0$ must be substituted into them; and for the case in which $\omega_y = \omega_0$, the values of δx and δy from expressions (5.118) should be used.

For small values of ω_y, such that $\omega_y^2 << \omega_0^2$, and simplifying the expressions for δy and δx in the same way as in the derivation of formulas (5.292) and (5.289) for δx_3 and δy_3, we obtain the equalities:

$$\left.\begin{aligned}
\theta_x &= -\frac{\Delta n_y}{r\omega_0^2} - \frac{1}{r}\left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t - \frac{\delta\dot{y}^0}{r\omega_0}\sin\omega_0 t, \\
\theta_y &= \frac{\Delta n_x}{r\omega_0^2} + \frac{1}{r}\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{r\omega_0}\sin\omega_0 t - \\
&\quad - \frac{2\omega_y}{3r\omega_0}\left[\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\frac{1}{\sqrt{2}}\sinh\omega_0\sqrt{2}t + \right. \\
&\qquad\qquad \left. + \frac{\delta\dot{z}^0}{\omega_0}\cosh\omega_0\sqrt{2}t\right], \\
\theta_z &= -\theta_{1z}^0 - \Delta m_z t.
\end{aligned}\right\} \qquad (5.308)$$

For the case in which $\omega_y = \omega_0$, we obtain:

$$\left.\begin{aligned}
\theta_x &= -\frac{\delta y^0}{r}\cos\omega_0 t - \frac{\delta\dot{y}^0}{r\omega_0}\sin\omega_0 t - \\
&\quad - \frac{1}{r\omega_0^2}(\Delta n_y - r\omega_0\Delta m_z)(1-\cos\omega_0 t).
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\theta_y &= \frac{1}{r}\Big\{\delta x^0 + \frac{\delta\dot{x}^0}{\omega_0}(4\sin\omega_0 t - 3\omega_0 t) + \\
&\quad + 6\,\delta z^0(\sin\omega_0 t - \omega_0 t) + \frac{2\delta\dot{z}^0}{\omega_0}(\cos\omega_0 t - 1) + \\
&\quad + \frac{\Delta n_z}{\omega_0^2}\left[4(1-\cos\omega_0 t) - \frac{3\omega_0^2 t^2}{2}\right] + \\
&\quad + \frac{4r\Delta m_y}{\omega_0}(\sin\omega_0 t - \omega_0 t) + \frac{2\Delta n_x}{\omega_0^2}(\sin\omega_0 t - \omega_0 t)\Big\}. \\
\theta_z &= -\theta_{1z}^0 - \Delta m_z t.
\end{aligned}\right\} \qquad (5.309)$$

The right sides of equalities (5.308) and (5.309) contain expressions δx and δy divided by r. These expressions enter into formulas (5.289), (5.292), (5.290) and (5.298). The analysis of these formulas performed above may therefore be extended to relations (5.308) and (5.309).

5.5.3. Motion along a parallel of latitude

For the case of motion at constant velocity along a parallel of latitude, the solution to the first group of error equations (5.121) is given by formulas (5.143). Trihedron xyz, in terms of projections on whose axes equations (5.121) and their solution (5.143) were found, are moving trihedrons oriented to the points of the compass. Therefore the total errors in the determination of the coordinates are also given by the final three equalities (5.5) together with formulas (5.262). From them we find:

$$\delta x_3 = \delta x + \theta_{1y} r, \quad \delta y_3 = \delta y - \theta_{1x} r, \quad \delta z_1 = \delta z. \tag{5.310}$$

The orientation errors θ_x and θ_y are found from the first two relations (5.267), and the error θ_z is found from expression (5.269). Once again, the formulas for these errors are:

$$\theta_x = -\frac{\delta y}{r}, \quad \theta_y = \frac{\delta x}{r}, \quad \theta_z = -\theta_{1z} + \frac{\delta x_3}{r} \operatorname{tg} \varphi. \tag{5.311}$$

In formulas (5.310) and (5.311) the quantities θ_{1x}, θ_{1y}, θ_{1z} are solutions to the second group of differential equations (5.3); for the case of motion along a parallel, these solutions will be given by formulas (5.65).

If we confine ourselves to the case in which

$$\omega_y^2 \ll \omega_0^2, \quad \omega_z^2 \ll \omega_0^2, \tag{5.312}$$

i.e., the case of relatively slow motion along a parallel not in the immediate vicinity of a pole, then, without loss of generality, we may consider only the case of an object which is stationary relative to the earth, such that

$$\omega_y = u \cos \varphi, \quad \omega_z = u \sin \varphi. \tag{5.313}$$

For the case of an object which is stationary relative to the earth, formulas (5.143) for the determination of δx, δy, δz may be replaced by the approximate equalities (5.159) and (5.164), in the right sides of which ω_y and ω_z should be replaced by their values from relations (5.313). Expressions (5.65) for θ_{1x}, θ_{1y}, θ_{1z} also simplify considerably for this case. Integrating the right sides of expressions (5.65) for constant instrument errors and noting that, for the case of an object stationary relative to the earth and in accordance with equality (5.63), ω becomes equal to the earth rate u, we obtain:

$$
\left.
\begin{aligned}
\theta_{1x} = {} & \frac{\Delta m_y \sin\varphi - \Delta m_z \cos\varphi}{u} + \\
& + \left(\theta^0_{1x} - \frac{\Delta m_y \sin\varphi - \Delta m_z \cos\varphi}{u}\right)\cos ut + \\
& + \left(\theta^0_{1y} \sin\varphi - \theta^0_{1z} \cos\varphi + \frac{\Delta m_x}{u}\right)\sin ut, \\
\theta_{1y} = {} & (\theta^0_{1y} \cos\varphi + \theta^0_{1z} \sin\varphi)\cos\varphi - \frac{\Delta m_x}{u}\sin\varphi + \\
& + \cos\varphi(\Delta m_y \cos\varphi + \Delta m_z \sin\varphi)t + \\
& + (\theta^0_{1y} \sin\varphi - \theta^0_{1z} \cos\varphi)\sin\varphi \cos ut + \\
& + \left(\frac{\Delta m_y}{u}\sin\varphi - \frac{\Delta m_z}{u}\cos\varphi - \theta^0_{1x}\right)\sin\varphi \sin ut, \\
\theta_{1z} = {} & \sin\varphi(\theta^0_{1y} \cos\varphi + \theta^0_{1z} \sin\varphi) + \frac{\Delta m_x}{u}\cos\varphi + \\
& + \sin\varphi(\Delta m_y \cos\varphi + \Delta m_z \sin\varphi)t - \\
& - \left(\theta^0_{1y} \sin\varphi - \theta^0_{1z} \cos\varphi + \frac{\Delta m_x}{u}\right)\cos\varphi \cos ut + \\
& + \left(\theta^0_{1x} - \frac{\Delta m_y}{u}\sin\varphi + \frac{\Delta m_z}{u}\cos\varphi\right)\sin ut.
\end{aligned}
\right\}
\tag{5.314}
$$

For $\varphi = 0$, these formulas reduce ($\omega_y = u$) to formulas (5.301). If ut is small it may be assumed that $\cos ut = 1$ and $\sin ut = ut$, formulas (5.314) simplify and take the form:

$$
\left.
\begin{aligned}
\theta_{1x} = \theta^0_{1x} + \Delta m_x t, \quad \theta_{1y} = \theta^0_{1y} + \Delta m_y t, \\
\theta_{1z} = \theta^0_{1z} + \Delta m_z t.
\end{aligned}
\right\}
\tag{5.315}
$$

It is easy to see that expressions (5.315) coincide with formulas (5.276), obtained for the case of a stationary object, since $\theta_{1x} = -\theta_x$, $\theta_{1y} = -\theta_y$ and $\theta_{1z} = -\theta_z$.

Let us now substitute for δx, δy and δz in equalities (5.310) the expressions which derive from relations (5.159) and (5.164), if ω_y and ω_z in the latter expressions are replaced by their values (5.313), and θ_{1x} and θ_{1y} are replaced by their expressions (5.314). We then obtain:

$$
\left.
\begin{aligned}
\delta x_3 ={} & [\delta x^0 \cos(u\sin\varphi)t + \delta y^0 \sin(u\sin\varphi)t]\cos\omega_0 t + \\
& + \frac{1}{\omega_0}[\delta\dot{x}^0 \cos(u\sin\varphi)t + \delta\dot{y}^0 \sin(u\sin\varphi)t]\sin\omega_0 t - \\
& - \frac{2\sqrt{2}\,u\cos\varphi}{3\omega_0}\left(\delta z^0 \sinh\omega_0\sqrt{2}\,t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\cosh\omega_0\sqrt{2}\,t\right) + \\
& + \frac{\Delta n_x - r\Delta m_x u\sin\varphi}{\omega_0^2}[1 - \cos\omega_0 t\cos(u\sin\varphi)t] - \\
& - \frac{\Delta n_y - r\Delta m_z u\cos\varphi - r\Delta m_y u\sin\varphi}{\omega_0^2}\cos\omega_0 t\sin(u\sin\varphi)t - \\
& - \frac{\sqrt{2}\,u\cos\varphi(\Delta n_z + 2r\Delta m_y u\cos\varphi)}{3\omega_0^3}\sinh\omega_0\sqrt{2}\,t + \\
& + r\left[(\theta_{1y}^0\cos\varphi + \theta_{1z}^0\sin\varphi)\cos\varphi - \frac{\Delta m_x}{u}\sin\varphi + \right. \\
& + \cos\varphi(\Delta m_y\cos\varphi + \Delta m_z\sin\varphi)t + \\
& + \left(\theta_{1y}^0\sin\varphi - \theta_{1z}^0\cos\varphi + \frac{\Delta m_x}{u}\right)\sin\varphi\cos ut + \\
& \left. + \left(\frac{\Delta m_x}{u}\sin\varphi - \frac{\Delta m_z}{u}\cos\varphi - \theta_{1x}^0\right)\sin\varphi\sin ut\right].
\end{aligned}
\right|
$$

$$
\left.
\begin{aligned}
\delta y_3 ={} & [\delta y^0 \cos(u\sin\varphi)t - \delta x^0 \sin(u\sin\varphi)t]\cos\omega_0 t + \\
& + \frac{1}{\omega_0}[\delta\dot{y}^0 \cos(u\sin\varphi)t - \delta\dot{x}^0 \sin(u\sin\varphi)t]\sin\omega_0 t + \\
& + \frac{5u^2\sin\varphi\cos\varphi}{9\omega_0^2}\left(\delta z^0\cosh\omega_0\sqrt{2}\,t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\sinh\omega_0\sqrt{2}\,t\right) + \\
& + \frac{\Delta n_x - r\Delta m_x u\sin\varphi}{\omega_0^2}\cos\omega_0 t\sin(u\sin\varphi)t + \\
& + \frac{\Delta n_y - r\Delta m_z u\cos\varphi - r\Delta m_y u\sin\varphi}{\omega_0^2}[1 - \cos\omega_0 t\cos(u\sin\varphi)t] + \\
& + \frac{5u^2\sin\varphi\cos\varphi(\Delta n_z + 2r\Delta m_y u\cos\varphi)}{18\omega_0^4}\cosh\omega_0\sqrt{2}\,t - \\
& - r\left[\frac{\Delta m_y\sin\varphi - \Delta m_z\cos\varphi}{u} + \left(\theta_{1x}^0 - \frac{\Delta m_y\sin\varphi - \Delta m_z\cos\varphi}{u}\right)\cos ut + \right. \\
& \left. + \left(\theta_{1y}^0\sin\varphi - \theta_{1z}^0\cos\varphi + \frac{\Delta m_x}{u}\right)\sin ut\right]. \\
\delta z_3 ={} & \delta z^0\cosh\omega_0\sqrt{2}\,t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\sinh\omega_0\sqrt{2}\,t + \\
& + \frac{\Delta n_z + 2r\Delta m_y u\cos\varphi}{2\omega_0^2}(\cosh\omega_0\sqrt{2}\,t - 1).
\end{aligned}
\right|
$$

(5.316)

Formulas (5.316) give the relation between δx_3, δy_3, δz_3 of the total errors in the determination by the inertial system of the coordinates and the instrument errors and the errors in the initial conditions for motion along a parallel.

For $\varphi = 0$, motion proceeds in the plane of the equator, i.e., in a plane containing the center of the earth. It is therefore natural that, for $\varphi = 0$, formulas (5.316) reduce to formulas (5.275), (5.288) and (5.294).

For $\varphi = \frac{\pi}{2}$, i.e., at a pole, the object, being stationary relative to the earth, is at the same time stationary in the $O_1\xi_*\eta_*\zeta_*$ coordinate system. Formulas (5.316) should then be derived from formulas (5.263) for the case of a stationary object. It is easy to show that this can indeed be done. In fact, for $\varphi = \frac{\pi}{2}$, the first two equalities (5.316) take the form:

$$
\left.
\begin{aligned}
\delta x_3 &= (\delta x^0 \cos ut + \delta y^0 \sin ut) \cos \omega_0 t + \\
&+ \frac{1}{\omega_0} (\delta \dot{x}^0 \cos ut + \delta \dot{y}^0 \sin ut) \sin \omega_0 t + \\
&+ \frac{\Delta n_x - r \Delta m_x u}{\omega_0^2} (1 - \cos \omega_0 t \cos ut) + \\
&+ r \left[-\frac{\Delta m_x}{u} + \left(\theta_{1y}^0 + \frac{\Delta m_x}{u} \right) \cos ut + \right. \\
&\left. + \left(\frac{\Delta m_y}{u} - \theta_{1x}^0 \right) \sin ut \right] - \frac{\Delta n_y - r \Delta m_y u}{\omega_0^2} \cos \omega_0 t \cos ut, \\
\delta y_3 &= (\delta y^0 \cos ut - \delta x^0 \sin ut) \cos \omega_0 t + \\
&+ \frac{1}{\omega_0} (\delta \dot{y}^0 \cos ut - \delta \dot{x}^0 \sin ut) \sin \omega_0 t + \\
&+ \frac{\Delta n_x - r \Delta m_x u}{\omega_0^2} \cos \omega_0 t \sin ut + \\
&+ \frac{\Delta n_y - r \Delta m_y u}{\omega_0^2} (1 - \cos \omega_0 t \cos ut) - \\
&- r \left[\frac{\Delta m_y}{u} + \left(\theta_{1x}^0 - \frac{\Delta m_y}{u} \right) \cos ut + \left(\theta_{1y}^0 + \frac{\Delta m_x}{u} \right) \sin ut \right].
\end{aligned}
\right\}
\tag{5.317}
$$

The third equality (5.316) does not change.

The expressions (5.317) for δx_3 and δy_3 are projections of the total error vector $\delta \vec{r}_3$ on the x and y axes of the xyz trihedron rotating with the earth (the z axis coinciding with the earth's axis of rotation). Expressions (5.263), on the other hand, are projections of the vector $\delta \vec{r}_3$ on the axes of a fixed trihedron. In order to distinguish these trihedra, we denote the latter by x'y'z'. The relative position of the x, y, z and x', y', z' axes will then be determined by the following direction cosines:

$$
\begin{array}{cccc}
 & x' & y' & z' \\
x & \cos ut & \sin ut & 0 \\
y & -\sin ut & \cos ut & 0 \\
z & 0 & 0 & 1.
\end{array}
\tag{5.318}
$$

In formulas (5.37) Δm_x, Δm_y, Δn_x, Δn_y are constant. If in the first two equalities (5.263) we substitute in place of Δm_x, Δm_y, Δn_x, Δn_y the following quantities

$$\begin{aligned}\Delta m'_x &= \Delta m_x \cos ut - \Delta m_y \sin ut,\\ \Delta m'_y &= \Delta m_x \sin ut + \Delta m_y \cos ut,\\ \Delta n'_x &= \Delta n_x \cos ut - \Delta n_y \sin ut,\\ \Delta n'_y &= \Delta n_x \sin ut + \Delta n_y \cos ut\end{aligned}$$

and integrate, we find $\delta x'_3$ and $\delta y'_3$. If we now convert from $\delta x'_3$ and $\delta y'_3$ to δx_3 and δy_3 in accordance with the formulas deriving from table (5.318)

$$\delta x_3 = \delta x'_3 \cos ut + \delta y'_3 \sin ut, \quad \delta y_3 = -\delta x'_3 \sin ut + \delta y'_3 \cos ut$$

and take into account the fact that u is small, we arrive at equalities (5.317).

Let us turn to formulas (5.294), (5.288) and (5.275) defining the errors δx_3, δy_3, δz_3 in the case of motion at constant velocity along an arc of a great circle, and compare them with formulas (5.316), giving the same errors for the case of motion along an arbitrary parallel (along an arc of a small circle).

The third formula (5.316) differs from (5.275) only in that it has $\Delta n_z + 2r\Delta m_y u \cos\varphi$ in place of Δn_z. The term $2r\Delta m_y u \cos\varphi$ may be ignored here, and so the third formula (5.316) coincides with formula (5.375).

According to the first two formulas (5.316), the errors δx_3 and δy_3 consist of: constant components, oscillations at two close frequencies $\omega_0 + u \sin\varphi$ and $\omega_0 - u \sin\varphi$, resulting in pulsations at a frequency $u \sin \varphi$, oscillations at a frequency u and components which increase exponentially. Moreover, the expression for δx_3 contains the component $r\cos\varphi(\Delta m_y \cos\varphi + \Delta m_z \sin\varphi)t$, which increases linearly with time. Formulas (5.294) and (5.288) differ from expression (5.316) for δx_3 and δy_3 not only in the coefficients, but also by the presence of pulsations at the two close frequencies

$\omega_0 + u \sin\varphi$ and $\omega_0 - u \sin\varphi$, and also by the fact that the exponential terms in the case of motion along a parallel enter into both δx_3 and δy_3, while in the case of motion along a great circle they enter into expression (5.294) for δx_3 only, and are absent from formula (5.288) for δy_3.

The latter difference is the more significant, since for extended operation of the inertial system it is the exponentially increasing terms which give rise to the largest error. The numerical calculation carried out in §5.5.2 for the preceding case, i.e., motion along a great circle of a fixed sphere surrounding the earth, showed[15] that these terms begin to decisively influence δx_3 in only 30-35 min from the moment at which the system begins to function (for an object which is stationary relative to the earth at the equator). The same time period, obviously, applies to motion along a parallel, since the angle ut remains small ($<7.3 \cdot 10^{-5} \times 35 \cdot 60 = 0.15$ rad), such that we may consider cos ut = 1, and sin ut = ut. Taking this into account, and also that, as a rule,

$$\left.\begin{array}{c} |\Delta n_x| \gg ru|\Delta m_x|, \; |\Delta n_y| \gg ru|\Delta m_z|, \; |\Delta n_y| \gg ru|\Delta m_y|, \\ |\Delta n_z| \gg ru|\Delta m_y|, \end{array}\right\} \tag{5.319}$$

formulas (5.316) may be simplified to the form

$$\left.\begin{aligned} \delta x_3 = {} & \frac{\Delta n_x}{\omega_0^2} + r\theta_{1y}^0 + r\Delta m_y t + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \\ & + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t - \frac{2\sqrt{2}u\cos\varphi}{3\omega_0}\left[\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\sinh\omega_0\sqrt{2}\,t + \right. \\ & \left. + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\cosh\omega_0\sqrt{2}\,t\right], \\ \delta y_3 = {} & \frac{\Delta n_y}{\omega_0^2} - r\theta_{1x}^0 - r\Delta m_x t + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \\ & + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t + \frac{5u^2\sin\varphi\cos\varphi}{9\omega_0^2}\left[\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\cosh\omega_0\sqrt{2}\,t + \right. \\ & \left. + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\sinh\omega_0\sqrt{2}\,t\right]. \end{aligned}\right\} \tag{5.320}$$

The first formula (5.320) coincides with formula (5.294), if in the latter we set $\omega_y = u\cos\varphi$; the second formula (5.320) differs from equality (5.289) or from the second equality (5.274) in its final term, the coefficient of which contains the square of the rate of rotation of the earth.

It is evident from relations (5.320) that the exponential terms influence δy_3 to a lesser extent than δx_3, and so the operational time of the system will be limited during motion along a parallel by the allowable magnitude of δx_3.

Approximate formulas for the orientation errors may be obtained from expressions (5.311), if values for δx, δy, δx_3 and θ_{1z} are substituted into them in accordance with equalities (5.164), (5.320) and (5.315). Then

$$
\left.
\begin{aligned}
\theta_x &= -\frac{1}{r}\left\{\frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \right. \\
&\quad + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t + \frac{5u^2\sin\varphi\cos\varphi}{9\omega_0^2}\times \\
&\quad \left.\times\left[\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\cosh\omega_0\sqrt{2}t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\sinh\omega_0\sqrt{2}t\right]\right\}. \\
\theta_y &= \frac{1}{r}\left\{\frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \right. \\
&\quad + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t - \frac{2\sqrt{2}u\cos\varphi}{3\omega_0}\times \\
&\quad \left.\times\left[\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\sinh\omega_0\sqrt{2}t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\cosh\omega_0\sqrt{2}t\right]\right\}. \\
\theta_z &= -\theta_{1z}^n - \Delta m_z t + \operatorname{tg}\varphi\left\{\theta_{1y}^n + \Delta m_y t + \frac{\Delta n_x}{r\omega_0^2} + \right. \\
&\quad + \frac{1}{r}\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{r\omega_0}\sin\omega_0 t - \\
&\quad - \frac{2\sqrt{2}u\cos\varphi}{3r\omega_0}\left[\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\sinh\omega_0\sqrt{2}t + \right. \\
&\quad \left.\left. + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\cosh\omega_0\sqrt{2}t\right]\right\}.
\end{aligned}
\right|
\tag{5.321}
$$

As is evident from these expressions for the projections θ_x, θ_y and θ_z, they all contain terms which increase exponentially with time. Quantitative analysis of formulas (5.321) may be carried out in a manner analogous to the analysis of formulas (5.316) and (5.320) for δx_3 and δy_3.

5.5.4. <u>Keplerian motion of an object. The case of an elliptic orbit with a small eccentricity</u>. Let us consider the relation between errors in the determination of coordinates and orientation errors, on the one hand, and instrument errors and errors in initial conditions, on the other, for Keplerian motion of an object. We will confine ourselves here to the case of elliptical orbits, for which a solution to the first group of the error equations was obtained in §5.5.

We note first of all that for the special case of a circular orbit, this question has already been considered in the analysis of errors for the case of motion at constant velocity along an arc of a great circle on a fixed sphere surrounding the earth. The projections δx_3, δy_3, δz_3 of the error vector $\delta \vec{r}_3$ on the axes of an orbital trihedron are expressed for the case of motion in a circular orbit by the formulas

$$\left.\begin{aligned} \delta x_3 = \delta x + r\theta_{1y}, \quad \delta y_3 = \delta y - r\theta_{1x}, \\ \delta z_3 = \delta z, \end{aligned}\right\} \tag{5.322}$$

where δx, δy, δz are given by equalities (5.177), and θ_{1x} and θ_{1y} by equalities (5.60) with ω_y replaced by ω_0. Expressions for the errors θ_x, θ_y, θ_z in the determination of the orientation parameters are given by formulas (5.300):

$$\theta_x = -\frac{\delta y}{r}, \quad \theta_y = \frac{\delta x}{r}, \quad \theta_z = -\theta_{1z}. \tag{5.323}$$

For constant instrument errors, the formulas for the determination of δx_3, δy_3 and δz_3 reduce to formulas (5.298), (5.290) and (5.287), and the formulas for the determination of θ_x, θ_y, θ_z reduce to formulas (5.309). Expressions (5.298), (5.290), (5.287) and (5.309) are characterized by the fact that they do not contain exponentially increasing terms.

The error δy_3 in the determination of the location of the object in a plane normal to the orbital plane, is, according to equality

(5.290), a harmonic oscillation at a frequency ω_0 about some displaced equilibrium position. The error δz_3 in the determination of the distance to the center of the earth, as follows from relation (5.287), includes, in addition to harmonic oscillations at a frequency ω_0, a component which increases linearly with time: $2\Delta n_x t/\omega_0$. The error δx_3, given by formula (5.298), contains the linearly increasing component $(-3\delta\dot{x}^0 - 6\delta z^0\omega_0 - 3r\Delta m_y - 2\Delta n_z/\omega_0)t$, as well as the component $-3\Delta n_x t^2/2$, proportional to the square of time.

The error θ_x in the orientation of the object in the plane normal to the plane of motion, according to the first formula (5.309) is, like δy_3, a harmonic oscillation. The error θ_y is determined by the second formula (5.309). It contains linear and quadratic functions of time. The error θ_z coincides with the corresponding error for the case of a stationary object.

For an elliptic object with arbitrary eccentricity, the general formulas (5.322) and (5.323), of course, remain valid, except that δx, δy, δz must be replaced by their expressions (5.258), in which the matrix elements A_{ij}, B_{ij}, D_{ij} and E_{ij} are determined by equalities (5.239), (5.240), (5.251) and (5.252). Initial values x_j^0 are related to δx^0, δy^0, δz^0, $\delta\dot{x}^0$, $\delta\dot{y}^0$, $\delta\dot{z}^0$ by equalities (5.255) and (5.256), and r and v are specified as functions of time by formulas (5.216) and (5.221).

Formulas (5.322) and (5.323) also contain the projections θ_{1x}, θ_{1y}, θ_{1z}. An expression for θ_{1y} is given by the second formula (5.71). As regards θ_{1x} and θ_{1z}, they are obtained from formulas (5.77), if in them ω_y is replaced by $\dot{v}$, in accordance with the second formula (5.227).

From the formulas for the matrix elements A_{ij}, B_{ij}, D_{ij}, and E_{ij} it follows that δy_3 and θ_x will be periodic functions of time, δz_3 and θ_z will contain linearly increasing terms, and δx_3 and θ_y will contain linear and quadratic functions of time.

The quantitative analysis of formulas (5.322) and (5.323) for the case of motion in an orbit with arbitrary eccentricity gives rise, in general, to considerable difficulties, since the integrals in the right sides of formulas (5.258) and (5.77) even for constant instrument errors may be taken only in series.

Let us consider the case in which the eccentricity of the elliptical orbit is small. We then find from equality (5.221), retaining only those terms on its right side containing e to the first degree:

$$\sin E = \sin M(1 + e\cos M). \tag{5.324}$$

Setting $E(t_0) = v(t_0) = 0$, we obtain $M_0' = 0$, and from the first relations (5.216) and equality (5.324) it follows that

$$\sin E = \sin \nu(t - t_0)[1 + e\cos \nu(t - t_0)]. \tag{5.325}$$

For the sake of simplicity we will henceforth consider that $t_0 = 0$, i.e., that at the initial moment the object is located at the perigee of its orbit (this, clearly, does not limit the generality of the analysis). Instead of (5.325) we will have:

$$\sin E = \sin \nu t(1 + e\cos \nu t). \tag{5.326}$$

from which

$$\cos E = \cos \nu t - e\sin^2 \nu t. \tag{5.327}$$

It now follows from the fourth formula (5.216) that

$$r = a(1 - e\cos \nu t). \tag{5.328}$$

and from the second equality (5.227) we obtain

$$\omega_y = \dot{\theta} = \dot{v} = \nu(1 + 2e\cos \nu t). \tag{5.329}$$

Integrating the latter equality and noting that $v(0) = 0$, we find:

$$v = \nu t + 2e\sin \nu t. \tag{5.330}$$

Formulas (5.239), (5.240), (5.251) and (5.252) for the matrix elements A_{ij}, B_{ij}, D_{ij}, and E_{ij}, contain the functions sin v and cos v. On the basis of formula (5.330) we obtain the following expressions for them:

$$\sin v = \sin \nu t + e \sin 2\nu t, \quad \cos v = \cos \nu t - 2e \sin^2 \nu t. \tag{5.331}$$

These expressions may also be obtained from relations (5.326), (5.237) and the next to the last equality (5.216).

We now substitute r, sin v, cos v and $t_0 = 0$ into expressions (5.239). Then, retaining only terms containing e to the first degree, we arrive at the following values of A_{ij}:

$$\left.\begin{aligned}
&A_{11} = -\frac{3\nu t(1 + e\cos \nu t)}{2}, \quad A_{12} = 2\sin \nu t + \frac{3}{2} e \sin 2\nu t,\\
&A_{13} = 2\cos \nu t - e(1 + 3\sin^2 \nu t), \quad A_{14} = 1 - e\cos \nu t;\\
&A_{21} = 1 - e\left(\cos \nu t + \frac{3\nu t}{2}\sin \nu t\right), \quad A_{22} = -\cos \nu t + 2e\sin^2 \nu t,\\
&A_{23} = \sin \nu t + e\sin 2\nu t, \quad A_{24} = 0;\\
&A_{31} = -\frac{\nu}{2}(1 + e\cos \nu t), \quad A_{32} = \nu(\cos \nu t + e\cos 2\nu t),\\
&A_{33} = -\nu(\sin \nu t + e\sin 2\nu t), \quad A_{34} = \nu e \sin \nu t;\\
&A_{41} = \nu\left[\frac{3\nu t}{2} + e\left(3\nu t\cos \nu t - \frac{\sin \nu t}{2}\right)\right],\\
&A_{42} = -\nu\left(\sin \nu t + \frac{3e}{2}\sin 2\nu t\right),\\
&A_{43} = -\nu[\cos \nu t + e(\cos^2 \nu t - 2\sin^2 \nu t)],\\
&A_{44} = -\nu(1 + e\cos \nu t).
\end{aligned}\right\} \tag{5.332}$$

Analogously, from expressions (5.240), (5.328) and (5.331) we find:

$$\left.\begin{aligned}
&B_{11} = \cos \nu t - e(1 + \sin^2 \nu t), \quad B_{12} = \sin \nu t + \frac{e}{2}\sin 2\nu t;\\
&B_{21} = -\nu(\sin \nu t + e\sin 2\nu t), \quad B_{22} = \nu(\cos \nu t + e\cos 2\nu t)
\end{aligned}\right\} \tag{5.333}$$

Further, from equalities (5.251), (5.328) and (5.331) we obtain:

$$
\left.\begin{aligned}
&D_{11} = 0, \quad D_{12} = 2(1 + 2e\cos\nu t),\\
&D_{13} = \frac{2}{\nu}(1 + e\cos\nu t), \quad D_{14} = \frac{2e}{\nu}\sin\nu t;\\
&D_{21} = \sin\nu t + e\sin 2\nu t, \quad D_{22} = \cos\nu t + e(\cos 2\nu t + \cos^2\nu t),\\
&D_{23} = \frac{1}{\nu}(2\cos\nu t - 3e\sin^2\nu t), \quad D_{24} = \frac{1}{\nu}(\sin\nu t + e\sin 2\nu t);\\
&D_{31} = \cos\nu t - 2e\sin^2\nu t, \quad D_{32} = -\sin\nu t + e\left(3\nu t - \frac{3}{2}\sin 2\nu t\right),\\
&D_{33} = \frac{1}{\nu}\left[-2\sin\nu t + e\left(3\nu t - \frac{3}{2}\sin 2\nu t\right)\right],\\
&D_{34} = \frac{1}{\nu}[\cos\nu t + e(\cos\nu t - 3 - \sin^2\nu t)];
\end{aligned}\right|
$$

$$
\left.\begin{aligned}
&D_{41} = -1, \quad D_{42} = 3\nu t + 2e(3\nu t\cos\nu t - \sin\nu t)],\\
&D_{43} = \frac{1}{\nu}[3\nu t + e(3\nu t\cos\nu t - 2\sin\nu t)],\\
&D_{44} = \frac{1}{\nu}[-2 + 3e(\nu t\sin\nu t + \cos\nu t)].
\end{aligned}\right\}
$$

(5.334)

Finally, from formulas (5.252), (5.328) and (5.331) we find the following approximate expressions for E_{ij}:

$$
\left.\begin{aligned}
&E_{11} = \cos\nu t + e(1 - 2\sin^2\nu t),\\
&E_{12} = -\frac{1}{\nu}\left[\sin\nu t + \frac{e}{2}\sin 2\nu t\right];\\
&E_{21} = \sin\nu t + e\sin 2\nu t,\\
&E_{22} = \frac{1}{\nu}[\cos\nu t - e(1 + \sin^2\nu t)].
\end{aligned}\right\}
$$

(5.335)

In order to obtain expressions for δx, δy, δz, it is necessary to substitute the values (5.332) -- (5.335) of the elements A_{ij}, B_{ij}, D_{ij}, E_{ij} of the matrices A, B, D, E into the right sides of equalities (5.258) and integrate them (as previously, we will confine ourselves to the case of constant instrument errors).

The integrands of formulas (5.258) contain, in addition to the elements of matrices D and E, the projections Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z of the instrument error vectors $\Delta\vec{n}$ and $\Delta\vec{m}$ on the axes of the orbital trihedron xyz, the magnitude r of the radius vector $\vec{r}$, its time-derivative $\dot{r}$, and the angular velocity ω_y of the orbital trihedron. Formulas (5.258) also contain the quantities x_j^0, which are expressed in terms of the initial values δx^0, δy^0, δz^0, $\delta\dot{x}^0$, $\delta\dot{y}^0$, $\delta\dot{z}^0$ by equalities (5.255) and (5.256).

The value of r is determined by expression (5.328), from which it follows that

$$\dot{r} = -ae\nu \sin \nu t. \tag{5.336}$$

Performing the indicated substitutions and the integration on the right sides of formulas (5.258), we arrive, after the appropriate transformations, at the following relations:

$$\left.\begin{aligned}
\delta x &= \delta x^0 + \frac{\delta \dot{x}^0}{\nu}(4\sin\nu t - 3\nu t) + 6\,\delta z^0(\sin\nu t - \nu t) + \\
&+ \frac{2\delta\dot{z}^0}{\nu}(\cos\nu t - 1) + \frac{\Delta n_z}{\nu^3}\left(-\frac{3(\nu t)^2}{2} + 4(1-\cos\nu t)\right) + \\
&+ \frac{4a\,\Delta m_y}{\nu}(\sin\nu t - \nu t) + \frac{2\Delta n_z}{\nu^3}(\sin\nu t - \nu t) + \\
&+ e\Big[\Delta n_x\Big(-\frac{3t^2}{2}\cos\nu t - \frac{5t}{\nu}\sin\nu t + \frac{1}{\nu^2}\cos\nu t - \\
&- \frac{1}{\nu^2} + \frac{6}{\nu^2}\sin^2\nu t\Big) + \Delta n_z\Big(-\frac{3t}{\nu} - \frac{5t}{\nu}\cos\nu t + \\
&+ \frac{7}{2\nu^2}\sin\nu t + \frac{5}{2\nu^2}\sin 2\nu t - \frac{1}{2\nu^2}\sin\nu t\cos 2\nu t\Big) + \\
&+ 2a\,\Delta m_y\Big(-6t - 6t\cos\nu t + \frac{15}{2\nu}\sin\nu t + \\
&+ \frac{5}{2\nu}\sin 2\nu t - \frac{1}{2\nu}\sin\nu t\cos 2\nu t\Big) + \delta x^0(1-\cos\nu t) - \\
&- \frac{3\delta\dot{x}^0}{\nu}(\nu t + \nu t\cos\nu t - \sin 2\nu t) - \\
&- 3\,\delta z^0\Big(5\nu t + 2\nu t\cos\nu t - \frac{3}{2}\sin 2\nu t - 4\sin\nu t\Big) + \\
&+ \frac{\delta\dot{z}^0}{\nu}(1 + \cos^2\nu t - 2\cos\nu t)\Big], \\
\delta y &= \frac{\Delta n_y - a\nu\,\Delta m_z}{\nu^2}(1 - \cos\nu t) + \delta y^0\cos\nu t + \\
&+ \frac{\delta\dot{y}^0}{\nu}\sin\nu t + e\Big[\frac{\Delta n_y}{\nu^2}\Big(1 - \cos\nu t + \sin^2\nu t - \\
&- \frac{3}{2}\nu t\sin\nu t\Big) + \frac{a\,\Delta m_z}{\nu}(\cos\nu t - 1 - \sin^2\nu t + \nu t\sin\nu t) + \\
&+ \frac{a\,\Delta m_z}{\nu}(\nu t\cos\nu t - \sin\nu t) + \delta y^0(\cos\nu t - 1 - \sin^2\nu t) + \\
&+ \frac{\delta\dot{y}^0}{\nu}(-\sin\nu t + \sin\nu t\cos\nu t)\Big].
\end{aligned}\right| \qquad (5.337)$$

$$
\begin{aligned}
\delta z = {} & \frac{2(\delta\dot{x}^0 + a\Delta m_y)}{\nu}(1-\cos\nu t) + \delta z^0(4-3\cos\nu t) + \\
& + \frac{\delta\dot{z}^0}{\nu}\sin\nu t + \frac{2\Delta n_x}{\nu^2}(\nu t-\sin\nu t) + \frac{\Delta n_z}{\nu^2}(1-\cos\nu t) + \\
& + e\Big[\Delta n_x\Big(-\frac{3t^2}{2}\sin\nu t - \frac{t\cos\nu t}{2\nu} + \frac{9}{2\nu^2}\sin\nu t - \\
& - \frac{2}{\nu^2}\sin 2\nu t\Big) + \Delta n_z\Big(-\frac{7t}{2\nu}\sin\nu t + \frac{3}{\nu^2}\sin^2\nu t - \\
& - \frac{2}{\nu^2} - \frac{2}{\nu^2}\cos\nu t - \frac{1}{2\nu^2}\sin^2\nu t\cos\nu t\Big) + \\
& + 2a\Delta m_y\Big(-4t\sin\nu t + \frac{4}{\nu} - \frac{21}{4\nu}\cos\nu t + \\
& + \frac{3}{\nu}\sin^2\nu t + \frac{5}{4\nu}\cos\nu t\cos 2\nu t\Big) + \frac{2\delta\dot{x}^0}{\nu}\Big(1-\cos\nu t - \\
& - \frac{3\nu t}{2}\sin\nu t\Big) + \delta z^0(-6\nu t\sin\nu t + 10 - 10\cos\nu t + \\
& + 6\sin^2\nu t) + \frac{\delta\dot{z}^0}{\nu}(\sin 2\nu t - 2\sin\nu t)\Big].
\end{aligned}
\qquad (5.337)
$$

The terms in formulas (5.337) which do not contain the orbital eccentricity e as a factor, characterize the errors δx, δy, δz for the case of motion in a circular orbit. These terms differ from formulas (5.118) above only in that formulas (5.337) contain ν in place of ω_0. For motion in a circular orbit $\omega_0 = \nu$.

The expression in brackets in formulas (5.337) characterize, clearly, the dependence of the errors δx, δy, δz on the eccentricity e of the orbit. Examination of these expressions shows that they do not contain the time t to a power higher than the second, and that t^2 is contained in the square brackets of the expressions defining δx and δz as a multiplier of Δn_x. For the case of motion in a circular orbit, t^2 enters only into the expression for δx.

If we compare the expression for δx for the case of a circular orbit with the first formula (5.337), we see that it is easily demonstrated that the extent to which the orbit differs from circularity, i.e., its ellipticity, does not give rise to a significant change in the dependence of δx on time. As for the case of a circular orbit, for the case of an elliptical orbit the time functions enter as factors in the corresponding instrument errors, are analogous in the sense that they contain time (outside of the trigonometric functions) to the same powers. The same is not true with regard to δy and δz.

Thus, the expression for δy for the case of a circular orbit contains time only in terms of a trigonometric function. For the case of an elliptical orbit, on the other hand, the expression for δy contains the term $ea\Delta m_x t \cos \nu t$. (It should be noted that for the case of a circular orbit, δy in general is not a function of Δm_x.) It is evident from the third formula (5.227) that for the case of an elliptical orbit, for all of the instrument errors and errors in initial conditions of which δz (after subtraction of $\delta \dot{z}^0$) is a function, the time t appears to a power greater by 1 than for the case of a circular orbit.

Let us now turn to the integration of the right sides of expression (5.71) for the case of motion in an elliptical orbit with small eccentricity. The first and third expressions are to be integrated, since the formula for θ_{1y} does not differ from the formula derived previously for the case of a stationary object. The first and third formulas (5.71) are equivalent to formulas (5.77). Let us substitute for ω_y on the right sides of formulas (5.77) its value from equality (5.329) and consider, as before, the instrument errors Δm_x and Δm_y to be constant. Then, noting that

$$\int_0^t \omega_y \, dt = \nu t + 2e \sin \nu t, \tag{5.338}$$

and therefore that, with accuracy to within terms of the first order of smallness relative to e,

$$\begin{aligned} \sin \int_0^t \omega_y \, dt &= \sin \nu t + e \sin 2\nu t, \\ \cos \int_0^t \omega_y \, dt &= \cos \nu t - 2e \sin^2 \nu t, \end{aligned} \tag{5.339}$$

we find formulas for θ_{1x} and θ_{1y} in the following form:

$$\left.\begin{aligned} \theta_{1x} = {} & \theta_{1x}^0 \cos \nu t - \theta_{1y}^0 \sin \nu t + \frac{\Delta m_x}{\nu} \sin \nu t - \\ & - \frac{\Delta m_y}{\nu}(1 - \cos \nu t) + e\Big[-2\theta_{1x}^0 \sin^2 \nu t - \theta_{1y}^0 \sin 2\nu t + \\ & + \Delta m_x\Big(t + \frac{\sin 2\nu t}{2\nu}\Big) - \frac{\Delta m_y}{2\nu}(1 - \cos 2\nu t)\Big], \\ \theta_{1y} = {} & \theta_{1x}^0 \sin \nu t + \theta_{1y}^0 \cos \nu t + \frac{\Delta m_x}{\nu}(1 - \cos \nu t) + \\ & + \frac{\Delta m_y}{\nu} \sin \nu t + e\Big[\theta_{1x}^0 \sin 2\nu t - 2\theta_{1y}^0 \sin^2 \nu t + \\ & + \frac{\Delta m_x}{2\nu}(1 - \cos 2\nu t) + \Delta m_y\Big(t + \frac{\sin 2\nu t}{2\nu}\Big)\Big]. \end{aligned}\right\} \tag{5.340}$$

The terms in formulas (5.340) which do not contain e characterize the errors θ_{1x} and θ_{1y} for the case of motion in a circular orbit.

If we now substitute expressions (5.337) and (5.340) together with equality (5.328) and the value

$$\theta_{1y} = \theta_{1y}^{0} + \Lambda m_y t \tag{5.341}$$

into formulas (5.322) and (5.323), we will find the relations between the total coordinate errors δx_3, δy_3, δz_3 and the errors θ_x, θ_y, θ_z in the determination of the orientation parameters for motion in an elliptical orbit with small eccentricity.

Notes

1. Lur'ye, A. I. Analiticheskaya mekhanika (Analytic Mechanics), Fizmatgiz, 1961; Appel', P. Teoreticheskaya mekhanika (Theoretical Mechanica), vol. 2, Fizmatgiz, 1960.

2. Compare for example, Gursa [Goursat], E. Kurs matematicheskogo analysis (Course in Mathematical Analysis), vol. 2, Gostekhizdat, 1933.

3. Merkin, D. P. Giroskopichesk'ye sistemy (Gyroscopic Systems), Gostekhizdat, 1956.

4. Compare, for example, Chetayev, N. G. Ustoychivost'dvizheniya (Stability of Motion), Gostekhizdat, 1955.

5. Bulgakov, B. V. Kolebaniya (Oscillations), Gostekhizdat, 1954.

6. Thomson, W. and Tait, P. Treatise on Natural Philosophy, vol. 1, Cambridge University Press, 1879.

7. This property of dynamic systems is sometimes called in the literature "stability in a finite interval of time" or "technical stability".

8. Compare, for example, Bulgakov, B. V., op. cit.

9. Compare, for example, Gursa, E., op. cit.

10. A detailed statement of the Theory of Keplerian motion can be found in Subbotin, M.F. Kurs nebesnoy mekhaniki (Course in Celestial Mechanics), vol. 2, ONTI, 1937; Duboshin, G. N. Nebesnoy mekhanika. Zadachi i metody (Celestial Mechanics, Basic problems and Methods), Fizmatgiz, 1963.

11. Compare, for example, Stepanov, V. V. Kurs differential'nykh uravneniy (Course in Differential Equations), Gostekhizdat, 1953.

12. Lur'ye, A. I. Free fall of a mass point in a spacecraft cabin, Prikladnaya matematika i mekhanika, vol. xxvii, Issue 1, 1963.

13. Andreyev, V. D. Integration of error equations of an inertial navigation system for Keplerian motion of the object, Prikladnaya matematika i mekhanika, vol. xxix, Issue 2, 1965.

14. Schuler, M. The disturbance of pendulum and gyroscope apparatus by the accleration of the vehicle, Physikalische Zeitschrift, vol. 24, Jahrgang No. 16, Leipsig, 1923.

15. See pp. 495-8.

Chapter 6

INERTIAL NAVIGATION ON THE SURFACE OF THE EARTH

§6.1. General Considerations

In the preceding chapters the theory of autonomous inertial systems was presented. These systems determine the parameters of motion required for purposes of navigation on the sole basis of the readings of inertial sensing elements: newtonometers and gyroscopes. No additional information is used for this purpose, with the exception, of course, of the initial conditions, which are considered as known.

The equations describing the ideal operation of an inertial system, i.e., the algorithms on the basis of which the functional diagrams an inertial system are constructed, as well as the error equations, i.e., the equations describing the perturbed operation of the inertial system, were obtained for arbitrary motion of the object. No limitations were imposed on the trajectory parameters. The only condition which might in a certain sense be considered as a limitation, was the assumption that the flight trajectory was sufficiently close to the earth so that gravitational attraction on the sensitive masses of the newtonometers caused by all celestial bodies except the earth could be ignored. This limitation, however, is insignificant, since it leads to vanishingly small errors even for the case in which the distance from the moving object to the surface of the earth is comparable to its radius. In addition, this restriction, which we used in deriving the fundamental equation (1.88) of inertial navigation, is not fundamental in nature. As was demonstrated, it could just as well not have been introduced.

In the preceding chapter the operational stability of an inertial system was investigated, and solutions to the error equations for several types of motion were obtained. These solutions give the relationship between the errors in the determination of the navigation parameters and instrument errors and errors in initial conditions. Analysis of the solutions to the error equations showed that operation of an inertial system of unlimited duration is impossible if a given level of accuracy in its determination of the navigation parameters is, in the general case, to be maintained. The total errors in the determination of coordinates and orientation increase with time. For motion at low velocities the coordinate errors increase exponentially, and for Keplerian motion they increase as a quadratic function of time. Errors in the determination of the orientation parameters increase, at best, as linear functions of time.

Let us assume that certain requirements have been placed on an inertial system with regard to its level of accuracy in the determination of coordinates during some specified period of continuous operation. Then, knowing how the functional errors of the system depend on the instrument errors and the errors in initial conditions, it is possible to impose requirements on the operational accuracy of the system elements and on the accuracy of the initial conditions, such that they will guarantee a given level of accuracy in the operation of the inertial system, taking into account the increase in the errors over time. However, these requirements on the accuracy of the system elements and the initial conditions may be so rigid that they cannot be satisfied.

This difficulty may be avoided by adducing additional information, i.e., through correction based on external sources of information This information could be the height of the object above the surface of the earth, as measured by means of a barometric altimeter for a radioaltimeter, the velocity of the object relative to the surface of the earth, measured by a Doppler velocity meter, the coordinates of the object relative to the earth, as determined by a radio navigation system or a panoramic radar, etc. Correction of the operation of the gyroscopic devices in an inertial system may be based on astronomical correction, i.e., comparison of the orientation of the gyroscopes with bearings to stars, planets, or artificial satellites, on the basis of bearings to orienting points on the earth's surface.

In the simplest case additional information may be used in the following manner. From time to time the readings from the inertial system are compared with the values of the navigation parameters as derived from other sources, and are corrected on the basis of these values. In this case, the sources of error and the dynamic processes in the inertial system do not affect one another. The interval between corrections is determined by the time during which the increase in the system errors does not exceed allowable limits. In this mode of correction the inertial system becomes, essentially, a device which stores, for a certain period of time (usually short), precise information on the navigation parameters obtained from the external sources. Continuous correction has no significance in this case. Of course, this correction procedure, i.e., simple periodic correction of the readings of the inertial system, does not in any way lead to new effects in its operation.

Of much greater interest are other means of using additional information, in which such information is actually used to alter the operational algorithm of the inertial system. The primary result of this approach is that, in addition to the algorithm (the equations describing ideal operation), the structure of the error equations changes, i.e., the basic character of the dependence of the errors on instrument errors and errors in initial conditions changes. Such correction procedures assume, of course, continuous use of external information during a relatively extended time period or even throughout the entire operational time of the inertial system.

Let us consider the following instance. We assume that an inertial system is determining the curvilinear coordinates $\varkappa^1$, $\varkappa^2$, $\varkappa^3$ of an object. Let us further assume that, on the basis of auxiliary information on board the object, one of the coordinates, for example the coordinate $\varkappa^1$, may be continuously computed. It is then possible, clearly, to use this value of the coordinate for the formation of the terms in the equations describing the ideal operation of the inertial system in which it appears. By differentiating $\varkappa^1$, we may also form terms containing the derivative $\dot{\varkappa}^1$. Conversely, if the derivative $\dot{\varkappa}^1$ is known from external information sources, the coordinate $\varkappa^1$ may be found by integrating this value.

Two possibilities arise here. All of the terms in the ideal equations containing $\varkappa^1$ may be formed. In this case the task of the inertial system becomes two-dimensional. Generally speaking, the newtonometer the direction of whose axes of sensitivity is normal to the coordinate surface $\varkappa^1$ = const., becomes superfluous. This newtonometer may be eliminated from the system. The equation containing $\delta\ddot{\varkappa}^1$ then falls out of the error equations. In the two remaining equations the terms containing $\delta\varkappa^1$ and $\delta\dot{\varkappa}^1$ move to the right sides. They are now known functions of time.

It is possible, on the other hand, to use auxiliary information not to form, in the ideal equations, all of the terms containing $\varkappa^1$ and $\dot{\varkappa}^1$, but rather only certain of them, namely those giving rise to terms containing $\delta\dot{\varkappa}^1$ and $\delta\varkappa^1$ in the error equations and lending these equations properties which it is useful for one reason or another to avoid. In this case the system remains three-dimensional, i.e., all three newtonometers are necessary. The system of equations describing errors in the determination of the coordinates retains its order. The only difference will be that now some of the terms containing $\delta\varkappa^1$ and $\delta\dot{\varkappa}^1$ move to the right side and no longer enter into the homogeneous error equations. It proves to be the case that both means of continuous use of auxiliary information on the magnitudes of $\varkappa^1$ and $\dot{\varkappa}^1$ may lead to interesting results.

It was assumed above that information on the coordinate $\varkappa^1$ was known from external sources. Clearly, the problem is in no way different if a relation between the three coordinates of the form

$$\Phi(\varkappa^1, \varkappa^2, \varkappa^3, t) = 0. \tag{6.1}$$

is known from external sources of information.

Clearly, this relation may be regarded as a specification of one of the coordinates, for example $\varkappa^1$, as a function of the other two.

The basic characteristics of this means of using auxiliary information in the operation of an inertial system are, as has already been noted, the continuous participation of this information in the formation of the equations describing the ideal operation of the inertial system and the dependence of the structure of the error equations on the means of using this information. The problem, therefore, is not so much that of the correction of an inertial system, as of the operation of a "complex" system, which includes devices by means of which auxiliary information is obtained from non-inertial sources. In the general case, a system of this sort loses the independence characteristic of a purely inertial system.

Systematic analysis of the problems involved in the correction of an inertial system is not part of our task here[1]. There is, however, a extremely important special case of the use of auxiliary information, in which the system remains autonomous and purely inertial: the case of motion along the surface of the ocean, i.e., the surface of the terrestrial spheroid. This case includes, for example, the motion of marine vessels of all types.

For motion along the surface of the terrestrial spheroid the position of the object in space is determined, clearly, by two coordinates on the spheroid. The three spatial coordinates of the objects are related by an equation of the form (6.1), which is the equation for a spheroid.

In terms of the rigid earth body-axis coordinate system, ξ, η, $\zeta(\eta^1, \eta^2, \eta^3)$ (with the ζ axis along the axis of symmetry of the earth), the equation of a spheroid has the form:

$$\Phi(\xi, \eta, \zeta) = \frac{\xi^2 + \eta^2}{a^2} + \frac{\zeta^2}{b^2} - 1 = 0. \qquad (6.2)$$

The spheroid is symmetrical about the ζ axis, and the vector $\vec{u}$ of the earth rate may be considered as
coinciding with the ζ axis. Therefore, if the ζ and ζ_* axes are considered as superposed, equations (6.2) retains its form in the coordinates ξ_*, η_*, ζ_*.

Below we will also require spheroidal equations in spherical (geocentric and geodetic) and geographic coordinates. In the geographic coordinates r, φ, λ, the spheroidal equation has the form:

$$\Phi(r, \varphi) = \frac{a\sqrt{1-e^2}}{\sqrt{1-e^2\sin^2\varphi}} - r = 0. \tag{6.3}$$

This equation is found from relations (2.20) and (2.21) or (2.20) and (2.23), if in these relations h = 0. It gives r in terms of the geocentric latitude φ:

$$r = \frac{a\sqrt{1-e^2}}{\sqrt{1-e^2\cos\varphi}}. \tag{6.4}$$

In order to obtain the spheroidal equation in the geodetic coordinates r, S, z, we have only to substitute for $\cos^2\varphi$ in equality (6.4) an expression in terms of S and z. The third equality (3.303) may be used for this purpose. According to this equality

$$\sin\varphi = \delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z. \tag{6.5}$$

Finally, the spheroidal equation in geographic coordinates h, λ, φ' has, clearly, the simplest form:

$$\Phi(h) = h = 0. \tag{6.6}$$

Relations (6.2), (6.3) and (6.6), relating the coordinates of the surface of the spheroid, enable us to express one of these coordinates in terms of the other two in the equations describing the ideal operation of the inertial system.

We note that for motion along the surface of a spheroid the distance r of the object from the center of the earth becomes a function of the coordinates of the object on the spheroid. In our analysis of the error equations it was shown that the greatest difficulty from the point of view of guaranteeing the operational stability of an inertial system arise as a result of the fact that part of the task which the inertial system must perform is the determination of r. The composition (in the ideal equations) of the gravitational field strength of the earth on the basis of the magnitude of r determined by the inertial system itself is precisely the factor which results in the appearance in the solutions to the error equations of rapidly growing exponentially and power functions of time. We will see below that use of relations (6.2), (6.3) and (6.6) will permit us to alter the ideal equations sufficiently such that these difficulties may for the most part be avoided.

§6.2. Systems With Two and Three Newtonometers

6.2.1. Derivation of the ideal and error equations on the assumption that the earth is a homogeneous sphere. Let us determine the character of the changes which may be introduced into the structure of the ideal equations and the error equations for motion along the surface of the earth. In order to avoid unwieldy calculations, it is expedient to first consider this problem under the assumption that the surface of the earth is a sphere of radius r_0, and that its gravitational field is spherical, i.e., that the strength of the gravitational field is

$$g = -\frac{\mu r}{r^3}. \tag{6.7}$$

Both of these assumptions are equivalent, clearly, to the assumption that the earth is a homogeneous sphere. Under this assumption, motion along the surface of the earth is equivalent to the condition

$$r = r_0 = \mathrm{const} \tag{6.8}$$

and there is no longer any need to determine r.

We will make use of this fact in order to vary the structure of the ideal equations. There are two possibilities in this regard[2]. The first possibility derives from the fact that two of the newtonometers of the inertial system are located in a plane tangent to the surface of the earth, while the third is completely superfluous. In the second case, all three newtonometers are retained in the system, and no restrictions are placed on their orientation, but the strength of the gravitational field is formed in accordance with the known value of $r = r_0$, i.e., in accordance with equality (6.7), it is assumed that

$$g = -\frac{\mu r}{r_0^3}. \tag{6.9}$$

If all three newtonometers are retained in the system, the ideal equations retain the form that they have for the case of general motion, except that

$$\operatorname{grad} V = -\frac{\mu r}{r_0^3}. \tag{6.10}$$

It was shown earlier that the error equations for an arbitrary inertial system with three newtonometers reduced to equations (5.1) -- (5.9), equations (5.1) being the projections on the x, y, z axes of the first vector equation (5.17):

$$\frac{d^2\delta r}{dt^2} + \frac{\mu\,\delta r}{r^3} - \frac{\mu r}{r^3}\,\frac{3r\cdot\delta r}{r^2} = \\ = \Delta n - 2\Delta m \times \frac{dr}{dt} + r \times \frac{d\Delta m}{dt}. \tag{6.11}$$

In deriving this equation, terms containing variations of the non-spherical component of the earth's gravitational field were considered sufficiently small to be ignored. This means, essentially, that in our analysis of the error equations the gravitational field of the earth was considered to be spherical. In the present case, this assumption is introduced from the beginning.

If r_0 is used only in the formation of the quantity μ/r_0^3 in the ideal equations, and it is precisely this variant which we are now considering, then in equations (6.11) the only term which changes is the last term on the left side, containing the factor

$$\delta\left(\frac{1}{r^3}\right) = \frac{1}{|r+\delta r|^3} - \frac{1}{r^3} = -\frac{3r\cdot\delta r}{r^5}. \tag{6.12}$$

This term will now be equal to 0. Equations (6.11) therefore takes the form:

$$\frac{d^2\delta r}{dt^2} + \frac{\mu}{r_0^3}\,\delta r = \Delta n - 2\Delta m \times \frac{dr}{dt} + r \times \frac{d\Delta m}{dt}. \tag{6.13}$$

In terms of projections on the x, y, z axes of the moving trihedron, the z axis of which is directed along the vector $\vec{r}$, we obtain the following equations:

$$
\left.\begin{aligned}
&\delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y - \\
&\quad - 2\omega_z\delta\dot{y} + (\omega_x\omega_z + \dot{\omega}_y)\delta z + 2\omega_y\delta\dot{z} = \\
&\qquad = \Delta n_x - \Delta\dot{m}_y r_0 - r_0\omega_x\Delta m_z - r_0\omega_z\Delta m_x. \\
&\delta\ddot{y} + (\omega_0^2 - \omega_x^2 - \omega_z^2)\delta y + (\omega_y\omega_z - \dot{\omega}_x)\delta z - \\
&\quad - 2\omega_x\delta\dot{z} + (\omega_y\omega_x + \dot{\omega}_z)\delta x + 2\omega_z\delta\dot{x} = \\
&\qquad = \Delta n_y + \Delta\dot{m}_x r_0 - r_0\omega_y\Delta m_z - r_0\omega_z\Delta m_y.
\end{aligned}\right|
$$

$$
\left.\begin{aligned}
&\delta\ddot{z} + (\omega_0^2 - \omega_x^2 - \omega_y^2)\delta z + (\omega_z\omega_x - \dot{\omega}_y)\delta x - \\
&\quad - 2\omega_y\delta\dot{x} + (\omega_z\omega_y + \dot{\omega}_x)\delta y + 2\omega_x\delta\dot{y} = \\
&\qquad = \Delta n_z + 2r(\omega_x\Delta m_x + \omega_y\Delta m_y).
\end{aligned}\right|
$$

(6.14)

where

$$\omega_0^2 = \mu/r_0^3.$$

It is evident that the remaining error equations of the system, except for the first group of equations obtained above, do not change, since they do not contain the gravitational field strength.

Let us now turn to the second alternative approach to the construction of an inertial navigation system for the case of motion along the surface of the earth, namely, to systems using two newtonometers oriented in the plane of the horizon, i.e., in the plane normal to the vector $\vec{r}$. Without decreasing the generality of our analysis we may consider, clearly, that the missing newtonometer is the one oriented along the axis coinciding with the vector $\vec{r}$.

Let us consider the cases of the determination of Cartesian and spherical curvilinear coordinates.

In the first case, the ideal equations are easily found from the general equations (3.59) -- (3.65). If we consider that the z axis of the platform of the inertial system coincides in the unperturbed position with the direction of the radius vector $\vec{r}$, then we must introduce the following changes into equations (3.59) -- (3.65):

we must everywhere set $x = y = 0$, $z = r_0$; drop equations (3.65) and the third equation (3.59); set $g_x = g_y = 0$, and $v_z = 0$ in the first two equations (3.59), in accordance with equality (6.7). Equations (3.59) will then take the form:

$$\left.\begin{aligned} v_x &= \int_0^t (n_x + m_z v_y)\,dt + v_x(0), \\ v_y &= \int_0^t (n_y - m_z v_x)\,dt + v_y(0), \\ v_x &= m_y r_0, \quad v_y = -m_x r_0, \quad z = r_0. \end{aligned}\right\} \tag{6.15}$$

Equations (3.60), (3.61) and (3.64) do not change, since they do not contain x, y, z. Relations (3.62) will take the form:

$$\xi_0 = \alpha_{13} r_0, \quad \eta_0 = \alpha_{23} r_0, \quad \zeta_0 = \alpha_{33} r_0. \tag{6.16}$$

Relations (3.63) will change in an analogous fashion:

$$\xi = \beta_{13} r_0, \quad \eta = \beta_{23} r_0, \quad \zeta = \beta_{33} r_0. \tag{6.17}$$

For the case of spherical curvilinear coordinates, in order to obtain the ideal equations we may also use the general equations obtained in §3.2. For oblique curvilinear coordinates, these will be equations (3.172), (3.163) or (3.174) and table (3.173). For orthogonal curvilinear coordinates, equations (3.210) -- (3.213) should be used. If the basic system is a maneuverable gyroplatform with fixed newtonometers, relations (3.205) and (3.207) by means of which the controlling moments are determined, should be used in place of equations (3.213).

For spherical curvilinear coordinates, for the case of motion on a sphere, we may take

$$\varkappa^1 = r = r_0, \tag{6.18}$$

such that relations (3.89) may now be written in the form:

$$\xi^s = r_0 f^s(\varkappa^2, \varkappa^3, t), \tag{6.19}$$

where the functions f^s are the direction cosines of the radius vector $\vec{r}$ in relation to the ξ^s axes, and $\varkappa^2$ and $\varkappa^3$ are the curvilinear coordinates of the sphere.

For the case of oblique curvilinear coordinates the newtonometers n_{e_1}, n_{e_2}, n_{e_3} are oriented along the vector $\vec{r}^s$. In accordance with equality (6.18), the vector $\vec{r}'$ has the same direction as the vector $\vec{r}_1$ and, consequently, the vector $\vec{r}$. For the case under consideration, the newtonometer n_{e_1} is absent. In equations (3.172), therefore, the first and fourth equations for $\dot{r}$ and $\vec{r}$, respectively, drop out. In the second and third equations (3.172) the sums

$$\operatorname{grad}^l V \eta^s_l, \quad \operatorname{grad}^l V \eta^1_l. \tag{6.20}$$

also drop out, since they are equal to zero, according to conditions (6.9). Equations (3.172) therefore take the form:

$$\left.\begin{aligned}
&\frac{\dot{x}^s}{\sqrt{a^{ss}}} = \int_0^t \Big[n_{e_s} - \frac{1}{\sqrt{a^{ss}}}\Big(\dot{x}^s \frac{d}{dt} \ln \sqrt{a^{ss}} + \\
&\qquad + \Gamma^s_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^s_{0n}\dot{x}^n + \Gamma^s_{00}\Big)\Big] dt + \frac{\dot{x}^s(0)}{\sqrt{a^{ss}(0)}}, \\
&\dot{x}^s = \left(\frac{\dot{x}^s}{\sqrt{a^{ss}}}\right)\sqrt{a^{ss}}, \quad x^s = \int_0^t \dot{x}^s dt + x^s(0), \\
&x^1 = r_0, \quad \dot{x}^1 = 0,
\end{aligned}\right\} \tag{6.21}$$

where s takes on the values 2 and 3.

As a result of equalities

$$\operatorname{grad}^l V \eta^s_l = 0, \quad \operatorname{grad}^l V \eta^1_l = 0 \tag{6.22}$$

the need to compute η^s_ℓ and η^k falls away, and therefore equations (3.163) drop out of the ideal equations. Table (3.173) retains its form, except that now, as in equations (6.21), the index s takes on only the values 2 and 3.

For the case of orthogonal curvilinear spherical coordinates, the changes in the ideal equations are analogous. The first equations from groups (3.210) and (3.212) drop out, equations (3.211) drops out entirely, and the three first equalities of equality (3.213), i.e., those corresponding to s = 1, drop out. Relations (3.205) and (3.207) remain unchanged.

As an example, let us write out the ideal equations for geodetic coordinates and the coordinates σ_1, σ_2, examined in §3.3. For geodetic coordinates, when the basis of the functional diagram is a maneuverable gyroplatform, by returning to equations (3.308), setting

$$g^1 = g^3 = 0, \quad r = r_0, \quad \dot{r} = 0 \tag{6.23}$$

and dropping the first equation (3.308), we obtain the system:

$$\left.\begin{aligned} v_x &= \int_0^t (n_x + v_y \omega_z)\,dt + v_x(0), \\ v_y &= \int_0^t (n_y - v_x \omega_z)\,dt + v_y(0). \end{aligned}\right|$$

$$\left.\begin{aligned} &\omega_x = -\frac{v_y}{r_0}, \quad \omega_y = \frac{v_x}{r_0}, \\ &S = \int_0^t \Big[\frac{\omega_y}{\cos z} - \frac{u}{\cos z}(\delta_{33}\cos z - \delta_{31}\sin z \cos S - \\ &\qquad\qquad - \delta_{32}\sin z \sin S)\Big]\,dt + S(0), \\ &z = \int_0^t [-\omega_x + u(-\delta_{31}\sin S + \delta_{32}\cos S)]\,dt + z(0), \\ &\sin\varphi = \delta_{31}\cos z \cos S + \delta_{32}\cos z \sin S + \delta_{33}\sin z, \\ &\omega_z = \omega_y \operatorname{tg} z + \frac{u}{\cos z}(\delta_{31}\cos S + \delta_{32}\sin S), \\ &M_{1y}^{1} = -H\omega_x, \quad M_{1x}^{1} = H\omega_y, \quad M_{1z}^{1} = H\omega_z. \end{aligned}\right| \tag{6.24}$$

If in formulas (6.24) we set $\delta_{33} = 1$, $\delta_{31} = \delta_{32} = 0$ and substitute φ and λ for z and S, respectively, we obtain the equations describing the ideal operation of a system determining geocentric coordinates.

In order to obtain the ideal equations for a system with two newtonometers in the curvilinear oblique angle coordinates σ_1, σ_2 on a sphere, the simplest procedure is to turn to equations (3.344). Dropping the first and fourth of these equations, setting $r = r_0$ and $\dot{r} = 0$ in the others, and taking into account equality (6.22), we arrive at the equations:

$$\left.\begin{aligned}
\dot{\sigma}_1 &= \int_0^t \Big[\frac{n_2}{r_0} + \frac{\operatorname{ctg}\sigma_1}{\sin^2\sigma_1 - \cos^2\sigma_2}(\dot{\sigma}_1\cos\sigma_1\cos\sigma_2 + \\
&\qquad + \dot{\sigma}_2\sin\sigma_1\sin\sigma_2)^2\Big]dt + \dot{\sigma}_1(0).\\
\dot{\sigma}_2 &= \int_0^t \Big[\frac{n_3}{r_0} + \frac{\operatorname{ctg}\sigma_2}{\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}}(\dot{\sigma}_1\sin\sigma_1\sin\sigma_2 + \\
&\qquad + \dot{\sigma}_2\cos\sigma_1\cos\sigma_2)^2\Big]dt + \dot{\sigma}_2(0).\\
\sigma_1 &= \int_0^t \dot{\sigma}_1\,dt + \sigma_1(0), \quad \sigma_2 = \int_0^t \dot{\sigma}_2\,dt + \sigma_2(0).
\end{aligned}\right\} \tag{6.25}$$

We must add to these equations the table of direction cosines between the unit vectors $\vec{e}_2$ and $\vec{e}_3$ of the axes of sensitivity of the newtonometers n_2, n_3 and the axes ξ_*, η_* and ζ_*, deriving from (3.350):

$$\left.\begin{array}{cccc}
 & \xi_* & \eta_* & \zeta_* \\
e_2 & -\sin\sigma_1 & \cos\sigma_2\operatorname{ctg}\sigma_1 & \operatorname{ctg}\sigma_1\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2} \\
e_3 & \cos\sigma_1\operatorname{ctg}\sigma_2 & -\sin\sigma_2 & \operatorname{ctg}\sigma_2\sqrt{\sin^2\sigma_1 - \cos^2\sigma_2}.
\end{array}\right\} \tag{6.26}$$

Let us now obtain the error equations for systems with two newtonometers. The first group of error equations in terms of projections on the axes of the moving trihedron xyz, the z axis of which coincides with the radius vector $\vec{r}$, may, clearly, be obtained immediately from equations (6.14). In order to do this we must drop the third of these equations and set $\delta z = 0$ in the first two. We then obtain a system of two fourth-order differential equations:

$$\left.\begin{aligned}
&\delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y - 2\omega_z\delta\dot{y} = \\
&\qquad = \Delta n_x - \Delta\dot{m}_y r_0 - \omega_x\Delta m_z r_0 - \omega_z\Delta m_z r_0.\\
&\delta\ddot{y} + (\omega_0^2 - \omega_z^2 - \omega_x^2)\delta y + (\omega_y\omega_x + \dot{\omega}_z)\delta x + 2\omega_z\delta\dot{x} = \\
&\qquad = \Delta n_y + \Delta\dot{m}_x r_0 - \omega_y\Delta m_z r_0 - \omega_z\Delta m_y r_0
\end{aligned}\right\} \tag{6.27}$$

There is one point which needs to be clarified with regard to these equations. We obtained them by setting

$$\delta z = 0 \tag{6.28}$$

in the first two equations (6.14).

This equality is valid for small values of δx and δy, such that their squares (like the square of δz) may be ignored. Indeed, from the identity

$$r_0^2 = x^2 + y^2 + z^2 \tag{6.29}$$

it follows that

$$\delta x^2 + \delta y^2 + \delta z^2 + 2r_0 \delta z = 0. \tag{6.30}$$

from which, if δx^2, δy^2, δz^2 are dropped, equalities (6.28) follows.

If more than a first approximation is desired, the following non-linear system should be used in place of equations (6.27):

$$\left.\begin{aligned} &\delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y - \\ &\quad - 2\omega_z \delta\dot{y} + (\omega_x\omega_z + \dot{\omega}_y)\delta z + 2\omega_y \delta\dot{z} = \\ &\qquad = \Delta n_x - \Delta\dot{m}_y r_0 - \omega_x \Delta m_z r_0 - \omega_z \Delta m_x r_0. \\ &\delta\ddot{y} + (\omega_0^2 - \omega_z^2 - \omega_x^2)\delta y + (\omega_y\omega_x + \dot{\omega}_z)\delta x + \\ &\quad + 2\omega_z \delta\dot{x} + (\omega_y\omega_z - \dot{\omega}_x)\delta z - 2\omega_x \delta\dot{z} = \\ &\qquad = \Delta n_y + \Delta\dot{m}_x r_0 - \omega_y \Delta m_z r_0 - \omega_z \Delta m_y r_0. \\ &\qquad \delta x^2 + \delta y^2 + \delta z^2 + 2r_0 \delta z = 0. \end{aligned}\right\} \tag{6.31}$$

6.2.2. <u>Motion on the surface of a terrestrial ellipsoid. Taking into account the non-sphericity of the earth's gravitational field</u>. We will now discard the assumption that the earth and its gravitational field are spherical in form, and consider the case of motion on the surface of the Clairaut ellipsoid given by equations (6.2), (6.4) and (6.6) and the expressions for the strength of the regularized gravitational field of the earth obtained in Chapter 2.

Let us again begin with a system using three newtonometers. Here, clearly, the ideal equations may be retained in the same form as for the case of arbitrary motion, the only change being in the means of forming the projections of the strength of the gravitational field. Thus, for example, for the determination of Cartesian coordinates equations (3.59) -- (3.65) remain valid. The only difference is that now in equations (3.65) for the formation of the spherical component of the earth's gravitational field the value for r obtained from equations (6.4) or equivalent equations must be used. We must proceed in the same fashion in the formation of the sums $\operatorname{grad}^{\ell} V \eta_{\ell}^{s}$ for the case of curvilinear coordinates. If in the first group of error equations we ignore small variations in the non-spherical component of the gravitational field, as we did previously in deriving equations (5.1), then the error equations will be the same equations (6.14) as in the case of motion on a sphere, except that now r from relation (6.4) must be substituted for r_0. Since the products of the instrument errors times the square of the eccentricity of the Clairaut ellipsoid are quantitites of the second order of smallness, it is permissible to substitute r = a for r_0.

We will illustrate these considerations by means of two examples. The first example is that of a three-newtonometer system determining geodetic coordinates. Let us assume that this system is constructed on the basis of a maneuverable platform such that in the unperturbed position the orthogonal trihedron xyz of the platform along whose axes the newtonometers are oriented is a moving trihedron of the geodetic reference grid (the z axis of which is directed along the vector $\vec{r}$). The ideal equations are then obtained from equations (3.308), if in the integrands on the right sides of the first three equations (3.308) in the functions $g^1(r, \varphi)$ and $g^3(r, \varphi)$ $\vec{r}$ is expressed in terms of φ by means of formula (6.4).

The functions $g^1(r, \varphi)$ and $g^3(r, \varphi)$ are the projections of the strength of the earth's gravitational field on the axes of the geocentric moving trihedron:

$$g^1(r, \varphi) = F_{z_2}, \quad g^3(r, \varphi) = F_{y_2}, \tag{6.32}$$

where F_{z2} and F_{y2} are determined by equalities (2.98).

In order to get rid of r in the arguments of the functions F_{z_2} and F_{y_2}, we must substitute into the right sides of equality (2.98) the expression for r in terms of φ given by formulas (6.4). However, we may also use results which have already been obtained: by turning to formulas (2.111) and setting h = 0, we obtain

$$\begin{aligned} g^1(\varphi) = -g_e\Big[1 - \frac{1}{2}e^2\sin^2\varphi + q\left(1 + \frac{3}{2}\sin^2\varphi\right) + \\ + e^4\left(-\frac{1}{8}\sin^2\varphi - \frac{11}{32}\sin^2 2\varphi\right) + \\ + e^2q\left(-\frac{17}{28}\sin^2\varphi + \frac{13}{16}\sin^2 2\varphi\right)\Big]. \\ g^3(\varphi) = \frac{1}{2}g_e(q - e^2)\sin 2\varphi\left[1 - \frac{e^4}{q - e^2} + e^2\frac{3e^2 - 6q}{2(q - e^2)}\sin^2 2\varphi\right]. \end{aligned} \tag{6.33}$$

Considering, for the sake of simplicity, only terms of the order of e^2, we will have:

$$
\left.\begin{aligned}
g^1(\varphi) &= -g_e\left[1+q+\frac{1}{2}(3q-e^2)\sin^2\varphi\right], \\
g^3(\varphi) &= \frac{g_e(q-e^2)}{2}\sin 2\varphi.
\end{aligned}\right\} \tag{6.34}
$$

On the basis of these equalities, equations (3.308) take the form:

$$
\left.\begin{aligned}
\dot r &= \int_0^t \Big[n_z + v_x\omega_y - v_y\omega_x - \\
&\qquad - g_e\Big(1+q+\frac{1}{2}(3q-e^2)\sin^2\varphi\Big)\Big]\,dt + \dot r(0), \\
v_x &= \int_0^t [n_x + v_y\omega_z - \dot r\omega_y + \\
&\qquad + g_e(q-e^2)\sin\varphi(-\delta_{31}\sin S + \delta_{32}\cos S)]\,dt + v_x(0), \\
v_y &= \int_0^t [n_y - v_x\omega_z + \dot r\omega_x + \\
&\qquad + g_e(q-e^2)\sin\varphi(-\delta_{31}\sin z\cos S - \\
&\qquad - \delta_{32}\sin z\sin S + \delta_{33}\cos z)]\,dt + v_y(0), \\
r &= \int_0^t \dot r\,dt + r(0), \quad \omega_x = -\frac{v_y}{r}, \quad \omega_y = \frac{v_x}{r}, \\
S &= \int_0^t \Big[\frac{\omega_y}{\cos z} - \frac{u}{\cos z}(\delta_{33}\cos z - \delta_{31}\sin z\cos S - \\
&\qquad - \delta_{32}\sin z\sin S)\Big]\,dt + S(0), \\
z &= \int_0^t [-\omega_x + u(-\delta_{31}\sin S + \delta_{32}\cos S)]\,dt + z(0), \\
\sin\varphi &= \delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z, \\
\omega_z &= \omega_y \operatorname{tg} z + \frac{u}{\cos z}(\delta_{31}\cos S + \delta_{32}\sin S), \\
M_{1y}^{0} &= -H\omega_x, \quad M_{1x}^{1} = H\omega_y, \quad M_{1z}^{0} = H\omega_z.
\end{aligned}\right| \tag{6.35}
$$

Setting $\delta_{33} = 1$, $\delta_{31} = \delta_{32} = 0$ and replacing S and z by λ and φ, respectively, we see that equations (6.35) convert into equations for the determination of geocentric coordinates.

We will take as our second example a system determining the geographic coordinates h, λ, φ'. Here we can begin with equations (3.333). For the case in question, they retain their basic form. Only the terms containing $\tilde{g}_0^1$ and $\tilde{g}_0^3$, in which we must set h = 0, are different. Since $\tilde{g}_0^1$ and $\tilde{g}_0^3$ are projections of the strength of the earth's gravitational field on the x, y, z axes of a geographic moving trihedron,

$$\tilde{g}_0^1 = F_{z_1}, \quad \tilde{g}_0^3 = F_{y_1}, \tag{6.36}$$

where F_{z_1} and F_{y_1} are determined from equalities (2.117). Setting in equalities (2.117) the altitude h = 0 and retaining only terms containing e^2 and q, we obtain:

$$\left.\begin{aligned} \tilde{g}_0^1 &= -g_e\left[1 + q + \frac{1}{2}(3q - e^2)\sin^2\varphi'\right], \\ \tilde{g}_0^3 &= \frac{g_e q}{2}\sin 2\varphi'. \end{aligned}\right\} \tag{6.37}$$

Substituting these expressions into the first and third equalities (3.333) and leaving the rest unchanged, we obtain the equations describing the operation of a three-newtonometer system determining geographic coordinates.

We emphasize again that both the system operating in accordance with equations (6.35) and the system structured in accordance with equations (3.333) and (6.37) have as their first group of error equations the system of differential equations (6.14) or the equivalent vector equations (6.13), while equations (3.308) and (3.333) correspond to the vector error equations (6.11). This difference in the error equations arises as a result of the fact that, when equations (6.35), (3.333) and (6.37) are used, the quantitites r and h, computed by the inertial system, do not take part in the formation of the projections g^1, g^3, $\tilde{g}_0^1$, $\tilde{g}_0^3$ of the strength of the earth's gravitational field, as is the case in equations (3.308) and (3.333).

The claim that equations (6.14) are to a first approximation the error equations of the systems in question derives from the following considerations. Equations (6.35), (3.333) and (6.37) for the case of motion on the surface of a spheroid differ from equations (6.24) (and the equations deriving from them for geographic coordinates) in that the former contain terms of the first order of smallness (containing the factor e^2). The variations of these equations will therefore differ by terms of the second order of smallness, i.e., they will, to a first approximation, coincide. This consideration is, of course, easily supported by formal calculations. We will not, however, perform these calculations here, since we have analyzed this question in detail elsewhere,[3] in an analysis of inertial navigation on the surface of the earth as a special case of navigation close to the earth's surface with altimeter correction.

Let us now consider the operation of two-newtonometer systems for motion on the surface of the terrestrial spheroid, namely the operation of systems in which the newtonometers are situated either in the plane of the geocentric horizon (a plane normal to the radius vector $\vec{r}$), or in the plane of the geographic horizon (a plane tangent to the Clairaut ellipsoid). The general case of the determination of Cartesian spherical curvilinear coordinates may be regarded here in the same way as the case of motion on a sphere. We will therefore not deal with the general formulas, but will cite them only for the most important cases, geodetic and geographic coordinates.

For geodetic coordinates the ideal equations are obtained from equations (6.35), dropping the first and fourth equations and substituting in the remaining equations for $\vec{r}$ and $\dot{r}$ the expressions for these quantities deriving from equality (6.4). Thus, for this case we obtain the following equations:

$$
\left.
\begin{aligned}
v_x &= \int_0^t [n_x + v_y\omega_z - \dot{r}\omega_y + \\
&\quad + g_e(q - e^2)\sin\varphi\,(-\delta_{31}\sin S + \delta_{32}\cos S)]\,dt + v_x(0), \\
v_y &= \int_0^t [n_y - v_x\omega_z + \dot{r}\omega_x + \\
&\quad + g_e(q - e^2)\sin\varphi\,(\delta_{31}\sin z\cos S - \\
&\quad - \delta_{32}\sin z\sin S + \delta_{33}\cos z)]\,dt + v_y(0).
\end{aligned}
\right|
$$

$$
\left.
\begin{aligned}
r &= \frac{a\sqrt{1-e^2}}{\sqrt{1-e^2\cos^2\varphi}}, \\
\sin\varphi &= \delta_{31}\cos z\cos S + \delta_{32}\cos z\sin S + \delta_{33}\sin z, \\
\omega_x &= -\frac{v_y}{r}, \quad \omega_y = \frac{v_x}{r}, \\
S &= \int_0^t \left[\frac{\omega_y}{\cos z} - \frac{u}{\cos z}(\delta_{33}\cos z - \delta_{31}\sin z\cos S - \right. \\
&\quad \left. - \delta_{32}\sin z\sin S)\right]dt + S(0), \\
z &= \int_0^t [-\omega_x + u(-\delta_{31}\sin S + \delta_{32}\cos S)]\,dt + z(0), \\
\omega_z &= \omega_y \operatorname{tg}\varphi + \frac{u}{\cos z}(\delta_{31}\cos S + \delta_{32}\sin S), \\
M_{1y}^{\circ} &= -H\omega_x, \quad M_{1x}^{\circ} = H\omega_y, \quad M_{1z}^{\circ} = H\omega_z.
\end{aligned}
\right\} \qquad (6.38)
$$

Let us now assume that the coordinates being determined are the geographic coordinates λ and φ', and that the x and y axes of the platform of the inertial system, along which the newtonometers are oriented, are situated in the plane of the geographic horizon. The ideal equations are then obtained from equations (3.333) and (6.37). Dropping the first and eighth equations (3.333), setting h = 0 in the remaining equations and substituting for $\tilde{g}_0^3$ in accordance with the second equalities (6.37), we obtain the ideal equations in the following form:

$$\left.\begin{aligned}
&v_x = \int_0^t (n_x + v_y\omega_z)\,dt + v_x(0),\\
&v_y = \int_0^t \left[n_y - v_x\omega_z + \frac{g_e q}{2}\sin 2\varphi'\right] dt + v_y(0),\\
&\omega_x = -\frac{v_y}{r_3}, \quad \omega_y = \frac{v_x}{r_2},\\
&r_2 = \frac{a}{\sqrt{1 - e^2\sin^2\varphi'}}, \quad r_3 = \frac{a(1-e^2)}{(1 - e^2\sin^2\varphi')^{3/2}},\\
&\varphi' = -\int_0^t \omega_x\,dt + \varphi'(0),\\
&\lambda = \int_0^t \left(\frac{\omega_y}{\cos\varphi} - u\right) dt + \lambda(0),\\
&\omega_z = \omega_y \operatorname{tg}\varphi',\\
&M_{1x}^0 = -H\omega_x, \quad M_{1y}^0 = H\omega_y, \quad M_{1z}^0 = H\omega_z.
\end{aligned}\right| \tag{6.39}$$

Because e^2 is small, we may expand the right sides of the fifth and sixth equations (6.39) into series in powers of e^2. Retaining only terms of the order of e^2, we obtain the approximate expressions:

$$\left.\begin{aligned}
&r_2 = a\left(1 + \frac{1}{2}e^2\sin^2\varphi'\right),\\
&r_3 = a\left[1 + e^2\left(\frac{3}{2}\sin^2\varphi' - 1\right)\right].
\end{aligned}\right\} \tag{6.40}$$

These allow us also to write the third and fourth equalities (6.39) in the following form:

$$\left.\begin{aligned}
&\omega_x = -\frac{v_y}{a}\left[1 + e^2\left(1 - \frac{3}{2}\sin^2\varphi'\right)\right],\\
&\omega_y = \frac{v_x}{a}\left(1 - \frac{1}{2}e^2\sin^2\varphi'\right).
\end{aligned}\right\} \tag{6.41}$$

Analogously, the third equality (6.39) may be replaced by the approximation

$$r = a\left(1 - \frac{1}{2}e^2\sin^2\varphi\right), \tag{6.42}$$

after which substitution the fifth and sixth equalities (6.38) reduce to the following forms:

$$\omega_x = -\frac{v_y}{a}\left(1 + \frac{e^2}{2}\sin^2\varphi\right), \quad \omega_y = \frac{v_x}{a}\left(1 + \frac{e^2}{2}\sin^2\varphi\right). \tag{6.43}$$

In the preceding subsection, in which we considered inertial navigation on the surface of the earth under the assumption that the earth is a homogeneous sphere, the first group of error equations (6.27) were obtained for two-newtonometer systems. These equations will be to a first approximation the error equations for motion on the surface of the terrestrial spheroid, independent of whether the newtonometers are oriented in the plane of the geographic or geocentric horizon. This fact derives from considerations analogous to those presented above with regard to three-newtonometer systems used for navigation on the terrestrial sphere and ellipsoid.

§6.3. Schuler Gyroscopic-Pendulum Systems

6.3.1. The compound Schuler pendulum. We will use the term compound pendulum to denote a solid body having an axis of dynamic symmetry and suspended from a point on this axis not corresponding to the center of mass, which, clearly, also lies on the axis of dynamic symmetry.

If the suspension point of the pendulum moves arbitrarily in a coordinate system $O_1\xi_*\eta_*\zeta_*$, the axis of dynamic symmetry will at each moment of time occupy in this coordinate system a position determined by the initial conditions of the motion of the pendulum, its parameters, the gravitational field of the earth, and the law of motion of the suspension point of a pendulum.

Max Schuler, investigating the plane motion of the suspension point of a pendulum at a constant distance from the center of the earth, under the assumption of the centrality of the earth's gravitational field, established[4] that the axis of dynamic symmetry of a pendulum, given that its parameters have been selected in a specific fashion, may lie on a line passing through the center of the earth at all times during its motion, if it lay on this line at the moment at which its motion began. A number of works were devoted to the generalization of this theorem of theoretical mechanics to the case of arbitrary motion of the point of support of a pendulum, including works by B. V. Bulgakov and A. Y. Ishlinskiy. Ishlinskiy gave a rigorous solution to the problem for arbitrary motion of the suspension point at a constant distance from the center of the earth.[5]

The Schuler theorem was the source of the ideas on which inertial navigation is based. Inertial navigation systems developed at first as mechanical devices modeling Schuler's physical pendulum. The classical examples of this type of model are the Anschutz-Heckeler pitch control gyrocompass and the twin vertical gyro.[6]

It is obvious that a mechanical device giving an alignment to the center of the earth is equivalent from the point of view of final results to the two-newtonometer inertial systems considered in the preceding section. Indeed, knowing this alignment, it is possible to solve the problem of navigation on the surface of the earth. For this purpose it is necessary only to include in the system gyroscopes for determining the orientation of the alignment to the center of the earth in the $O_1\xi_*\eta_*\zeta_*$ coordinate system, and, for navigation in an earth body-axis coordinate system, a timer, in order to compute the change in time of the position of the $O_1\xi_*\eta_*\zeta_*$ coordinate system relative to the earth. These two approaches, however, are more than superficially similar. The analogy between them, as we will see, proves to be much deeper.

Let us now consider Schuler's compound pendulum. We have two problems to solve: to determine the conditions under which the position of the axis of dynamic symmetry of the pendulum coinciding with the alignment to the center of the earth is a position of relative equilibrium, and to investigate the motion of the axis of dynamic symmetry about the position of relative equilibrium.

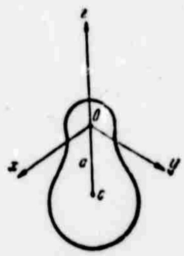

Figure 6.1

Let us consider the first of these problems. Let us attach to the body of the pendulum the trihedron Oxyz (Figure 6.1), the origin of which we place at the suspension point O, and the z axis of which we align along the axis of dynamic symmetry of the pendulum away from the center of mass C, which, therefore, will have the coordinates

$$x_C = y_C = 0, \quad z_C = -a. \tag{6.44}$$

The x, y, z axes are aligned along the major axes of the ellipsoid of inertia of the pendulum, which is an ellipsoid of revolution. The moments of inertia of the pendulum are therefore equal:

$$J_{xy} = J_{yz} = J_{zx} = 0, \quad J_{xx} = J_{yy} = A, \quad J_{zz} = C. \tag{6.45}$$

The equations of motion of the pendulum are most conveniently formulated in the $O\xi_*\eta_*\zeta_*$ coordinate system, the origin of which coincides with the point O, and the orientation of the axes of which coincides with the orientation with the axes of the $O_1\xi_*\eta_*\zeta_*$ coordinate system, i.e., fixed relative to bearings to distant stars. In formulating the equations of motion we will use the moment of momentum theorem. Since the coordinate system $O\xi_*\eta_*\zeta_*$ moves by translation in inertial space, the moment of momentum theorum leads to the well-known Euler equations (in terms of projections on the x, y, z axes):

$$
\left.\begin{aligned}
A\frac{d\omega_x}{dt}+(C-A)\omega_y\omega_z&=M_x,\\
A\frac{d\omega_y}{dt}-(C-A)\omega_z\omega_x&=M_y,\\
C\frac{d\omega_z}{dt}&=M_z.
\end{aligned}\right|
\qquad (6.46)
$$

Here ω_x, ω_y, ω_z are projections of the absolute rate of rotation of trihedron xyz about its axis, and M_x, M_y, M_z are projections on these same axes of the total moment about point O of the forces applied to the pendulum.

Since the coordinate system $O\xi_*\eta_*\zeta_*$, in which the equations of motion have been written, is mobile, the inertial forces of translational motion and Coriolis forces should be taken into account, in addition to gravitational forces and the reaction force of the fulcrum, in calculating the moments. The moment of the reaction force of the fulcrum is equal to zero, since the linear action of the force passes through point O (we will consider the fulcrum to be frictionless). The Coriolis forces are also zero, since the motion of the $O\xi_*\eta_*\zeta_*$ coordinate system is translational. For this reason the inertial forces acting on the elementary masses of the pendulum are parallel to one another and reduce to the single force Q, applied to the center of mass C.

We will consider the earth's gravitational field to be homogeneous within the amplitude of the pendulum. The gravitational forces reduce, therefore, to the resultant force

$$
F=mg, \qquad (6.47)
$$

applied at the center of mass C of the pendulum and directed along the strength vector $\vec{g}$ of the gravitational field at point O (m denoting the mass of the pendulum).

Taking into account these remarks and equalities (6.44) we find the following expressions for the moments:

$$
M_x=a(F_y+Q_y),\quad M_y=-a(F_x+Q_x),\quad M_z=0. \qquad (6.48)
$$

Let us now assume that the axis of dynamic symmetry of the pendulum always coincides with the direction to the center of the earth, that the point O moves freely on the surface of the earth, taken as a sphere of radius r_0, and that the earth's gravitational field is central. Then

$$F_x = F_y = 0. \tag{6.49}$$

It is obvious that

$$Q_x = -mw_x, \quad Q_y = -mw_y, \quad Q_z = -mw_z, \tag{6.50}$$

where

$$\left.\begin{aligned} w_x = \dot{v}_x + \omega_y v_z - \omega_z v_y, \quad w_y = \dot{v}_y + \omega_z v_x - \omega_x v_z, \\ w_z = \dot{v}_z + \omega_x v_y - \omega_y v_x. \end{aligned}\right\} \tag{6.51}$$

Since the z axis coincides with the direction to the center of the earth, and r_0 = const, it follows that

$$v_x = r_0\omega_y, \quad v_y = -r_0\omega_x, \quad v_z = 0. \tag{6.52}$$

Substituting equalities (6.49) -- (6.52) into expressions (6.48) and also into equations (6.46), we arrive at the following equations describing the equilibrium of the physical pendulum:

$$\left.\begin{aligned} (A - mar_0)\left(\frac{d\omega_x}{dt} - \omega_y\omega_z\right) + C\omega_y\omega_z = 0, \\ (A - mar_0)\left(\frac{d\omega_y}{dt} + \omega_x\omega_z\right) - C\omega_x\omega_z = 0, \\ C\frac{d\omega_z}{dt} = 0. \end{aligned}\right| \tag{6.53}$$

Equalities (6.53) should be satisfied identically. For arbitrary ω_x, ω_y, ω_z, this can occur if the following conditions are satisfied at the same time:

$$A/ma = r_0, \quad C = 0. \tag{6.54}$$

The first condition is the well known Schuler condition: the <u>reduced length of a compound pendulum should be equal to the radius of the earth</u>. The second condition requires that the entire mass of the pendulum be situated on its axes. This condition may be replaced by another one. For $C \neq 0$, it follows from the third equality (6.53) that

$$\omega_z = \omega_z^0 = \text{const.} \tag{6.55}$$

If we now take $\omega_z^0 = 0$, the first two equalities (6.53) are satisfied only if the first condition (6.54) is observed for arbitrary C. In this case the projection of the absolute rate of the rotation of the pendulum on its axis is

$$\omega_z = \omega_z^0 = 0. \tag{6.56}$$

Condition (6.56) and the second condition (6.54) may be combined into a single condition:

$$K_z = C\omega_z^0 = 0. \tag{6.57}$$

We note that condition (6.56) has to do with the selection of the initial conditions of the motion of the pendulum. Two additional analogous conditions should be added to it: a) at the initial moment of time the z axis should coincide with the direction to the center of the earth, and b) the projections ω_x^0 and ω_y^0 of the absolute angular velocity of the pendulum at the initial moment of time should be such that the rate of change in orientation of the z axis of the pendulum should coincide with the initial velocity of the change in the orientation of the radius vector $\vec{r}$ from the center of the earth to the suspension point of the pendulum.

Let us now consider perturbed motion of a Schuler pendulum, i.e., swinging of the pendulum about its position of relative equilibrium. This swinging occurs if the parameters of the pendulum or the initial[7] conditions do not precisely satisfy the conditions cited above.

Let us designate the trihedron xyz, attached to the pendulum at its unperturbed position, by $x_0y_0z_0$. The perturbed position of the pendulum (or, equivalently, the trihedron xyz attached to it) we will define by means of the angles α, β, and γ (Figure 6.2). The direction cosines between the x, y, z and x_0, y_0, z_0 axes will then form the table:

$$\left.\begin{array}{cccc} & x & y & z \\ x_0 & \cos\beta\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\alpha\sin\gamma & \cos\alpha\cos\gamma + \sin\alpha\cos\beta\sin\gamma \\ y_0 & \cos\beta\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma & \sin\beta\sin\gamma - \sin\alpha\cos\beta\cos\gamma \\ z_0 & -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta \end{array}\right\} \tag{6.58}$$

Let us consider first the case of small α, β and γ, for which table (6.58) simplifies and takes the form:

$$\begin{array}{cccc} & x & y & z \\ x_0 & 1 & -\gamma & \beta \\ y_0 & \gamma & 1 & -\alpha \\ z_0 & -\beta & \alpha & 1 \end{array} \tag{6.59}$$

To describe the perturbed motion of the pendulum, it is sufficient to obtain equations satisfied by α, β, and γ. We will use equations (6.46) for this purpose.

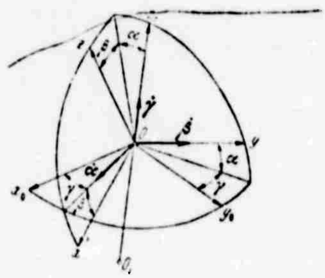

Figure 6.2

Let us denote the projections of the absolute angular velocity of trihedron $x_0y_0z_0$ on its axes by ω_{x_0}, ω_{y_0}, and ω_{z_0}, respectively. In accordance with table (6.59) we then obtain:

$$\left.\begin{aligned} \omega_x &= \omega_{x_0} + \omega_{y_0}\gamma - \omega_{z_0}\beta + \dot{\alpha}, \\ \omega_y &= \omega_{y_0} - \omega_{x_0}\gamma + \omega_{z_0}\alpha + \dot{\beta}, \\ \omega_z &= \omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta + \dot{\gamma}. \end{aligned}\right\} \tag{6.60}$$

To calculate the moments M_x, M_y and M_z on the right sides of equations (6.46), we require the projections ω_x, ω_y and ω_z of the acceleration of point O on the x, y, z axes. They are:

$$\left.\begin{aligned} w_x &= w_{x_0} - w_{z_0}\beta + w_{y_0}\gamma, \\ w_y &= w_{y_0} + w_{z_0}\alpha - w_{x_0}\gamma, \\ w_z &= w_{x_0}\beta - w_{y_0}\alpha + w_{z_0}. \end{aligned}\right\} \tag{6.61}$$

where, in accordance with equalities (6.51) and (6.52),

$$\left.\begin{gathered} w_{x_0} = r_0(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}), \quad w_{y_0} = r_0(-\dot{\omega}_{x_0} + \omega_{y_0}\omega_{z_0}), \\ w_{z_0} = -r_0(\omega_{x_0}^2 + \omega_{y_0}^2). \end{gathered}\right\} \tag{6.62}$$

Further, assuming that the earth's gravitational field is central, we find:

$$F_x = -F_{z_0}\beta, \quad F_y = F_{z_0}\alpha. \tag{6.63}$$

From expressions (6.48), (6.50), (6.61), (6.62) and (6.63), we obtain the following values of the moments M_x, M_y and M_z:

$$\left.\begin{aligned} M_x &= aF_{z_0}\alpha + amr_0(\omega_{x_0}^2 + \omega_{y_0}^2)\alpha + amr_0(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0})\gamma, \\ M_y &= aF_{z_0}\beta + amr_0(\omega_{x_0}^2 + \omega_{y_0}^2)\beta + \\ &\qquad + amr_0(-\dot{\omega}_{x_0} + \omega_{y_0}\omega_{z_0})\gamma, \\ M_z &= 0. \end{aligned}\right| \tag{6.64}$$

Let us now substitute the values (6.64) of the moments, together with expressions (6.60) for the projections ω_x, ω_y, and ω_z in equations (6.46). Performing the obvious groupings and ignoring terms of the second order of smallness in α, β and γ, we obtain the equations:

$$\left.\begin{aligned} &(A - mar_0)(\dot{\omega}_{x_0} - \omega_{y_0}\omega_{z_0}) + C\omega_{y_0}\omega_{z_0} + A(\omega_{y_0}\dot{\gamma} + \\ &+ \dot{\omega}_{y_0}\gamma - \omega_{z_0}\dot{\beta} - \dot{\omega}_{z_0}\beta + \ddot{\alpha}) + (C - A)(-\omega_{y_0}^2\alpha + \omega_{x_0}\omega_{y_0}\beta + \\ &\qquad + \omega_{y_0}\dot{\gamma} - \omega_{x_0}\omega_{z_0}\gamma + \omega_{z_0}^2\alpha + \omega_{z_0}\dot{\beta}) = aF_{z_0}\alpha + \\ &\qquad\qquad + amr_0(\omega_{x_0}^2 + \omega_{y_0}^2) + amr_0(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}). \\ &(A - mar_0)(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}) - C\omega_{x_0}\omega_{z_0} + \\ &\qquad + A(-\omega_{z_0}\dot{\gamma} - \dot{\omega}_{x_0}\gamma + \dot{\omega}_{z_0}\alpha + \omega_{z_0}\dot{\alpha} + \ddot{\beta}) - \\ &\qquad - (C - A)(-\omega_{x_0}\omega_{y_0}\alpha + \omega_{z_0}^2\beta + \omega_{x_0}\dot{\gamma} + \\ &+ \omega_{y_0}\omega_{z_0}\gamma - \omega_{z_0}^2\beta + \omega_{z_0}\dot{\alpha}) = aF_{z_0}\beta + amr_0(\omega_{x_0}^2 + \omega_{y_0}^2)\beta + \\ &\qquad\qquad + amr_0(-\dot{\omega}_{x_0} + \omega_{y_0}\omega_{z_0})\gamma. \\ &C\frac{d}{dt}(\omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta + \dot{\gamma}) = 0. \end{aligned}\right| \tag{6.65}$$

The last equation (6.65) is integrable. From it we obtain for $C \neq 0$:

$$\omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta + \dot{\gamma} = \omega_z = \omega_z^0, \tag{6.66}$$

where the superscript "0" designates, as before, the initial value of ω_z.

Now let the parameters of the pendulum satisfy conditions (6.54). The following equations then follow from equations (6.65):

$$\left.\begin{aligned} &\ddot{\alpha} + (\omega_0^2 - \omega_{x_0}^2 - \omega_{z_0}^2)\alpha - (\dot{\omega}_{z_0} + \omega_{x_0}\omega_{y_0})\beta - 2\omega_{z_0}\dot{\beta} = 0, \\ &\ddot{\beta} + (\omega_0^2 - \omega_{y_0}^2 - \omega_{z_0}^2)\beta + (\dot{\omega}_{z_0} - \omega_{x_0}\omega_{y_0})\alpha + 2\omega_{z_0}\dot{\alpha} = 0. \end{aligned}\right\} \tag{6.67}$$

where

$$\omega_0^2 = -\frac{F_{z_0}}{r_0 m}. \tag{6.68}$$

For a spherical gravitational field, such that

$$F_{z_0} = -\frac{m\mu}{r_0^2}, \tag{6.69}$$

the expression for ω_0^2 takes the form:

$$\omega_0^2 = \frac{\mu}{r_0^3} = \frac{g}{r_0}. \tag{6.70}$$

If the parameters of the pendulum satisfy the first condition (6.54) and condition (6.56), the following equations are obtained in place of equations (6.67):

$$\left.\begin{aligned} &\ddot{\alpha} + (\omega_0^2 - \omega_{x_0}^2)\alpha - \omega_{x_0}\omega_{y_0}\beta = 0, \\ &\ddot{\beta} + (\omega_0^2 - \omega_{y_0}^2)\beta - \omega_{x_0}\omega_{y_0}\alpha = 0. \end{aligned}\right\} \tag{6.71}$$

Equations (6.67) and (6.71) have trivial solutions. This demonstrates once again that the conditions derived above guarantee the existence of a position of relative equilibrium, in which its axis coincides with the direction to the center of the earth.

At first sight equations (6.67) and (6.71) are different. In fact, however, they are the same, but written in terms of projections on different axes. Indeed, the orientation of trihedron $x_0y_0z_0$, in terms of projections on the axes of which equations (6.67) are written, is not subjected to any conditions other than the requirement that the z axis of this trihedron should coincide with the direction to the center of the earth at the fulcrum point of the pendulum. Trihedron $x_0y_0z_0$, on the other hand, in terms of projections on the axes of which equations (6.71) are written, was selected such that the projection ω_{z_0} of its absolute rate of rotation on the z axis is

$$\omega_{z_0} = \omega_{y_0}\alpha - \omega_x\beta - \dot{\gamma}, \tag{6.72}$$

which follows from equalities (6.66) and (6.56). It is evident that, if expression (6.72) is substituted into equations (6.67) in place of ω_{z_0} and only terms of the first order of smallness in α, β and γ and their derivatives are retained, equations (6.71) are obtained.

Equations (6.67) and (6.71) describing the motion of the pendulum about its position of relative equilibrium were obtained under the assumption that the angles α, β and γ are small, i.e., these equations describe free swinging of the pendulum near its position of relative equilibrium. Let us now derive the equations for natural oscillation of the pendulum for finite magnitudes of angles α, β and γ.

Let us define angle γ by the position of trihedron $x_0y_0z_0$, i.e., we will consider that the position of trihedron xyz relative to trihedron $x_0y_0z_0$ is defined by angles α and β only. This does not diminish the generality of our analysis. We will then have in place of table (6.58):

$$\begin{array}{cccc} & x & y & z \\ x_0 & \cos\beta & 0 & \sin\beta \\ y_0 & \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta \\ z_0 & -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta. \end{array} \tag{6.73}$$

As before, let ω_{x_0}, ω_{y_0} and ω_{z_0} be the projections of the absolute angular velocity of trihedron $x_0y_0z_0$ on its axes. Trihedron $x_0y_0z_0$ is now not attached to the pendulum at its unperturbed position, since angle γ is defined by the position of this trihedron. But this, as will become evident below, will not constitute a difficulty in derivation of the required equations.

In accordance with table (6.73) we may write:

$$\left.\begin{aligned}
\omega_x &= \omega_{x_0}\cos\beta + \omega_{y_0}\sin\alpha\sin\beta - \\
&\qquad - \omega_{z_0}\cos\alpha\sin\beta + \dot{\alpha}\cos\beta, \\
\omega_y &= \omega_{y_0}\cos\alpha + \omega_{z_0}\sin\alpha + \dot{\beta}, \\
\omega_z &= \omega_{x_0}\sin\beta - \omega_{y_0}\sin\alpha\cos\beta + \\
&\qquad + \omega_{z_0}\cos\alpha\cos\beta + \dot{\alpha}\sin\beta.
\end{aligned}\right\} \tag{6.74}$$

Analogously,

$$\left.\begin{aligned}
F_x + Q_x &= Q_{x_0}\cos\beta + Q_{y_0}\sin\alpha\sin\beta - \\
&\qquad - (F_{z_0} + Q_{z_0})\cos\beta\sin\beta, \\
F_y + Q_y &= Q_{x_0}\cos\alpha + (F_{z_0} + Q_{z_0})\sin\alpha.
\end{aligned}\right\} \tag{6.75}$$

Let us assume that conditions (6.54) are satisfied. We then obtain from relations (6.46), (6.75), (6.74), (6.89), (6.70), (6.50), (6.51) and (6.52) the following equations for the motion of the z axis of the pendulum about its position of relative equilibrium: 8

$$\left.\begin{aligned}
&\ddot{\alpha}\cos\beta - 2\dot{\alpha}\dot{\beta}\sin\beta + 2\dot{\beta}(-\omega_{x_0}\sin\beta + \omega_{y_0}\sin\alpha\cos\beta - \\
&\quad - \omega_{z_0}\cos\alpha\cos\beta) + \dot{\omega}_{x_0}(\cos\beta - \cos\alpha) + \\
&\quad + \dot{\omega}_{y_0}\sin\alpha\sin\beta - \dot{\omega}_{z_0}\cos\alpha\sin\beta + \\
&\quad + (\omega_0^2 - \omega_{x_0}^2 - \omega_{y_0}^2(1 - \cos\alpha\cos\beta) - \\
&\quad - \omega_{z_0}^2\cos\alpha\cos\beta)\sin\alpha - \omega_{x_0}\omega_{y_0}\cos\alpha\sin\beta - \\
&\quad - \omega_{x_0}\omega_{z_0}\sin\alpha\sin\beta + \\
&\quad + \omega_{y_0}\omega_{z_0}(\sin^2\alpha\cos\beta - \cos^2\alpha\cos\beta + \cos\alpha) = 0. \\
&\ddot{\beta} + \dot{\alpha}^2\sin\beta\cos\beta + 2\dot{\alpha}(\omega_{x_0}\sin\beta\cos\beta - \\
&\quad - \omega_{y_0}\sin\alpha\cos^2\beta + \omega_{z_0}\cos\alpha\cos^2\beta) + \\
&\quad + \dot{\omega}_{y_0}(\cos\alpha - \cos\beta) + \dot{\omega}_{z_0}\sin\alpha + \dot{\omega}_{x_0}\sin\alpha\sin\beta + \\
&\quad + (\omega_0^2 - \omega_{z_0}^2\cos\alpha\cos\beta - \omega_{y_0}^2)\sin\beta\cos\alpha + \\
&\quad + \omega_{x_0}^2(\cos\beta - \cos\alpha)\sin\beta - \omega_{y_0}^2\sin^2\alpha\sin\beta\cos\beta + \\
&\quad + \omega_{x_0}\omega_{y_0}(\sin^2\beta - \cos^2\beta)\sin\alpha + \\
&\quad + \omega_{x_0}\omega_{z_0}(\cos^2\beta\cos\alpha - \cos\beta - \sin^2\beta\cos\alpha) + \\
&\qquad\qquad + \omega_{y_0}\omega_{z_0}(2\cos\alpha\cos\beta - 1)\sin\alpha\sin\beta = 0.
\end{aligned}\right\} \tag{6.76}$$

It is evident that for small α and β, equations (6.76) reduce to equations (6.67). Equations (6.67) obtain if conditions (6.54) are satisfied. For the case in which the first condition (6.54) and condition (6.56) are satisfied, the corresponding equations are obtained from equations (6.76), if, for ω_{z_0}, the following expression, deriving from (6.56) and the third equality (6.74), is substituted:

$$\omega_{z_0} = \omega_{y_0} \operatorname{tg} \alpha - (\omega_{x_0} + \dot{\alpha}) \frac{\operatorname{tg} \beta}{\cos \alpha}. \tag{6.77}$$

6.3.2. Schuler's gyroscopic-pendulum systems. In §6.3.1 we obtained conditions (6.54) and (6.56) for the existence of a position of relative equilibrium for a compound pendulum under the assumption of motion of its fulcrum on the surface of the earth, regarded as a sphere of radius r_0. If the non-sphericity of the earth is taken into account, the distance from the center of the earth to the fulcrum of the pendulum will be variable.

If in the first condition (6.54), r_0 is assumed to be variable (henceforth denoted by r), then A should also be a variable, i.e., in this case the pendulum is not regarded as a rigid body, but as a variable mechanical system. As a result of the variability of A, the following equations must take the place of equations (6.46):

$$\left.\begin{aligned} \frac{d}{dt}(A\omega_x) + (C - A)\omega_y\omega_z &= M_x, \\ \frac{d}{dt}(A\omega_y) - (C - A)\omega_x\omega_z &= M_y, \\ C\frac{d\omega_z}{dt} &= 0. \end{aligned}\right\} \tag{6.78}$$

From equations (6.78) and (6.48) -- (6.51) and the equalities

$$v_x = r\omega_y, \quad v_y = -r\omega_x, \quad v_z = \dot{r}, \tag{6.79}$$

analogous to equalities (6.52), we obtain in place of equations (6.53) the following equations for the conditions for the existence of a position of relative equilibrium:

$$\left.\begin{aligned} \dot{A}\omega_x + (A - mar)(\dot{\omega}_x - \omega_y\omega_z) + C\omega_y\omega_z - 2\dot{r}am\omega_x &= 0, \\ \dot{A}\omega_y + (A - mar)(\dot{\omega}_y + \omega_x\omega_z) - C\omega_x\omega_z - 2\dot{r}am\omega_y &= 0, \\ C\dot{\omega}_z &= 0. \end{aligned}\right\} \tag{6.80}$$

If

$$A \dot{} = mar, \quad C\omega_z = 0, \tag{6.81}$$

then in each of the two first equalities (6.80) there remains:

$$\dot{A} - 2\dot{r}am = 0, \tag{6.82}$$

or

$$\frac{d}{dt}(mar) - 2\dot{r}am = 0. \tag{6.83}$$

Equalities (6.83) may, of course, be satisfied, if we take[9]

$$a = kr. \tag{6.84}$$

Conditions (6.84) complements the conditions

$$A = mar, \quad C\omega_z = 0 \tag{6.85}$$

obtained above.

The simultaneous fulfillment of requirements (6.84) and (6.85) insures the existence of a position of relative equilibrium when r is variable, which is the case, specifically, for motion on the surface of the terrestrial spheroid.

Let us now generalize the foregoing analysis by considering an arbitrary mechanical system suspended on a moving object in a three-degree-of-freedom suspension such that the center of mass of the system does not coincide with the center of the suspension. We will denote the variable distance between the center of the suspension and the center of mass of the system by a. The system may include various mechanical devices which move relative to one another, including gyroscopes. Therefore, a system of this sort may be termed a gyroscopic-pendulum system. The compound pendulum analyzed above is a special case of a system of this sort. Such well known devices as the Anschutz-Heckeler pitch control gyrocompass and the multi-vertical gyro fall into this class.

Let us attach to this system a trihedron Oxyz (Figure 6.1), with its origin at the center of the suspension, and the z axis along the line connecting the center of mass with the center of the suspension, directed away from the center of mass. The coordinates of the center of mass will then be:

$$z_C = -a, \quad x_C = y_C = 0. \tag{6.86}$$

Let us find the conditions under which the z axis of the system in the position of relative equilibrium will coincide with the direction from the center of the earth, and also let us study the perturbed motion of the z axis about this position.[10]

Let us apply the moment of momentum law to the system as a whole in its motion about the center of mass:

$$\frac{dK}{dt} = M. \tag{6.87}$$

As before, in calculating the moments we will consider the gravitational pull of the earth as reducing to the action of the force

$$F = m \operatorname{grad} V, \tag{6.88}$$

applied to the center of mass of the body and coinciding in direction with the direction of the gravitational field strength . point O. We will ignore the non-homogeneity of the gravitational field within the compass of the system. Adding to the moment of the gravitational force the moment of the inertial forces of translational motion, as well as an artificially formed moment M^*, we obtain:

$$M = a \times \left[F - m\frac{d^2r}{dt^2}\right] + M^*. \tag{6.89}$$

Here, as previously, $\vec{r}$ denotes the radius vector of point O from the center of the earth O_1 and $\vec{a}$ denotes the radius vector of the point C from point O.

Substituting equations (6.89) into equalities (6.87), we obtain the following equations:

$$\frac{dK}{dt} = a \times \left[F - m\frac{d^2 r}{dt^2}\right] + M^{\circ}. \tag{6.90}$$

We will denote the trihedron xyz at the position of relative equilibrium by $x_0 y_0 z_0$. Then

$$\left.\begin{aligned} F &= F_{x_0} x_0 + F_{y_0} y_0 + F_{z_0} z_0, \\ M^{\circ} &= M^{\circ}_{x_0} x_0 + M^{\circ}_{y_0} y_0 + M^{\circ}_{z_0} z_0, \end{aligned}\right\} \tag{6.91}$$

where $\vec{x}_0$, $\vec{y}_0$, $\vec{z}_0$ are the unit vectors of the corresponding axes.

Since at the point of relative equilibrium vector $\vec{a}$ is collinear with vector $\vec{r}$, we find from equations (6.90):

$$\begin{aligned} \frac{dK}{dt} = -a \times m\frac{d^2 r}{dt^2} + (M^{\circ}_{x_0} - aF_{y_0}) x_0 + \\ + (M^{\circ}_{y_0} + aF_{x_0}) y_0 + M^{\circ}_{z_0} z_0. \end{aligned} \tag{6.92}$$

Let

$$M^{\circ}_{y_0} = -aF_{x_0}, \quad M^{\circ}_{x_0} = aF_{y_0}, \quad M^{\circ}_{z_0} = 0. \tag{6.93}$$

Equality (6.92) then takes the form:

$$\frac{dK}{dt} = -a \times m\frac{d^2 r}{dt^2}. \tag{6.94}$$

We now require that the condition

$$a = kr, \tag{6.95}$$

be fulfilled, i.e., we require that the distance from the center of mass of the system to the center of the suspension should change proportionally to the change in the distance to the center of the earth. For this to occur, of course, the system must receive information regarding the vector $\vec{r}$. For the case of motion on the surface of the terrestrial spheroid, this information will be, for example, a priori knowledge of relation (6.4) between $\vec{r}$ and the latitude φ.

If condition (6.95) is satisfied, then in the position of relative equilibrium

$$a = -kr. \tag{6.96}$$

which means that

$$\frac{da}{dt} \times \frac{dr}{dt} = 0, \tag{6.97}$$

and therefore equality (6.94) may be written in the form

$$\frac{dK}{dt} = km \frac{d}{dt}\left(r \times \frac{dr}{dt}\right). \tag{6.98}$$

This expression may now be integrated, as a result of which we obtain:

$$K - kmr \times \frac{dr}{dt} = h. \tag{6.99}$$

ere $\vec{h}$ is a constant vector.

Setting $\vec{h} = 0$, introducing the designation $\vec{v} = d\vec{r}/dt$ and projecting equality (6.99) on the x_0, y_0, z_0 axes, we find:

$$K_{x_0} + mav_{y_0} = 0, \quad K_{y_0} - mav_{x_0} = 0, \quad K_{z_0} = 0. \tag{6.100}$$

Conditions (6.100), (6.93) and (6.95) will be sufficient for the existence of a position of relative equilibrium for an arbitrary gyroscopic-pendulum system, the z axis of which coincides with the direction to the center of the earth.

From these conditions we obtained the above conditions of relative equilibrium for a compound Schuler pendulum, as well as analogous conditions for a twin vertical gyro and a pitch control gyrocompas.[11]

Let us now derive the equations describing the oscillation of an arbitrary gyroscopic-pendulum system satisfying conditions (6.93), (6.95) and (6.100), about its position of relative equilibrium. To do this we will again use the moment of momentum law. Varying equation (6.87) in the vicinity of the position of relative equilibrium, we obtain:

$$\frac{d\delta K}{dt} = \delta M. \tag{6.101}$$

Let us find expressions for the variations $\delta\vec{K}$ and $\delta\vec{M}$. The moment of momentum K of the system should be formed in accordance with equalities (6.99), and therefore

$$\delta K = km\,\delta r \times \frac{dr}{dt} + kmr \times \frac{d\,\delta r}{dt}, \tag{6.102}$$

where

$$\left.\begin{aligned} r = rz_0, \quad \delta r = \delta x x_0 + \delta y y_0 + \delta z z_0, \\ \delta x^2 + \delta y^2 + \delta z^2 + 2r\,\delta z = 0. \end{aligned}\right\} \tag{6.103}$$

To obtain an expression for $\delta\vec{M}$, we must take into account that the variation of the moment is determined only by the variation $\delta\vec{a}$, and also by the variation $\delta\vec{M}^*$ of the corrective moment. Therefore, taking into account relations (6.89), we obtain:

$$\delta M = \delta a \times \left(F - m\frac{d^2\,\delta r}{dt^2}\right) + \delta M^*. \tag{6.104}$$

In this equality $\delta\vec{a}$ may be expressed, in accordance with equality (6.96), in terms of $\delta\vec{r}$:

$$\delta a = -k\,\delta r. \tag{6.105}$$

Differentiating equality (6.102) and noting that

$$km\frac{d\,\delta r}{dt} \times \frac{dr}{dt} + km\frac{dr}{dt} \times \frac{d\,\delta r}{dt} = 0, \tag{6.106}$$

we obtain:

$$\frac{d\,\delta K}{dt} = km\,\delta r \times \frac{d^2 r}{dt^2} + kmr \times \frac{d^2\,\delta r}{dt^2}. \tag{6.107}$$

We must now substitute expression (6.104) and (6.107) in equations (6.101). Before doing this we can simplify expression (6.104) for the variation of the moment. To begin with we may ignore the variation $\delta\vec{M}^*$ of the moment $\vec{M}^*$, which corrects for the action of the horizontal component of the gravitational field. Further, turning to the first equality (6.91), we find that

$$\delta a \times F = \delta a \times (F_{x_0} x_0 + F_{y_0} y_0 + F_{z_0} z_0). \tag{6.108}$$

On the right side of this expression we can ignore the product of $\delta\vec{a}$ times the sum of the first and second terms in parentheses, since this product is of the second order of smallness. With the same level of accuracy we can set

$$F_{z_0} = -\frac{\mu m}{r^2}. \tag{6.109}$$

Then

$$\delta M = \delta a \times \left(-\frac{\mu m}{r^3} z_0 - m \frac{d^2 r}{dt^2} \right). \tag{6.110}$$

Substituting here expression (6.105) for $\delta\vec{a}$, we obtain the formula

$$\delta M = mk\,\delta r \times \left(\frac{\mu}{r^3} z_0 - \frac{d^2 r}{dt^2} \right). \tag{6.111}$$

Let us now substitute expressions (6.111) and (6.107) in equations (6.101). After combining and grouping terms using the first equality (6.103) we obtain:

$$kmr \times \left(\frac{d^2 \delta r}{dt^2} + \frac{\mu\,\delta r}{r^3} \right) = 0. \tag{6.112}$$

This equation is a vector equation describing the oscillations of the z axis of a gyroscopic-pendulum system satisfying the generalized Schuler conditions (6.93), (6.95) and (6.100) about its position of relative equilibrium.

To obtain the scaler form of these equations, we must substitute into (6.112) the expressions for $\vec{r}$ and $\delta\vec{r}$ from equality (6.103) and project the resulting vector relation on the x_0 and y_0 axes. In doing this we must take into account the equalities

$$\left.\begin{aligned} &\frac{dz_0}{dt} \cdot x_0 = \omega_{y_0}, \quad \frac{d^2 z_0}{dt^2} \cdot x_0 = \dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}, \\ &\frac{dz_0}{dt} \cdot y_0 = -\omega_{x_0}, \quad \frac{d^2 z_0}{dt^2} \cdot y_0 = -\dot{\omega}_{x_0} + \omega_{y_0}\omega_{z_0}, \end{aligned}\right\} \tag{6.113}$$

in which ω_{x_0}, ω_{y_0} and ω_{z_0} designate the projections of the absolute angular velocity of trihedron $x_0 y_0 z_0$ on its axes.

Performing the indicated transformations, after obvious simplifications and groupings of terms, we arrive at the equations:

$$
\left.
\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_{y_0}^2 - \omega_{z_0}^2\right)\delta x + \left(\omega_{x_0}\omega_{y_0} - \dot{\omega}_{z_0}\right)\delta y - \\
&\qquad - 2\omega_{z_0}\delta\dot{y} + \left(\omega_{x_0}\omega_{z_0} + \dot{\omega}_{y_0}\right)\delta z + 2\omega_{y_0}\delta\dot{z} = 0. \\
&\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{z_0}^2\right)\delta y + \left(\omega_{x_0}\omega_{y_0} + \dot{\omega}_{z_0}\right)\delta x + \\
&\qquad + 2\omega_{z_0}\delta\dot{x} + \left(\omega_{y_0}\omega_{z_0} - \dot{\omega}_{x_0}\right)\delta z - 2\omega_{x_0}\delta\dot{z} = 0. \\
&\qquad \delta x^2 + \delta y^2 + \delta z^2 + 2r\,\delta z = 0.
\end{aligned}
\right|
\qquad (6.114)
$$

If in equations (6.114) we retain only terms which are linear in δx, δy and δz, then, according to the third equality (6.114), $\delta z = 0$, and in the first two equations (6.114) the last two terms drop out. Equations then take the form:

$$
\left.
\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_{y_0}^2 - \omega_{z_0}^2\right)\delta x + \left(\omega_{x_0}\omega_{y_0} - \dot{\omega}_{z_0}\right)\delta y - 2\omega_{z_0}\delta\dot{y} = 0, \\
&\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{z_0}^2\right)\delta y + \left(\omega_{x_0}\omega_{y_0} + \dot{\omega}_{z_0}\right)\delta x + 2\omega_{z_0}\delta\dot{x} = 0.
\end{aligned}
\right\}
\qquad (6.115)
$$

§6.4. The Analogy Between Gyroscopic-Pendulum Systems and Inertial Systems with Two Newtonometers

At the beginning of the preceding section it was stated that, from the point of view of final results, Schuler pendulum systems are analogous to inertial systems using two newtonometers. Both types of systems permit determination of the bearing to the center of the earth at the current position of the object, i.e., the vertical at this point.

Now that the equations describing the perturbed motion of Schuler gyroscopic-pendulum systems have been compiled, we can see that this analogy reaches much further.

Let us compare the first group of the error equations for an inertial system with two newtonometers and the equations describing the perturbed motion of a Schuler pendulum system about its position of relative equilibrium.

Let us begin with equations valid to a first approximation. For the general case of pendulum systems, these will be equations (6.115), for a compound pendulum, equations (6.67), and for an inertial system with two horizontally oriented newtonometers, equations (6.27). It is evident that the homogeneous equations (6.27) and equations (6.115) coincide fully. It remains to be demonstrated that equations (6.67) for a compound pendulum are also equivalent to equations (6.27) or the homogeneous equation (6.115). To do this it is sufficient to perform the following change of variables in (6.67):

$$\delta x = r_0\beta, \quad \delta y = -r_0\alpha. \tag{6.116}$$

Since in equations (6.67) r_0 is constant, this substitution immediately reduces them to equations (6.27) and (6.115).

Let us now compare equations (6.114) and (6.76) with equations (6.31). Equations (6.114) are exact equations for the natural oscillations of a Schuler pendulum system under conditions of arbitrary motion of its point of support and under the assumption that the earth's gravitational field is spherical. Equations (6.31) are exact error equations derived under the same assumptions for an inertial system with two newtonometers. Equations (6.114) and the homogeneous equations (6.31) fully coincide. Since (6.114) describe the natural vibrations of an arbitrary gyroscopic-pendulum system about its position of relative equilibrium, they clearly also encompass equations (6.76) for the oscillations of a compound pendulum. It is therefore clear that equations (6.76) are also equivalent to the homogeneous equations (6.31). To demonstrate directly that this is the case, it is sufficient to perform the following change of variables in equation (6.31):

$$\left.\begin{array}{c}\delta x = r_0\cos\alpha\sin\beta, \quad \delta y = -r_0\sin\alpha, \\ \delta z = r_0(\cos\alpha\cos\beta - 1),\end{array}\right\} \tag{6.117}$$

the geometrical significance of which is obvious. After solving for $\ddot{\alpha}\cos\beta$ and $\ddot{\beta}$, the homogeneous equations (6.31) reduce to equations (6.76).

It was noted above that equations (6.114) and (6.31) are exact equations if the earth's gravitational field is considered to be spherical. The significance of this remark is as follows. Equations (6.31) and (6.114) were derived for a non-spherical gravitational field.

Later, however, variations of the non-spherical component of the gravitational field strength were disregarded on the basis of their smallness. The disregarding of these terms has the same character in each case. It can be shown that retention of the variation of the non-spherical component of the gravitational field in equations (6.31) and (6.114) leads to the appearance in these equations of additional terms which are totally identical, such that the analogy between pendulum systems and inertial systems with two newtonometers according to the error equations remains complete in this case as well.

The comparison was made above between the equations describing the natural oscillations of pendulum systems and the homogeneous error equations of an inertial system. This comparison is the only one which has significance. If we take into account the instrument errors of pendulum systems,[12] then the right sides of equations (6.67), (6.76), (6.114) and (6.115) assume a form which differs from that of the right sides of equations (6.27) and (6.31), since the sources of error are different. It is useful to recall in this regard that the right sides of equations (6.27) and (6.31) are also to a certain degree conditional, since they contain only the instrument errors of the sensing elements.

We further note the following. Equations (6.114) and (6.115), and equations (6.27) and (6.31), characterize the errors $\delta\vec{r}$ in the determination of the vertical (the direction to the center of the earth) at the current location of the object, since in both cases the error in the modulus $|\vec{r}|$ is assumed equal to zero. This is especially clear from equations (6.67) and (6.76) for the motion of a physical pendulum, in which α and β are the angles of the deviation of its z axis from the direction to the center of the earth. In §4.5 it was also shown that the angles θ_x and θ_y of the deviation of the z axis of the platform of an inertial system are related to the errors δx and δy by the equalities

$$-r\theta_x = \delta y, \quad r\theta_y \doteq \delta x, \tag{6.118}$$

which are fully analogous to equalities (6.116).

Equations (6.27) and (6.31) are only the first group of the error equations. There is in addition to them a second group characterizing the errors in the orientation of the gyroscopes of the system. Thus, the total errors in the determination of the coordinates consist of the errors in the determination of the vertical and of gyroscope errors (drifts).

It has already been noted that a single pendulum system is insufficient for the determination of the coordinates of the object. This requires that gyroscopes be added to the pendulum system. The angles formed by the vertical (the z axis of the pendulum system) and the fixed axes of the gyroscopes will define the coordinates. Consequently, the total coordinate errors will contain the errors in the orientation of the gyroscopes, i.e., the same second group of the error equations. A pendulum system without gyroscopes cannot determine a reference bearing in the plane of the horizontal, for example, the meridian. A gyroscopic-pendulum system can do this. An example of a system of this sort is the pitch control gyrocompass. In this case, the error in the determination of the bearing to the north will be determined by the gyroscope and vertical errors, as in the case of an inertial system using newtonometers.

The above discussion implies a complete dynamic analogy between a navigation system based on a Schuler gyroscopic-pendulum system and an inertial system using two newtonometers. These systems are theoretically equivalent. Indeed, although the pendulum system does not contain newtonometers as such, the pendulum itself, the axis of which coincides with the direction to the center of the earth, is a two-component newtonometer measuring horizontal accelerations. Integration is also effected by the pendulum system itself as a result of its motion, since motion represents the integration of the differential equations which describe it.

§6.5. Analysis of the Error Equations of Two-Newtonometer Systems

6.5.1. General properties. Analysis of stability. As has already been noted, the second group of error equations for two-newtonometer systems is the same as for three-newtonometer systems, and therefore the results obtained in §5.2 are valid here also.

The problem reduces, consequently, to analysis of the first group of the error equations, i.e., equations (6.31), in which r is to be substituted for r_0, considering r to be given by equalities (6.4). They then take the form:

$$\left.\begin{aligned}
&\delta\ddot{x}+\left(\frac{\mu}{r^3}-\omega_y^2-\omega_z^2\right)\delta x+(\omega_x\omega_y-\dot{\omega}_z)\delta y-\\
&\quad-2\omega_z\delta\dot{y}+(\omega_x\omega_z+\dot{\omega}_y)\delta z+2\omega_y\delta\dot{z}=\\
&\qquad=\Delta n_x-\Delta\dot{m}_y r-\omega_z\Delta m_z r-\omega_x\Delta m_x r.\\
&\delta\ddot{y}+\left(\frac{\mu}{r^3}-\omega_z^2-\omega_x^2\right)\delta y+(\omega_y\omega_x-\dot{\omega}_z)\delta x+\\
&\quad+2\omega_z\delta\dot{z}+(\omega_y\omega_z-\dot{\omega}_x)\delta z-2\omega_x\delta\dot{z}=\\
&\qquad=\Delta n_y+\Delta\dot{m}_x r-\omega_y\Delta m_z r-\omega_z\Delta m_y r.\\
&\delta x^2+\delta y^2+\delta z^2+2r\delta z=0.
\end{aligned}\right| \qquad (6.119)$$

To a first approximation these equations reduce, as has already been noted, to the equations

$$\left.\begin{aligned}
&\delta\ddot{x}+\left(\frac{\mu}{r^3}-\omega_y^2-\omega_z^2\right)\delta x+(\omega_x\omega_y-\dot{\omega}_z)\delta y-\\
&\quad-2\omega_z\delta\dot{y}=\Delta n_x-\Delta\dot{m}_y r-\omega_x\Delta m_z r-\omega_z\Delta m_x r.\\
&\delta\ddot{y}+\left(\frac{\mu}{r^3}-\omega_z^2-\omega_x^2\right)\delta y+(\omega_y\omega_x-\dot{\omega}_z)\delta x+\\
&\quad+2\omega_z\delta\dot{x}=\Delta n_y+\Delta\dot{m}_x r-\omega_y\Delta m_z r-\omega_z\Delta m_y r.
\end{aligned}\right| \qquad (6.120)$$

Equations (6.119) and (6.120) retain their form under rotation of trihedron xyz through an arbitrary angle ϑ about the z axis, i.e., allow a group of rotations about this axis.[13] This property derives directly from the fact that equations (6.119) and (6.120) describe the deviation of the z axis of the platform of the inertial system from the direction to the center of the earth. This may also be demonstrated directly by performing the change of variables

$$\left.\begin{aligned}
&\delta x=\delta x'\cos\vartheta-\delta y'\sin\vartheta,\\
&\delta y=\delta x'\sin\vartheta+\delta y'\cos\vartheta,\\
&\delta z=\delta z'
\end{aligned}\right\} \qquad (6.121)$$

and simultaneously converting from the projections ω_x, ω_y, ω_z, Δm_x, Δm_y, Δm_z, Δn_x, Δn_y of the vectors $\vec{\omega}$, $\Delta\vec{m}$, and $\Delta\vec{n}$ on the xyz axes to the projections $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$, $\Delta m_{x'}$, $\Delta m_{y'}$, $\Delta m_{z'}$, $\Delta n_{x'}$ and $\Delta n_{y'}$ of these vectors on the x', y', z' axes, rotated relative to the x, y, z axes through an angle ϑ about the z axis. The formulas for the conversion from the projections ω_x, ω_y, Δm_x, Δm_y, Δn_x and Δn_y to the projections $\omega_{x'}$, $\omega_{y'}$, $\Delta m_{x'}$, $\Delta m_{y'}$, $\Delta n_{x'}$, and $\Delta n_{y'}$ are analogous to formulas (6.121) for the conversion from δx and δy to $\delta x'$ and $\delta y'$. The conversion from ω_z and Δm_z to $\omega_{z'}$ and $\Delta m_{z'}$, respectively, is given by the relations

$$\omega_{z'} = \omega_z + \dot{\theta}, \quad \Delta m_{z'} = \Delta m_{z'}. \tag{6.122}$$

This property of equations (6.119) and (6.120) permits selection of the orientation of the xyz trihedron relative to the points of the compass in such a way as to facilitate the analysis as much as possible.

The most interesting of the possible orientations of this trihedron is the azimuth-free trihedron, for which

$$\dot{\theta} = -\omega_z \quad \text{или} \quad \omega_{z'} = 0. \tag{6.123}$$

For example, conversion to this trihedron causes equation (6.120) to reduce to the following equations:

$$\left.\begin{aligned} \delta\ddot{x}' + \left(\frac{\mu}{r^3} - \omega_{y'}^2\right)\delta x' + \omega_{x'}\omega_{y'}\delta y' &= \\ = \Delta n_{x'} - \Delta\dot{m}_{y'} r - \omega_{x'}\Delta m_z r&, \\ \delta\ddot{y}' + \left(\frac{\mu}{r^3} - \omega_{x'}^2\right)\delta y' + \omega_{x'}\omega_{y'}\delta x' &= \\ = \Delta n_{y'} + \Delta\dot{m}_{x'} r - \omega_{y'}\Delta m_z r&. \end{aligned}\right| \tag{6.124}$$

These equations are obtained, clearly, from equations (6.120), if in the latter $\omega_z = 0$. Equations (6.124) are distinguished by the absence in them of the projection ω_z, which can be large even for small velocities of the object over the surface of the earth. The quantities $\omega_{x'}$ and $\omega_{y'}$ are related to the horizontal projections $v_{x'}$ and $v_{y'}$ of the velocity of the object by the equalities

$$v_{x'} = r\omega_{y'}, \quad v_{y'} = -r\omega_{x'} \tag{6.125}$$

and as a result are always bounded. This permits the use of approximation methods in solving equations (6.124).

In the general case of arbitrary motion on the surface of the earth, equations (6.119), (6.120) and (6.124) are equations with variable coefficients and their analysis entails insuperable difficulties. Let us consider, for example, motion along the equator. If the x axis is directed along the equator, then $\omega_x = \omega_z = 0$ and the homogeneous equations (6.120) take the form:

$$\left.\begin{aligned} \delta\ddot{x} + \left(\frac{\mu}{a^3} - \omega_y^2\right)\delta x = 0, \\ \delta\ddot{y} + \frac{\mu}{a^3}\delta y = 0. \end{aligned}\right\} \qquad (6.126)$$

For special selection of the range of $\omega_y(t)$, the first equation (6.126) can be reduced, for example, to the Mathieu-Hill equation[14]. In the case of more general motion the equations can, of course, be even more complex.

It is possible to fully analyze equations (6.119) and (6.120) only for the case of motion at a constant velocity v_0 along a parallel of latitude, for which the coefficients of these equations may be considered as constant. Indeed, if the xyz trihedron is oriented to the points of the compass, with the y axis, for example, directed toward the north,

$$\left.\begin{aligned} &\omega_x = 0, \quad \omega_y = u\cos\varphi + \frac{v_0}{r} = \text{const}, \\ &\omega_z = u\sin\varphi + \frac{v_0}{r}\operatorname{tg}\varphi = \text{const}, \quad \frac{\mu}{r^3} = \omega_0^2 = \text{const}, \end{aligned}\right\} \qquad (6.127)$$

and the coefficients of equations (6.119) and (6.120) in fact become constant.

Let us analyze the stability of a two-newtonometer inertial system. To do this we will begin by using the first-approximation equations (6.120). These equations are linear. For motion at constant velocity along a parallel, they take the form:

$$\left.\begin{aligned} &\delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x - 2\omega_z\delta\dot{y} = \\ &\qquad = -\Delta n_x - \Delta\dot{m}_y r - \Delta m_x\omega_z r, \\ &\delta\ddot{y} + (\omega_0^2 - \omega_z^2)\delta y + 2\omega_z\delta\dot{x} = \\ &\qquad = -\Delta n_y + \Delta\dot{m}_x r - \omega_y\Delta m_z r - \Delta m_y\omega_z r. \end{aligned}\right\} \qquad (6.128)$$

Here the coefficients are constant.

The characteristic equation of the homogeneous system (6.128) is the biquadratic equation

$$p^4 + p^2(2\omega_0^2 - \omega_y^2 + 2\omega_z^2) + (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2) = 0. \tag{6.129}$$

For the motion to be stable (of course, only non-asymptotic stability is being considered here), it is necessary that the roots of the characteristic equation (6.129) be zero or purely imaginary. For this it is necessary that the roots of the quadratic equation

$$q^2 + q(2\omega_0^2 - \omega_y^2 + 2\omega_z^2) + (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2) = 0 \tag{6.130}$$

be real and non-positive. This, in turn, will be the case if the discriminant Λ of equation (6.130) is non-positive:

$$\Lambda = 2\omega_z^2\omega_y^2 - \frac{\omega_y^4}{4} - 4\omega_0^2\omega_z^2 \leqslant 0. \tag{6.131}$$

at the same time that it's coefficients are non-negative:

$$\left.\begin{aligned} &2\omega_0^2 - \omega_y^2 + 2\omega_z^2 \geqslant 0, \\ &(\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2) \geqslant 0. \end{aligned}\right\} \tag{6.132}$$

Let us consider the plane of the parameters ω_y^2 and ω_z^2 (Figure 6.3). It follows from equalities (6.132) that the first coefficient of equations (6.130) is positive above the line

$$\omega_z^2 = -\omega_0^2 + \frac{\omega_y^2}{2} \tag{6.133}$$

and equal to zero on this line.

The second coefficient is positive above the line

$$\omega_z^2 = \omega_0^2. \tag{6.134}$$

beneath the line

$$\omega_z^2 = \omega_0^2 - \omega_y^2 \tag{6.135}$$

and equal to zero on these lines.

The equality (6.131) divides the plane of parameters ω_y^2 and ω_z^2 into three regions. The curve forming the boundaries of these regions is the hyperbola

$$\omega_z^2 = \frac{\omega_y^4}{8(\omega_y^2 - 2\omega_0^2)}. \tag{6.136}$$

The asymptotes of the hyperbola are the lines

$$\omega_y^2 = 2\omega_0^2, \quad \omega_z^2 = \frac{\omega_0^2}{4} + \frac{\omega_y^2}{8}. \tag{6.137}$$

One of the branches of the hyperbola passes through the point $(\omega_y^2 = 0, \omega_z^2 = 0)$ and lies in the lower half-plane. The second branch

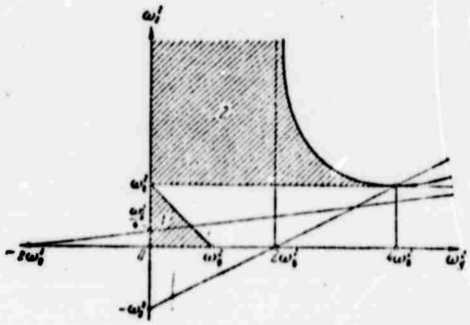

Figure 6.3

lies entirely in the upper half-plane and is tangent to the line $\omega_z^2 = \omega_0^2$ at the point where

$$\omega_y^2 = 4\omega_0^2. \tag{6.138}$$

The discriminant (6.131) of equations (6.130) is non-positive in the region between the branches of the hyperbola. Considering that, in view of the fact that ω_y^2 and ω_z^2 are positive, we are interested only in the first quadrant of the ω_y^2 -- ω_z^2 plane, we arrive at the conclusion that the motion defined by equation (6.128) can be stable only in two regions. Region 1 (Figure 6.3) is bounded by the lines

$$\omega_y^2 = 0, \quad \omega_z^2 = 0, \quad \omega_0^2 - \omega_y^2 - \omega_z^2 = 0 \tag{6.139}$$

and is defined by the inequality:

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0. \tag{6.140}$$

Region 2 is bounded by the lines

$$\omega_y^2 = 0, \quad \omega_z^2 = \omega_0^2 \tag{6.141}$$

and by part of the upper branch of the hyperbola between the points $(2\omega_0^2, \infty)$ and $(4\omega_0^2, \omega_0^2)$.

Within these regions the roots of the characteristic equation (6.129) are purely imaginary.

At the boundaries of the regions of stability at the point $(\omega_y^2 = 0, \omega_z^2 = 0)$,

$$p_{1,2} = p_{3,4} = \pm j\omega_0, \tag{6.142}$$

i.e., two double imaginary roots appear.

On the section of line $\omega_z^2 = 0$ between the points

$$\omega_y^2 = 0, \quad \omega_y^2 = \omega_0^2 \tag{6.143}$$

we obtain:

$$p_{1,2} = \pm j\omega_0, \quad p_{3,4} = \pm j\sqrt{\omega_0^2 - \omega_y^2}, \tag{6.144}$$

i.e., distinct imaginary roots.

At the point

$$\omega_z^2 = 0, \quad \omega_y^2 = \omega_0^2 \tag{6.145}$$

we obtain:

$$p_{1,2} = \pm j\omega_0, \quad p_{3,4} = 0, \tag{6.146}$$

i.e., a double zero root appears.

On the line segment

$$\omega_y^2 = 0, \quad 0 < \omega_z^2 < \omega_0^2, \tag{6.147}$$

we have:

$$p_{1,2,3,4} = \pm j(\omega_0 \pm \omega_z). \tag{6.148}$$

i.e., here, as on segment (6.143), the roots are imaginary and distinct.

At the point

$$\omega_y^2 = 0, \quad \omega_z^2 = \omega_0^2 \tag{6.149}$$

the roots are the following:

$$p_{1,2} = 0, \quad p_{3,4} = \pm j2\omega_0, \tag{6.150}$$

i.e., another double zero root.

Further, on the line

$$\omega_0^2 - \omega_y^2 - \omega_z^2 = 0 \tag{6.151}$$

between the points $(0, \omega_0^2)$ and $(\omega_0^2, 0)$, we obtain:

$$p_{1,2} = 0, \quad p_{3,4} = \pm j\sqrt{4\omega_0^2 - 3\omega_y^2}, \tag{6.152}$$

i.e., a double zero root and two imaginary conjugate roots.

Let us now consider the half line

$$\omega_y^2 = 0, \quad \omega_z^2 > \omega_0^2. \tag{6.153}$$

Here

$$p_{1,2,3,4} = \pm j(\omega_0 \mp \omega_z), \tag{6.154}$$

i.e., all of the roots are distinct.

On the segment

$$\omega_z^2 = \omega_0^2, \quad 0 < \omega_y^2 < 4\omega_0^2, \tag{6.155}$$

we have:

$$p_{1,2} = 0, \quad p_{3,4} = \pm j\sqrt{4\omega_0^2 - \omega_y^2}. \tag{6.156}$$

At the point

$$\omega_z^2 = \omega_0^2, \quad \omega_y^2 = 4\omega_0^2, \tag{6.157}$$

clearly,

$$p_{1,2,3,4} = 0, \tag{6.158}$$

i.e., a quadruple zero root.

Finally, on the section of the hyperbola where

$$2\omega_0^2 < \omega_y^2 < 4\omega_0^2, \tag{6.159}$$

the discriminant (6.131) of equation (6.130) is equal to zero, and therefore

$$p_{1,2} = i\sqrt{\omega_0^2 + \omega_z^2 - \omega_y^2/2}, \quad p_{3,4} = -i\sqrt{\omega_0^2 + \omega_z^2 - \omega_y^2/2}, \tag{6.160}$$

i.e., two double imaginary roots appear here.

In all cases in which multiple roots appear it is necessary to check whether the elementary divisors of the characteristic matrix of system (6.128) remain linear.

Using the normal procedure for finding the elementary divisors of a characteristic matrix,[15] we arrive at the following results.

The portion of the boundaries of the region of stability formed by the lines $\omega_y^2 = 0$ and $\omega_z^2 = 0$ with the exception of the point ($\omega_y^2 = \omega_0^2$, $\omega_z^2 = 0$), is included in the region of stability, since linear elementary divisors correspond to the multiple roots.

The boundaries by the hyperbola, the straight line $\omega_z^2 = \omega_0^2$ with the exception of point $(0, \omega_0^2)$ and the straight line $\omega_0^2 - \omega_y^2 - \omega_z^2 = 0$, with the exception of the same point, do not fall within the region of stability. On the hyperbola the multiple roots (6.160) correspond to second-degree elementary divisors of the characteristic matrix; on the line $\omega_z^2 = \omega_0^2$, the null root (6.156) corresponds to second-degree elementary divisors; on the line $\omega_0^2 - \omega_y^2 - \omega_z^2 = 0$ there is also a double zero root with second-degree elementary divisors. At the point $(4\omega_0^2, \omega_0^2)$ the quadruple zero root corresponds to second-degree elementary divisors.

In analyzing the stability of the system according to equations (6.128), it is possible to avoid analysis of the characteristic equation and the characteristic matrix by making use of the fact that equations (6.128) may be regarded as equations describing two-dimensional motion in the vicinity of the equilibrium position

$\delta x = 0$ and $\delta y = 0$ of a mechanical system under the influence of potential and gyroscopic forces.

In this case the force function will have the form:

$$U = -\frac{1}{2}\left[(\omega_0^2 - \omega_y^2 - \omega_z^2)(\delta x)^2 + (\omega_0^2 - \omega_y^2)(\delta y)^2\right]. \tag{6.161}$$

The gyroscopic forces may be represented by the expressions

$$-2\omega_z \delta \dot{y}, \quad 2\omega_z \delta \dot{x}. \tag{6.162}$$

If we discard the gyroscopic forces in the homogeneous system (6.128), only potential forces will remain. For equilibrium stability under the influence of only potential forces, the force function (6.161) should have a maximum at the equilibrium point.

In this case a maximum will occur, clearly, if

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0. \tag{6.163}$$

If this condition is satisfied, the system is stable even with the addition of gyroscopic forces, as follows from the well-known Kelvin theorem.[16]

Comparison of inequality (6.163) with the results obtained previously from direct analysis of the characteristic equation (6.129) shows that condition (6.163) is only one of the two regions of stability found from analysis of the characteristic equation.

The second region appears as a result of the gyroscopic forces (6.162).

The following statement is valid: if the degree of instability of a system under the influence only of potential forces is even, then the introduction of appropriate gyroscopic forces may render the system stable.[17]

If the system in question is unstable under the influence of potential forces, the inverse inequality

$$\omega_0^2 - \omega_z^2 < 0. \tag{6.164}$$

is valid in place of inequality (6.163).

The degree of instability must be even if, in addition to condition (6.164), the equality

$$\omega_0^2 - \omega_y^2 - \omega_z^2 < 0. \tag{6.165}$$

obtains.

But the second region of stability obtained as a result of analysis of the characteristic equation, satisfies, in addition to the other conditions, inequality (6.165). This shows that it appears as a result of the gyroscopic forces.

The above stability analysis was carried out on the basis of equations (6.120). These, however, are linearized first approximations. The characteristic equation has zero or purely imaginary roots in the regions of stability obtained as a result of this analysis. Therefore, we cannot draw any final conclusions with regard to stability on the basis of this first approximation. The conditions of stability obtained from these equations are only necessary conditions. To obtain the sufficient conditions, we must analyze the exact equations (6.119).

Here we can use the analogy with Schuler's compound pendulum, i.e., we can analyze, instead of the homogeneous equation (6.119), equations (6.76), which for the case of motion of an object along a parallel, take the form:

$$\left.\begin{aligned}
&\ddot{\alpha}\cos\beta - 2\dot{\alpha}\dot{\beta}\sin\beta + \\
&\quad + 2\dot{\beta}(\omega_{y_0}\sin\alpha\cos\beta - \omega_{z_0}\cos\alpha\cos\beta) + \\
&+ [\omega_0^2 - \omega_{y_0}^2(1-\cos\alpha\cos\beta) - \omega_{z_0}^2\cos\alpha\cos\beta]\sin\alpha + \\
&\quad + \omega_{y_0}\omega_{z_0}(\sin^2\alpha\cos\beta - \cos^2\alpha\cos\beta + \cos\alpha) = 0, \\
&\ddot{\beta} + \dot{\alpha}^2\sin\beta\cos\beta - 2\dot{\alpha}(\omega_{y_0}\sin\alpha\cos^2\beta - \\
&- \omega_{z_0}\cos\alpha\cos^2\beta) + (\omega_0^2 - \omega_{z_0}^2\cos\alpha\cos\beta - \omega_{y_0}^2)\times \\
&\quad \times\sin\beta\cos\alpha - \omega_{y_0}^2\sin^2\alpha\sin\beta\cos\beta + \\
&\quad + \omega_{y_0}\omega_{z_0}(2\cos\alpha\cos\beta - 1)\sin\alpha\sin\beta = 0.
\end{aligned}\right| \tag{6.166}$$

Equations (6.166) have a first integral -- the energy integral. To obtain it it is sufficient to multiple the first equation (6.166) by $\dot\beta$, the second by $\dot\alpha\cos\beta$, and then to add them. Integration of the sum yields:

$$\left.\begin{aligned} V = {} & (\dot\alpha\cos\beta)^2 + \dot\beta^2 - 2\omega_0^2\cos\alpha\cos\beta + \\ & + \omega_{z_0}^2\cos^2\alpha\cos^2\beta + \omega_{y_0}^2(\sin^2\alpha\cos^2\beta + 2\cos\alpha\cos\beta) + \\ & + 2\omega_{y_0}\omega_{z_0}(\sin\alpha\cos\beta - \sin\alpha\cos\alpha\cos^2\beta) = \text{const.} \end{aligned}\right\} \tag{6.167}$$

The function

$$W = V - V(0) \tag{6.168}$$

may be taken as a Lyapunov function for purposes of analysis of the stability of the position of relative equilibrium.

Expanding the function W in a series of powers of $\dot\alpha$, $\dot\beta$, α and β, we find that the expansion begins with the quadratic terms:

$$W = \dot\alpha^2 + \dot\beta^2 + \alpha^2(\omega_0^2 - \omega_{z_0}^2) + \beta^2(\omega_0^2 - \omega_{y_0}^2 - \omega_{z_0}^2) + \cdots \tag{6.169}$$

The quadratic form on the right side of this equality, and also, therefore, the function W, will be positively defined if the condition [18]

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0. \tag{6.170}$$

is fulfilled.

At the same time the total time-derivative of the function W is equal to zero by virtue of equations (6.166). Therefore, conditions (6.170) will be, according to the well-known Lyapunov theorem, a sufficient condition of stability.

Condition (6.170) coincides with condition (6.140) and (6.163). In the plane of the parameters ω_z^2 and ω_y^2, the area defined by condition (6.170) coincides with the first region of stability obtained from analysis of the characteristic equation (6.129). The second region cannot be obtained from stability analysis on the basis of the Lyapunov function and the energy integral. The appearance of this region is associated, as was noted above, with gyroscopic forces which do not enter into the energy integral, since, according to the definition of gyroscopic forces, they do not perform work over real displacements.

Let us compare the results of the stability analysis of the two-newtonometer system with the results of the analysis of the three-newtonometer system performed in §5.3.

Stability of a three-newtonometer system for motion along a parallel requires that the following conditions be satisfied:

$$\left.\begin{aligned} \omega_0^2 - \omega_y^2 - \omega_z^2 < 0, \\ 2\omega_0^2 - 2\omega_z^2 + \omega_y^2 > 0. \end{aligned}\right\} \tag{6.171}$$

The first of these conditions excludes a region of the ω_y^2 -- ω_z^2 plane which is important for certain applications (region 1 in Figure 6.3). The fulfillment of condition (6.171) renders stability attainable only by means of gyroscopic forces.

The two-newtonometer system is stable in the region

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0, \tag{6.172}$$

and stability is guaranteed here by potential forces.[19] This system also has a second region of stability, in which stabilization is guaranteed by gyroscopic forces.

It is necessary in this regard to note the following. It is well known[20] that if a system under the influence of potential forces is unstable, and its stability (for an even degree of instability) is guaranteed by gyroscopic forces, then the resulting stability is disturbed by forces of overall internal dissipation. This stability has, therefore, an intermittent character, as distinct from stabilization by potential forces, for which stabilization has a permanent character. In the latter case the presence of forces of overall internal dissipation results in increased stability.

The introduction of forces of overall internal dissipation into the error equations (5.121) of a three-newtonometer system therefore destroys the region of stability characterized by (6.171), and the introduction of dissipation into equations (6.120) results in the destruction of the second region of stability of the two-newtonometer system. To demonstrate this we may use the Hurwitz criterion[21] or the direct Lyapunov methods, taking as the Lyapunov function the energy integral and introducing a dissipation function into the analysis.

On the basis of the fact that energy-dissipating forces act on a particle system over any real displacement, regions of stability, in which stabilization is guaranteed by gyroscopic forces, are sometimes ignored. It is necessary here to keep in mind, however, that for small dissipative forces and limited system operating time, the process of the disruption of gyroscopic stability may not be able to develop.

6.5.2. Solution of the second group of error equations for the case of motion of an object along a parallel. Let us solve the error equations of the first group for motion along a parallel, i.e., equations (6.128), in which the coefficients are given by relations (6.127).

If the condition

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0$$

is satisfied, the characteristic equation (6.129) of system (6.128) has two pairs of purely imaginary conjugate roots:

$$p_{1,2} = \pm j\mu, \quad p_{3,4} = \pm j\nu. \qquad (6.173)$$

where

$$
\left.\begin{aligned}
\mu &= \left\{ -\frac{\omega_y^2}{2} + \omega_z^2 + \omega_0^2 + \sqrt{\left(\frac{\omega_y^2}{2} - \omega_z^2 - \omega_0^2\right)^2 - (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2)} \right\}^{1/2}, \\
\nu &= \left\{ -\frac{\omega_y^2}{2} + \omega_z^2 + \omega_0^2 - \sqrt{\left(\frac{\omega_y^2}{2} - \omega_z^2 - \omega_0^2\right)^2 - (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2)} \right\}^{1/2}
\end{aligned}\right| \qquad (6.174)
$$

The following functions, therefore, constitute the general solution to the homogeneous equations (6.128):

$$
\left.\begin{aligned}
\delta x &= \frac{C_1}{b\mu - a\nu}(b\mu\cos\nu t - a\nu\cos\mu t) + \frac{abC_2}{b\nu - a\mu}(\mu\sin\nu t - \nu\sin\mu t) + \\
&\quad + \frac{C_3}{b\nu - a\mu}(b\sin\nu t - a\sin\mu t) + \frac{abC_4}{b\mu - a\nu}(\cos\mu t - \cos\nu t). \\
\delta y &= \frac{C_1}{b\mu - a\nu}(\mu\sin\nu t - \nu\sin\mu t) + \frac{C_2}{b\nu - a\mu}(b\nu\cos\mu t - a\mu\cos\nu t) + \\
&\quad + \frac{C_3}{b\nu - a\mu}(\cos\mu t - \cos\nu t) + \frac{C_4}{b\mu - a\nu}(b\sin\mu t - a\sin\nu t).
\end{aligned}\right| \qquad (6.175)
$$

Here C_1, C_2, C_3 and C_4 are arbitrary constants corresponding to δx^0, δy^0, $\delta\dot{x}^0$, $\delta\dot{y}^0$. The quantities a and b are expressed in terms of the moduli (6.174) of the roots of the characteristic equation and the coefficients of the initial system (6.128):

$$
a = \frac{\omega_0^2 - \omega_z^2 - \mu^2}{2\mu\omega_z}, \quad b = \frac{\omega_0^2 - \omega_z^2 - \nu^2}{2\nu\omega_z}. \qquad (6.176)
$$

The solution to the non-homogeneous system (6.128) may be obtained from formulas (6.175) by varying the arbitrary constants C_1, C_2, C_3 and C_4. It has the following form:

$$
\left.\begin{aligned}
\delta x &= \int_0^t \left\{ \frac{f_1(\tau)}{b\nu - a\mu}[-a\sin\mu(t-\tau) + b\sin\nu(t-\tau)] + \frac{abf_2(\tau)}{b\mu - a\nu}[\cos\mu(t-\tau) - \cos\nu(t-\tau)] \right\} d\tau + \\
&\quad + \frac{\delta x^0}{b\mu - a\nu}(b\mu\cos\nu t - a\nu\cos\mu t) + \frac{\delta\dot{x}^0}{b\nu - a\mu}(b\sin\nu t - a\sin\mu t) + \\
&\quad + \frac{\delta y^0 ab}{b\nu - a\mu}(\mu\sin\nu t - \nu\sin\mu t) + \frac{\delta\dot{y}^0 ab}{b\mu - a\nu}(\cos\mu t - \cos\nu t),
\end{aligned}\right| \qquad (6.177)
$$

$$\begin{aligned}\delta y = \int_0^t \Big\{ & \frac{f_1(\tau)}{b\nu - a\mu}[\cos\mu(t-\tau) - \cos\nu(t-\tau)] + \\ & + \frac{f_2(\tau)}{b\mu - a\nu}[b\sin\mu(t-\tau) - a\sin\nu(t-\tau)]\Big\} d\tau + \\ & + \frac{\delta y^0}{b\nu - a\mu}(b\nu\cos\mu t - a\mu\cos\nu t) + \\ & + \frac{\delta \dot{y}^0}{b\mu - a\nu}(b\sin\mu t - a\sin\nu t) + \\ & + \frac{\delta x^0}{b\mu - a\nu}(\mu\sin\nu t - \nu\sin\mu t) + \\ & + \frac{\delta \dot{x}^0}{b\nu - a\mu}(\cos\mu t - \cos\nu t).\end{aligned} \tag{6.177}$$

where f_1 and f_2 denote the right sides of equations (6.128), such that

$$\left.\begin{aligned} f_1 &= \Delta n_x - \Delta \dot{m}_y r - \Delta m_x \omega_z r, \\ f_2 &= \Delta n_y + \Delta \dot{m}_x r - \Delta m_z \omega_y r - \Delta m_y \omega_z r. \end{aligned}\right\} \tag{6.178}$$

If $\omega_z = 0$, as, for example, in the case of motion along the equator, then, according to relations (6.173) and (6.174),

$$\mu = \omega_0, \quad \nu = \sqrt{\omega_0^2 - \omega_y^2}. \tag{6.179}$$

Noting that, according to equalities (6.176)

$$\lim_{\omega_z \to 0} a = 0, \quad \lim_{\omega_z \to 0} b = \infty, \tag{6.180}$$

by passage to the limit in formulas (6.177) we obtain in this case the following expressions for δx and δy:

$$\left.\begin{aligned} \delta x = {} & \delta x^0 \cos\sqrt{\omega_0^2 - \omega_y^2}\, t + \\ & + \frac{\delta \dot{x}^0 + r\Delta m_y^0}{\sqrt{\omega_0^2 - \omega_y^2}} \sin\sqrt{\omega_0^2 - \omega_y^2}\, t + \\ & + \frac{1}{\sqrt{\omega_0^2 - \omega_y^2}} \int_0^t \Delta n_x(\tau) \sin\sqrt{\omega_0^2 - \omega_y^2}\,(t-\tau)\, d\tau - \\ & - r\int_0^t \Delta m_y(\tau) \cos\sqrt{\omega_0^2 - \omega_y^2}\,(t-\tau)\, d\tau, \\ \delta y = {} & \delta y^0 \cos\omega_0 t + \frac{\delta \dot{y}^0 - r\Delta m_x^0}{\omega_0} \sin\omega_0 t + \\ & + \frac{1}{\omega_0}\int_0^t (\Delta n_y - \omega_y \Delta m_z r) \sin\omega_0 (t-\tau)\, d\tau + \\ & + r\int_0^t \Delta m_x(\tau) \cos\omega_0 (t-\tau)\, d\tau. \end{aligned}\right| \tag{6.181}$$

For constant Δn_x, Δn_y, Δm_x, Δm_y and Δm_z, formulas (6.181) reduce after intergration to the following expressions for δx and δy:

$$\left.\begin{aligned} \delta x &= \frac{\Delta n_x}{\omega_0^2-\omega_y^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2-\omega_y^2}\right)\cos\sqrt{\omega_0^2-\omega_y^2}\,t + \\ &\quad + \frac{\delta\dot{x}^0}{\sqrt{\omega_0^2-\omega_y^2}}\sin\sqrt{\omega_0^2-\omega_y^2}\,t, \\ \delta y &= \frac{\Delta n_y - \omega_y r\,\Delta m_z}{\omega_0^2} + \\ &\quad + \left(\delta y^0 - \frac{\Delta n_y - \omega_y r\,\Delta m_z}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t. \end{aligned}\right| \tag{6.182}$$

Finally, if the object is fixed in the $O_1\xi_*\eta_*\zeta_*$ coordinate ystem, then $\omega_y = 0$. It then follows from expressions (6.181) that

$$\left.\begin{aligned} \delta x &= \delta x^0\cos\omega_0 t + \frac{\delta\dot{x}^0 + r\,\Delta m_y}{\omega_0}\sin\omega_0 t + \\ &\quad + \frac{1}{\omega_0}\int_0^t \Delta n_x(\tau)\sin\omega_0(t-\tau)\,d\tau - \\ &\quad - r\int_0^t \Delta m_y(\tau)\cos\omega_0(t-\tau)\,d\tau, \\ \delta y &= \delta y^0\cos\omega_0 t + \frac{\delta\dot{y}^0 - r\,\Delta m_x^0}{\omega_0}\sin\omega_0 t + \\ &\quad + \frac{1}{\omega_0}\int_0^t \Delta n_y(\tau)\sin\omega_0(t-\tau)\,d\tau + \\ &\quad + r\int_0^t \Delta m_x(\tau)\cos\omega_0(t-\tau)\,d\tau. \end{aligned}\right| \tag{6.183}$$

and for constant instrument errors we correspondingly obtain:

$$\left.\begin{aligned} \delta x &= \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t, \\ \delta y &= \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t. \end{aligned}\right| \tag{6.184}$$

If $\omega_y = 0$, but $\omega_z \neq 0$, equations (6.182) reduce by the change variables (6.121) to the form which they have when $\omega_z = 0$.

Although the exact solution (6.177) to equations (6.128) for the case of motion along an arbitrary parallel can be inspected, it is extremely unwieldy. If we assume that

$$\omega_y^2 \ll \omega_0^2, \tag{6.185}$$

we can obtain a simpler approximate solution to equations (6.128) which is analogous to the solution to equations (5.121) obtained in §5.3.

Performing in equations (6.128) the change of variables (6.121), ·e obtain the system (6.1•2). Since it follows from conditions (6.185)

$$\omega_0^2 \gg \omega_x^2, \quad \omega_0^2 \gg \omega_y^2, \quad \omega_0^2 \gg |\omega_x \omega_y|, \tag{6.186}$$

e may consider the following equations as a first approximation:

$$\left.\begin{aligned} \delta\ddot{x}' + \omega_0^2 \delta x' &= \Delta n_x - \Delta\dot{m}_y r - \omega_x \Delta m_z r, \\ \delta\ddot{y}' + \omega_0^2 \delta y' &= \Delta n_y + \Delta\dot{m}_x r - \omega_y \Delta m_z r. \end{aligned}\right\} \tag{6.187}$$

We note that the use of equations (6.187) as a first approximation is also possible for arbitrary motion, for which ω_x, ω_y and ω_z are not constant, but rather functions of time. This requires only that conditions (6.186), which apply also to variable ω_x, ω_y, and ω_z, be satisfied. This is an especially useful possibility, because, for motion on the surface of the earth or in the immediate vicinity of its surface, velocities are generally low.

If f_1' and f_2' designate the right sides of equation (6.187), the solution to these equations may be obtained in the form

$$\left.\begin{aligned} \delta x' = \frac{1}{\omega_0}\int_0^t f_1'(\tau)\sin\omega_0(t-\tau)\,d\tau + {} \\ + \delta x'^0 \cos\omega_0 t + \frac{\delta\dot{x}'^0}{\omega_0}\sin\omega_0 t, \\ \delta y' = \frac{1}{\omega_0}\int_0^t f_2'(\tau)\sin\omega_0(t-\tau)\,d\tau + {} \\ + \delta y'^0 \cos\omega_0 t + \frac{\delta\dot{y}'^0}{\omega_0}\sin\omega_0 t. \end{aligned}\right\} \tag{6.188}$$

According to equalities (6.121), here

$$\left.\begin{aligned} \delta x'^0 = \delta x^0, \quad \delta y'^0 = \delta y^0, \\ \delta\dot{x}'^0 = \delta\dot{x}^0 + \delta y^0 \omega_z^0, \quad \delta\dot{y}'^0 = \delta\dot{y}^0 - \delta x^0 \omega_z^0. \end{aligned}\right\} \tag{6.189}$$

Now converting back from δx' and δy' to δx and δy, we obtain the following approximate solutions to equations (6.128) for the case of motion of an object at constant velocity along a parallel:

$$\left.\begin{aligned}
\delta x = \frac{1}{\omega_0}\int_0^t [f_1(\tau)\cos\omega_z(t-\tau) + \\
+ f_2(\tau)\sin\omega_z(t-\tau)]\sin\omega_0(t-\tau)\,d\tau + \\
+ (\delta x^0\cos\omega_z t + \delta y^0\sin\omega_z t)\cos\omega_0 t + \\
+ \frac{1}{\omega_0}[(\delta\dot{x}^0 + \delta y^0\omega_z)\cos\omega_z t + \\
+ (\delta\dot{y}^0 - \delta x^0\omega_z)\sin\omega_z t]\sin\omega_0 t, \\
\delta y = \frac{1}{\omega_0}\int_0^t [-f_1(\tau)\sin\omega_z(t-\tau) + \\
+ f_2(\tau)\cos\omega_z(t-\tau)]\sin\omega_0(t-\tau)\,d\tau + \\
+ (-\delta x^0\sin\omega_z t + \delta y^0\cos\omega_z t)\cos\omega_0 t + \\
+ \frac{1}{\omega_0}[-(\delta\dot{x}^0 + \delta y^0\omega_z)\sin\omega_z t + \\
+ (\delta\dot{y}^0 - \delta x^0\omega_z)\cos\omega_z t]\sin\omega_0 t.
\end{aligned}\right| \tag{6.190}$$

We will show that formulas (6.190) may be obtained by direct simplification of the exact solution (6.177) to equations (6.128). Without loss of generality we may here limit ourselves to the case of an object which is motionless relative to the earth, for which

$$\omega_y = u\cos\varphi, \quad \omega_z = u\sin\varphi. \tag{6.191}$$

For $\varphi = 0$, i.e., on the equator, equations (6.128) split: $\mu = \omega_0$, $\nu = \sqrt{\omega_0^2 - u^2}$. The corresponding solution is obtained from relations (6.177) by passage to the limit, since

$$\lim_{\varphi\to 0} a = 0, \quad \lim_{\varphi\to 0} b = \infty. \tag{6.192}$$

For $\varphi = \pi/2$, i.e., at a pole,

$$\left.\begin{aligned}
\mu = \omega_0 + u, \quad \nu = \omega_0 - u, \\
b\mu - a\nu = b\nu - a\mu = 2\omega_0, \quad a = -1, \quad b = 1.
\end{aligned}\right\} \tag{6.193}$$

Expansion of expressions (6.174) for μ and ν in powers of u/ω_0 gives:

$$\begin{aligned}
\mu, \nu = \omega_0\left(1 - \frac{u^2\cos^2\varphi}{4\omega_0^2}\right) \pm \\
\pm \frac{u}{4\omega_0^2}\sqrt{u^2\cos^2\varphi - 9u^2\sin^2\varphi\cos^2\varphi + 16\omega_0^2\sin^2\varphi}.
\end{aligned} \tag{6.194}$$

The second term under the radical is small relative to the sum of the first and third, since

$$\max\left\{\frac{9u^2\sin^2\varphi\cos^2\varphi}{u^2\cos^2\varphi+16\omega_0^2\sin^2\varphi}\right\}\approx\frac{9u^3}{16\omega_0^2}. \tag{6.195}$$

his maximum is reached at

$$\operatorname{tg}\varphi=\frac{1}{2}\sqrt{\frac{u}{\omega_0}}. \tag{6.196}$$

The first term of the expression under the radical influences the magnitude of the root only when the object is in the immediate vicinity of the equator, where equations (6.128) split. It is therefore possible to retain only the last term under the radical, after which, and also having disregarded the second term in parentheses before the radical, we obtain:

$$\mu,\ \nu=\omega_0\pm u\sin\varphi. \tag{6.197}$$

If we now substitute these expressions for μ and ν into equalities (6.176) and then into solution (6.177) and retain only the first terms of the expansions of the coefficients of solution (6.177) in powers of u/ω_0, we obtain expressions for δx and δy which differ from solutions (6.190) only in that, u sin φ will replace $_z$, as required.

To conclude our analysis of motion along a parallel, we will write out the approximate solution (6.190) for constant instrument errors, for which, according to equalities (6.178),

$$\left.\begin{aligned}f_1&=\Delta n_x-\Delta m_y\omega_z r,\\ f_2&=\Delta n_y-\Delta m_z\omega_y r-\Delta m_y\omega_z r.\end{aligned}\right\} \tag{6.198}$$

To simplify the formulas we will assume that, in addition to inequality (6.185), the condition $\omega_0^2>>\omega_z^2$ is satisfied. Then

$$
\begin{aligned}
\delta x = {} & \frac{\Delta n_x - \Delta m_z \omega_x r}{\omega_0^2} (1 - \cos\omega_0 t \cos\omega_z t) - \\
& - \frac{\Delta n_y - \Delta m_z \omega_y r - \Delta m_y \omega_z r}{\omega_0^2} \cos\omega_0 t \sin\omega_z t + \\
& + \delta x^0 \cos\omega_0 t \cos\omega_z t + \delta y^0 \cos\omega_0 t \sin\omega_z t + \\
& + \frac{\delta \dot{x}^0}{\omega_0} \sin\omega_0 t \cos\omega_z t + \frac{\delta \dot{y}^0}{\omega_0} \sin\omega_0 t \sin\omega_z t, \\
\delta y = {} & \frac{\Delta n_x - \Delta m_z \omega_x r}{\omega_0^2} \cos\omega_0 t \sin\omega_z t + \\
& + \frac{\Delta n_y - \Delta m_z \omega_y r - \Delta m_y \omega_z r}{\omega_0^2} (1 - \cos\omega_0 t \cos\omega_z t) + \\
& + \delta y^0 \cos\omega_0 t \cos\omega_z t - \delta x^0 \cos\omega_0 t \sin\omega_z t + \\
& + \frac{\delta \dot{y}^0}{\omega_0} \sin\omega_0 t \cos\omega_z t - \frac{\delta \dot{x}^0}{\omega_0} \sin\omega_0 t \sin\omega_z t.
\end{aligned}
\qquad (6.199)
$$

6.5.3. The relation between inertial system errors and instrument errors and errors in initial conditions. The only difference between the error equations of two-newtonometer inertial systems for the case of motion on the surface of the earth and the error equations of three-newtonometer systems for arbitrary motion lies in the error equations of the first group. Specifically, in place of equations (5.1), equations (6.27) or (6.31) were obtained. The remaining error equations are the same. Therefore, the difference in the dependency of the total errors in the determination of coordinates and errors in the orientation parameters on instrument errors and errors in the initial conditions occurs in these two cases only as a result of different solutions to the first group of error equations. We may therefore confine ourselves here to comparison of the solutions to the first group of the error equations.

Equations (5.1) indicate the instability of the three-newtonometer system, at least in the practically important region

$$\omega_0^2 - \omega_x^2 - \omega_y^2 - \omega_z^2 > 0.$$

Among the roots of the characteristic equation of system (5.1) for motion along a parallel, there is a positive root which is, at low velocities, close to the value $\omega_0\sqrt{2}$. As a result the errors δx, δy and δz grow exponentially, rapidly moving away from the initial values. The numerical evaluations carried out in §5.6 indicate that maintenance of a level of accuracy comparable with that of the initial conditions is possible here only for a period of 10--15 min from the time at which the system begins operating.

In the case of the two-newtonometer system the problem of stability is solved much more easily. The region

$$\omega_0^2 - \omega_x^2 - \omega_y^2 - \omega_z^2 > 0$$

emerges here as the region of stability. Within this region the roots of the characteristic equation of system (6.27) are purely imaginary. At low velocities they are close to $\pm i\omega_0$. The errors δx and δy therefore have an oscillatory character. There are no exponentially increasing terms in them.

Thus, in the two-newtonometer system the most unacceptable errors for extended operation are those deriving from the solutions to the second group of equations, i.e., those associated with the gyroscope errors Δm_x, Δm_y, and Δm_z. These errors lead to the appearance of components of the total errors in the determination of coordinates which grow linearly with time. Errors in the initial conditions lead only to harmonic oscillations. The instrument errors in the newtonometers are integrable with the weights $\sin \omega_0(t - \tau)$, $\cos \omega_0(t - \tau)$, and for constant values of the newtonometer instrument errors lead to harmonic oscillations in the errors in the determination of the coordinates about certain constant average values of these errors.

We will limit ourselves here to the above remarks. A detailed analysis of the difference between two- and three-newtonometer systems may be performed by direct term-wise comparison of the solutions to equations (5.1) obtained in §5.3, and the solutions to equations (6.27) obtained in §6.5.2.

§6.6. Analysis of the Error Equations of Three-Newtonometer Systems

6.6.1. General properties. Stability. Let us consider the first group of error equations for a system with three arbitrarily oriented newtonometers, using in the ideal equations, for the case of motion on the surface of the earth, relation (6.4) between $\vec{r}$ and the latitude φ for the formation of the quantity μ/r^3.

It was shown in §6.2 that the error equations (6.14) are in this case equivalent to the vector equation (6.13)

$$\frac{d^2 \delta r}{dt^2} + \omega_0^2 \delta r = \Delta n - 2\Delta m \times \frac{dr}{dt} + r \times \frac{d\Delta m}{dt}. \tag{6.200}$$

in which the differentiation is performed in the $O_1\xi_*\eta_*\zeta_*$ coordinate system, and ω_0^2 may be considered constant for motion on the surface of the earth.

The simplest form of the scaler equations corresponding to the vector equations (6.200) is obtained by projecting it on the ξ_*, η_*, ζ_* axes:

$$\left.\begin{aligned}
\delta\ddot{\xi}_* + \omega_0^2\delta\xi_* &= \Delta n_{\xi_*} - 2(\Delta m_{\eta_*}\dot{\zeta}_* - \Delta m_{\zeta_*}\dot{\eta}_*) + \\
&\quad + \eta_*\Delta\dot{m}_{\zeta_*} - \zeta_*\Delta\dot{m}_{\eta_*}.\\
\delta\ddot{\eta}_* + \omega_0^2\delta\eta_* &= \Delta n_{\eta_*} - 2(\Delta m_{\zeta_*}\dot{\xi}_* - \Delta m_{\xi_*}\dot{\zeta}_*) + \\
&\quad + \zeta_*\Delta\dot{m}_{\xi_*} - \xi_*\Delta\dot{m}_{\zeta_*}.\\
\delta\ddot{\zeta}_* + \omega_0^2\delta\zeta_* &= \Delta n_{\zeta_*} - 2(\Delta m_{\xi_*}\dot{\eta}_* - \Delta m_{\eta_*}\dot{\xi}_*) + \\
&\quad + \xi_*\Delta\dot{m}_{\eta_*} - \eta_*\Delta\dot{m}_{\xi_*}.
\end{aligned}\right| \tag{6.201}$$

The left sides of equations (6.201) represent harmonic oscillations. Non-asymptotic stability is, therefore, obvious here. The stability of equations (6.14) for arbitrary ω_x, ω_y and ω_z, i.e., for the case of arbitrary motion on the surface of the earth, also follows from equations (6.201).

If ω_x, ω_y and ω_z in equations (6.14) are constant, stability appears even without reference to equations (6.201). Let us consider, for example, the case of motion along a parallel, for which $\omega_x = 0$, $\omega_y = \text{const}$, and $\omega_z = \text{const}$. In this case the homogeneous equations (6.14) take the form:

$$\left.\begin{aligned}
\delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x - 2\omega_z\delta\dot{y} + 2\omega_y\delta\dot{z} &= 0,\\
\delta\ddot{y} + (\omega_0^2 - \omega_z^2)\delta y + \omega_y\omega_z\delta z + 2\omega_z\delta\dot{x} &= 0,\\
\delta\ddot{z} + (\omega_0^2 - \omega_y^2)\delta z + \omega_z\omega_y\delta y - 2\omega_y\delta\dot{x} &= 0.
\end{aligned}\right| \tag{6.202}$$

These equations may be treated as the equations of motion of a mechanical system near the equilibrium position

$$\delta x = \delta y = \delta z = 0 \tag{6.203}$$

under the influence of potential forces with a force function

$$U = -\frac{1}{2}\left[(\omega_0^2 - \omega_y^2 - \omega_z^2)(\delta x)^2 + (\omega_0^2 - \omega_z^2)(\delta y)^2 + (\omega_0^2 - \omega_y^2)(\delta z)^2 + 2\omega_y\omega_z\,\delta y\,\delta z\right] \tag{6.204}$$

and the gyroscopic forces

$$-2\omega_z\,\delta\dot{y} + 2\omega_y\,\delta\dot{z}, \quad 2\omega_z\,\delta\dot{x}, \quad -2\omega_y\,\delta\dot{x}. \tag{6.205}$$

For the system to be stable, the force functions should have a maximum at the equilibrium point. Application of the Sylvester criteria of positive definiteness to the quadratic form on the right side of expression (6.204) shows that the condition for a maximum of the force function reduces here to the single inequality

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0. \tag{6.206}$$

Outside of the region defined by inequality (6.206), the degree of instability is even and equilibrium is stabilized by the gyroscopic forces. This is easily demonstrated by examining the characteristic equation, which, if written in terms of the square of the unknown $p^2 = q$, has the form:

$$q^3 + (3\omega_0^2 + 2\omega^2)q^2 + (3\omega_0^4 + \omega^4)q + \omega_0^2(\omega_0^2 - \omega^2)^2 = 0, \tag{6.207}$$

where for brevity the rotation

$$\omega^2 = \omega_y^2 + \omega_z^2. \tag{6.208}$$

is introduced.

Polynomial (6.207) satisfies the Hurwitz conditions, since it is always the case that

$$(3\omega_0^2 + 2\omega^2)(3\omega_0^4 + \omega^4) - \omega_0^2(\omega_0^2 - \omega^2)^2 > 0. \tag{6.209}$$

The discriminant Δ of the cubic equation

$$y^3 + 3by + 2c = 0, \tag{6.210}$$

deriving from equations (6.207) by the substitution

$$y = q + \frac{3\omega_0^2 + 2\omega^2}{3}, \tag{6.211}$$

is non-positive:

$$\Delta = -\frac{4}{27}\omega_0^2\omega^6(4\omega_0^2 - \omega^2)^2 \leq 0. \tag{6.212}$$

If

$$\omega \neq 0, \quad 4\omega_0^2 - \omega^2 \neq 0, \quad \omega_0^2 - \omega_\nu^2 \neq 0, \tag{6.213}$$

then equation (6.207) has three distinct real negative roots. Consequently, the characteristic equation of system (6.202) has three pairs of distinct purely imaginary roots.

For

$$\omega_0^2 = \omega^2 \tag{6.214}$$

equation (6.207) has, in addition to two real negative roots, a zero root, and the characteristic equation of system (6.202), consequently, has a multiple zero root.

If

$$\omega = 0, \tag{6.215}$$

then equation (6.207) has the triple root

$$q_{1,2,3} = -\omega_0^2, \tag{6.216}$$

and the characteristic equation, correspondingly, has a pair of conjugate imaginary roots of the same multiplicity.

Finally, for

$$4\omega_0^2 - \omega^2 = 0 \tag{6.217}$$

equation (6.207) has the multiple root

$$q_{2,3} = -\omega_0^2, \tag{6.218}$$

and the characteristic equation has a pair of imaginary multiple roots.

It is easily demonstrated that, in all of the cases cited above in which the roots of the characteristic equation of system (6.202) are multiple, the elementary divisors of the characteristic matrix of this system remain linear.

6.6.2. <u>Integration of the second group of the error equations.</u> The solution of equations (6.201) is obvious:

$$
\left.
\begin{aligned}
\delta\xi_* &= \delta\xi_*^0 \cos\omega_0 t + \frac{1}{\omega_0}\delta\dot{\xi}_*^0 \sin\omega_0 t + \\
&+ \frac{1}{\omega_0}\int_0^t [\Delta n_{\xi_*} - 2(\Delta m_{\eta_*}\dot{\zeta}_* - \Delta m_{\zeta_*}\dot{\eta}_*) + \\
&\qquad + \eta_*\Delta\dot{m}_{\zeta_*} - \zeta_*\Delta\dot{m}_{\eta_*}] \sin\omega_0(t-\tau)\,d\tau, \\
\delta\eta_* &= \delta\eta_*^0 \cos\omega_0 t + \frac{1}{\omega_0}\delta\dot{\eta}_*^0 \sin\omega_0 t + \\
&+ \frac{1}{\omega_0}\int_0^t [\Delta n_{\eta_*} - 2(\Delta m_{\zeta_*}\dot{\xi}_* - \Delta m_{\xi_*}\dot{\zeta}_*) + \\
&\qquad + \zeta_*\Delta\dot{m}_{\xi_*} - \xi_*\Delta\dot{m}_{\zeta_*}] \sin\omega_0(t-\tau)\,d\tau, \\
\delta\zeta_* &= \delta\zeta_*^0 \cos\omega_0 t + \frac{1}{\omega_0}\delta\dot{\zeta}_*^0 \sin\omega_0 t + \\
&+ \frac{1}{\omega_0}\int_0^t [\Delta n_{\zeta_*} - 2(\Delta m_{\xi_*}\dot{\eta}_* - \Delta m_{\eta_*}\dot{\xi}_*) + \\
&\qquad + \xi_*\Delta\dot{m}_{\eta_*} - \eta_*\Delta\dot{m}_{\xi_*}] \sin\omega_0(t-\tau)\,d\tau.
\end{aligned}
\right| \qquad (6.219)
$$

We may proceed as follows in order to pass on to the solution of equations (6.14). Using the table of direction cosines between the ξ_*, η_*, ζ_* and x, y, z axes

	x	y	z
ξ_*	a_{11}	a_{12}	a_{13}
η_*	a_{21}	a_{22}	a_{23}
ζ_*	a_{31}	a_{32}	a_{33},

(6.220)

we can express Δn_{ξ_*}, Δn_{η_*}, Δn_{ζ_*}, Δm_{ξ_*}, Δm_{η_*}, Δm_{ζ_*} and their derivatives in terms of Δn_x, Δn_y, Δ_z, Δm_x, Δm_y and Δm_z, entering into the right sides of equations (6.14). For the given case of motion of the object, its coordinates $\xi_*(t)$, $\eta_*(t)$ and $\zeta_*(t)$ are known as functions of time. Specifically, when the xyz trihedron is, as in the case of equations (6.14), a moving trihedron on a sphere,

$$\xi_* = r a_{13}, \quad \eta_* = r a_{23}, \quad \zeta_* = r a_{33}. \qquad (6.221)$$

The integrands appearing in formula (6.219) are now known. Integrating them, we obtain $\delta\xi_*$, $\delta\eta_*$, $\delta\zeta_*$, from which according to the formulas:

$$
\left.
\begin{aligned}
\delta x &= a_{11}\delta\xi_* + a_{21}\delta\eta_* + a_{31}\delta\zeta_*, \\
\delta y &= a_{12}\delta\xi_* + a_{22}\delta\eta_* + a_{32}\delta\zeta_*, \\
\delta z &= a_{13}\delta\xi_* + a_{23}\delta\eta_* + a_{33}\delta\zeta_*,
\end{aligned}
\right| \qquad (6.222)
$$

we pass over to δx, δy and δz, i.e., to the desired solutions of equations (6.14). The initial conditions $\delta\xi_*^0$, $\delta\eta_*^0$, $\delta\zeta_*^0$, $\delta\dot{\xi}_*^0$, $\delta\dot{\eta}_*^0$

$\delta\dot{\zeta}_*^0$ are expressed here in terms of the initial conditions of equations (6.14) using the inverse of equalities (6.222).

This operation gives the solution of equations (6.14), at least in quadratic forms, for arbitrary motion at a constant distance from the center of the earth and arbitrary time-variation in the intrument errors.

In order to be able to compare these solutions with other alternatives, it is expedient to obtain explicit expressions for the solutions δx, δy and δz of equations (6.14) for those cases of motion for which solutions of equations (5.1) and (6.27) were obtained.

For an object which is stationary in the $O_1\xi_*\eta_*\zeta_*$ coordinate system, superposing the ξ_*, η_* and ζ_* and x, y, z axes and directing the z axis along the radius vector $\vec{r}$, we find from solutions (6.219):

$$\left.\begin{aligned}
\delta x &= \delta x^0 \cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t + \\
&\quad + \frac{1}{\omega_0}\int_0^t (\Delta n_x - r\Delta\dot{m}_y)\sin\omega_0(t-\tau)\,d\tau, \\
\delta y &= \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t + \\
&\quad + \frac{1}{\omega_0}\int_0^t (\Delta n_y + r\Delta\dot{m}_x)\sin\omega_0(t-\tau)\,d\tau, \\
\delta z &= \delta z^0 \cos\omega_0 t + \frac{\delta\dot{z}_0}{\omega_0}\sin\omega_0 t + \\
&\quad + \frac{1}{\omega_0}\int_0^t \Delta n_z \sin\omega_0(t-\tau)\,d\tau.
\end{aligned}\right| \qquad (6.223)$$

For constant instrument errors we have:

$$\left.\begin{aligned}
\delta x &= \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t, \\
\delta y &= \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t, \\
\delta z &= \frac{\Delta n_z}{\omega_0^2} + \left(\delta z^0 - \frac{\Delta n_z}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{z}^0}{\omega_0}\sin\omega_0 t.
\end{aligned}\right| \qquad (6.224)$$

The first two formulas (6.223) coincide with formulas (6.183), and the first two formulas (6.224) with the first two formulas (6.184), respectively.

Let us now consider motion in a fixed plane containing the point O_1. We will take the $\xi_* \eta_*$ plane as the plane of motion and will superpose on it the zx plane of the xyz trihedron. Table (6.220) then takes the form:

$$\begin{array}{cccc} & x & y & z \\ \xi_* & -\sin\omega_y t & 0 & \cos\omega_y t \\ \eta_* & \cos\omega_y t & 0 & \sin\omega_y t \\ \zeta_* & 0 & 1 & 0. \end{array} \tag{6.225}$$

Here $\omega_y = v/r$, where v is the velocity of the object. Table (6.225) assumes that at the initial moment of time the object is situated at a point on the ξ_* axis.

From table (6.225) and equality (6.221) the coordinates ξ_*, η_*, ζ_* are obtained as:

$$\xi_* = r\cos\omega_y t, \quad \eta_* = r\sin\omega_y t, \quad \zeta_* = 0. \tag{6.226}$$

Further,

$$\left.\begin{aligned} \Delta n_{\xi_*} &= -\Delta n_x \sin\omega_y t + \Delta n_z \cos\omega_y t, \\ \Delta n_{\eta_*} &= \Delta n_x \cos\omega_y t + \Delta n_z \sin\omega_y t, \\ \Delta n_{\zeta_*} &= \Delta n_y. \end{aligned}\right\} \tag{6.227}$$

Analogous formulas relate Δm_{ξ_*}, Δm_{η_*}, Δn_{ζ_*} with Δm_x, Δm_y and Δm_z.

Finally, for the initial conditions we obtain:

$$\left.\begin{aligned} &\delta\xi_*^0 = \delta z^0, \quad \delta\eta_*^0 = \delta x^0, \quad \delta\zeta_*^0 = \delta y^0, \\ &\delta\dot{\xi}_*^0 = \delta\dot{z}^0 - \omega_y \delta x^0, \quad \delta\dot{\eta}_*^0 = \delta\dot{x}^0 + \omega_y \delta z^0, \quad \delta\dot{\zeta}_*^0 = \delta\dot{y}^0. \end{aligned}\right\} \tag{6.228}$$

Substituting relations (6.226) -- (6.228) and the formulas for the projections of the vector $\Delta\vec{m}$ on the ξ_*, η_*, ζ_* axes into solutions (6.219) and then passing over, in accordance with formulas (6.222) and table (6.225) to the errors δx, δy and δz, we obtain the following expressions:

$$\left.\begin{aligned}
\delta x &= (\delta x^0 \cos\omega_y t - \delta z^0 \sin\omega_y t)\cos\omega_0 t + \\
&+ \frac{1}{\omega_0}[(\delta\dot{x}^0 + \delta z^0\omega_y)\cos\omega_y t - (\delta\dot{z}^0 - \omega_y\,\delta x^0)\sin\omega_y t]\sin\omega_0 t + \\
&+ \frac{1}{\omega_0}\int_0^t [(\Delta n_x - r\Delta\dot{m}_y)\cos\omega_y (t-\tau) - \\
&\qquad - (\Delta n_z + 2\Delta m_y r\omega_y)\sin\omega_y (t-\tau)]\sin\omega_0 (t-\tau)\,d\tau,
\end{aligned}\right|$$

$$\left.\begin{aligned}
\delta y &= \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0 - r\Delta m_x^0}{\omega_0}\sin\omega_0 t + \\
&+ \frac{1}{\omega_0}\int_0^t (\Delta n_y - r\omega_y\Delta m_z)\sin\omega_0 (t-\tau)\,d\tau + \\
&\qquad\qquad + r\int_0^t \Delta m_x \cos\omega_0 (t-\tau)\,d\tau, \\
\delta z &= (\delta x^0 \sin\omega_y t + \delta z^0 \cos\omega_y t)\cos\omega_0 t + \\
&+ \frac{1}{\omega_0}[(\delta\dot{x}^0 + \omega_y\delta z^0)\sin\omega_y t + (\delta\dot{z}^0 - \omega_y\,\delta x^0)\cos\omega_y t]\sin\omega_0 t + \\
&+ \frac{1}{\omega_0}\int_0^t [(\Delta n_x - r\Delta\dot{m}_y)\sin\omega_y (t-\tau) + \\
&\qquad + (\Delta n_z + 2\omega_y r\Delta m_y)\cos\omega_y (t-\tau)]\sin\omega_0 (t-\tau)\,d\tau.
\end{aligned}\right| \qquad (6.229)$$

The second formula (6.229) coincides with the second formula (6.181). This result was easily predictable, since for the case in question the second equation (6.14) takes the same form as the second equation (6.27). We note that the remaining system formed by the first and third equations (6.14) has the characteristic equation

$$p^4 + 2p^2(\omega_0^2 + \omega_y^2) + (\omega_0^2 - \omega_y^2)^2 = 0. \qquad (6.230)$$

the roots of which are

$$p_{1,2,3,4} = \pm j(\omega_0 \pm \omega_y). \qquad (6.231)$$

Therefore, for the case under consideration formulas (6.229) may be obtained fairly simply by direct solution of the differential equation (6.14), in which it is necessary to set $\omega_z = 0$.

If the instrument errors Δn_x, Δn_y, Δn_z, Δm_x, Δm_y and Δm_z are constant, the second formula (6.229) reduces to the second formula (6.182), and from the first and last formulas (6.229) we find:

$$\begin{aligned}
\delta x = {} & (\delta x^0 \cos\omega_y t - \delta z^0 \sin\omega_y t)\cos\omega_0 t + \\
& + \frac{1}{\omega_0}[(\delta\dot{x}^0 + \omega_y\,\delta z^0)\cos\omega_y t - (\delta\dot{z}^0 - \omega_y\,\delta x^0)\sin\omega_y t]\sin\omega_0 t + \\
& + \frac{1}{\omega_0^2 - \omega_y^2}\Big[\Delta n_x\Big(1 - \cos\omega_0 t\cos\omega_y t - \frac{\omega_y}{\omega_0}\sin\omega_0 t\sin\omega_y t\Big) + \\
& + (\Delta n_z + 2r\omega_y\,\Delta m_y)\Big(\cos\omega_0 t\sin\omega_y t - \frac{\omega_y}{\omega_0}\sin\omega_0 t\cos\omega_y t\Big)\Big], \\
\delta z = {} & (\delta x^0 \sin\omega_y t + \delta z^0 \cos\omega_y t)\cos\omega_0 t + \\
& + \frac{1}{\omega_0}[(\delta\dot{x}^0 + \omega_y\,\delta z^0)\sin\omega_y t + (\delta\dot{z}^0 - \omega_y\,\delta x^0)\cos\omega_y t]\sin\omega_0 t + \\
& + \frac{1}{\omega_0^2 - \omega_y^2}\Big[\Big(\Delta n_z + 2r\omega_y\,\Delta m_y\Big)\Big(1 - \cos\omega_0 t\cos\omega_y t - \\
& \qquad - \frac{\omega_y}{\omega_0}\sin\omega_0 t\sin\omega_y t\Big) - \\
& - \Delta n_x\Big(\cos\omega_0 t\sin\omega_y t - \frac{\omega_y}{\omega_0}\sin\omega_0 t\cos\omega_y t\Big)\Big].
\end{aligned} \tag{6.232}$$

For the case of motion at constant velocity along a parallel, the table of direction cosines (6.220) takes the form:

$$\begin{array}{c|ccc}
 & x & y & z \\
\xi_* & -\sin\omega t & -\sin\varphi\cos\omega t & \cos\varphi\cos\omega t \\
\eta_* & \cos\omega t & -\sin\varphi\sin\omega t & \cos\varphi\sin\omega t \\
\zeta_* & 0 & \cos\varphi\,. & \sin\varphi,
\end{array} \tag{6.233}$$

where

$$\omega = u + \frac{v}{r\cos\varphi}. \tag{6.234}$$

From formulas (6.219) and tables (6.233) and (6.220) we obtain:

$$\begin{aligned}
\delta x = {} & (\delta x^0 \cos\omega t + \delta y^0 \sin\varphi\sin\omega t - \delta z^0\cos\varphi\sin\omega t)\cos\omega_0 t + \\
& + \frac{1}{\omega_0}[\delta\dot{x}^0\cos\omega t + \delta\dot{y}^0\sin\varphi\sin\omega t - \delta\dot{z}^0\cos\varphi\sin\omega t + \\
& + \omega(\delta x^0\sin\omega t - \delta y^0\sin\varphi\cos\omega t + \delta z^0\cos\varphi\cos\omega t)]\sin\omega_0 t + \\
& + \frac{1}{\omega_0}\int_0^t \{\Delta n_x\cos\omega(t-\tau) + \Delta n_y\sin\varphi\sin\omega(t-\tau) - \\
& - \Delta n_z\cos\varphi\sin\omega(t-\tau) + r\omega[-\Delta m_x\sin\varphi\cos\omega(t-\tau) - \\
& \qquad - \Delta m_y(1+\cos^2\varphi)\sin\omega(t-\tau) - \\
& - \Delta m_z\sin\varphi\cos\varphi\sin\omega(t-\tau)] + r[\Delta\dot{m}_x\sin\varphi\sin\omega(t-\tau) - \\
& - \Delta\dot{m}_y\cos\omega(t-\tau)]\}\sin\omega_0(t-\tau)\,d\tau.
\end{aligned} \tag{6.235}$$

$$
\begin{aligned}
\delta y = {} & [-\delta x^0 \sin\varphi \sin\omega t + \delta y^0 (\cos^2\varphi + \sin^2\varphi\cos\omega t) + \\
& + \delta z^0 \sin\varphi\cos\varphi(1-\cos\omega t)]\cos\omega_0 t + \\
& + \frac{1}{\omega_0}[-\delta\dot{x}^0\sin\varphi\sin\omega t + \delta\dot{y}^0(\cos^2\varphi + \sin^2\varphi\cos\omega t) + \\
& + \delta\dot{z}^0 \sin\varphi\cos\varphi(1-\cos\omega t) + \omega\sin\varphi(\delta x^0\cos\omega t + \\
& + \delta y^0 \sin\varphi\sin\omega t - \delta z^0\cos\varphi\sin\omega t)]\sin\omega_0 t + \\
& + \frac{1}{\omega_0}\int_0^t [-\Delta n_x \sin\varphi\sin\omega(t-\tau) + \\
& + \Delta n_y[\cos^2\varphi + \sin^2\varphi\cos\omega(t-\tau)] + \\
& + \Delta n_z \sin\varphi\cos\varphi[1-\cos\omega(t-\tau)] + \\
& + r\omega[\Delta m_x \sin^2\varphi\sin\omega(t-\tau) + \\
& + \Delta m_y \sin\varphi(\cos^2\varphi - (1+\cos^2\varphi)\cos\omega(t-\tau)) - \\
& - \Delta m_z \cos\varphi(\cos^2\varphi + \sin^2\varphi\cos\omega(t-\tau))] + \\
& + r[\Delta\dot{m}_x(\sin^2\varphi\cos\omega(t-\tau) + \cos^2\varphi) + \\
& + \Delta\dot{m}_y \sin\varphi\sin\omega(t-\tau)]\}\sin\omega_0(t-\tau)\,d\tau.
\end{aligned}
$$

$$
\begin{aligned}
\delta z = {} & [\delta x^0\cos\varphi\sin\omega t + \delta y^0\sin\varphi\cos\varphi(1-\cos\omega t) + \\
& + \delta z^0(\sin^2\varphi + \cos^2\varphi\cos\omega t)]\cos\omega_0 t + \frac{1}{\omega_0}[\delta\dot{x}^0\cos\varphi\sin\omega t + \\
& + \delta\dot{y}^0\sin\varphi\cos\varphi(1-\cos\omega t) + \delta\dot{z}^0(\sin^2\varphi + \cos^2\varphi\cos\omega t) + \\
& + \omega\cos\varphi(-\delta x^0\cos\omega t - \delta y^0\sin\varphi\sin\omega t + \\
& + \delta z^0\cos\varphi\sin\omega t)]\sin\omega_0 t + \frac{1}{\omega_0}\int_0^t [\Delta n_x\cos\varphi\sin\omega(t-\tau) + \\
& + \Delta n_y\sin\varphi\cos\varphi[1-\cos\omega(t-\tau)] + \\
& + \Delta n_z[\sin^2\varphi + \cos^2\varphi\cos\omega(t-\tau)] + \\
& + r\omega\cos\varphi[-\Delta m_x\sin\varphi\sin\omega(t-\tau) + \\
& + \Delta m_y(\sin^2\varphi + (1+\cos^2\varphi)\cos\omega(t-\tau)) + \\
& + \Delta m_z\sin\varphi\cos\varphi(\cos\omega(t-\tau) - 1)] + \\
& + r\cos\varphi[\Delta\dot{m}_x\sin\varphi(1-\cos\omega(t-\tau)) - \\
& - \Delta\dot{m}_y\sin\omega(t-\tau)]\}\sin\omega_0(t-\tau)\,d\tau.
\end{aligned}
\tag{6.235}
$$

For $\varphi = 0$, as can easily be demonstrated, formulas (6.235) reduce to formulas (6.229).

For constant instrument errors and zero intial conditions, it follows from formulas (6.235) that

$$
\begin{aligned}
\delta x = {} & \frac{\Delta n_x - r\omega\Delta m_z\sin\varphi}{\omega_0^2-\omega^2}\left(1-\cos\omega_0 t\cos\omega t - \frac{\omega}{\omega_0}\sin\omega_0 t\sin\omega t\right) + \\
& + \frac{\Delta n_y\sin\varphi - \Delta n_z\cos\varphi - r\omega\Delta m_y(1+\cos^2\varphi) - r\omega\Delta m_z\sin\varphi\cos\varphi}{\omega_0^2-\omega^2}\times \\
& \times\left(-\cos\omega_0 t\cos\omega t + \frac{\omega}{\omega_0}\sin\omega_0 t\cos\omega t\right), \\
\delta y = {} & \frac{-\Delta n_x\sin\varphi + r\omega\Delta m_x\sin^2\varphi}{\omega_0^2-\omega^2}\times \\
& \times\left(-\cos\omega_0 t\sin\omega t + \frac{\omega}{\omega_0}\sin\omega_0 t\cos\omega t\right) +
\end{aligned}
\tag{6.236}
$$

$$
\begin{aligned}
&+\frac{1}{\omega_0^2-\omega^2}[\Delta n_y\sin^2\varphi-\Delta n_z\sin\varphi\cos\varphi-\\
&-r\omega\Delta m_y\sin\varphi(1+\cos^2\varphi)-r\Delta m_z\omega\cos\varphi\sin^2\varphi]\times\\
&\qquad\times\left(1-\cos\omega_0 t\cos\omega t-\frac{\omega}{\omega_0}\sin\omega_0 t\sin\omega t\right)+\\
&+\frac{1}{\omega_0^2}(\Delta n_y\cos^2\varphi+\Delta n_z\sin\varphi\cos\varphi+\\
&\qquad+r\omega\Delta m_y\sin\varphi\cos^2\varphi-r\omega\Delta m_z\cos^3\varphi)(1-\cos\omega_0 t).\\
\delta z&=\frac{\Delta n_x\cos\varphi-r\omega\Delta m_x\sin\varphi\cos\varphi}{\omega_0^2-\omega^2}\left(-\cos\omega_0 t\sin\omega t+\right.\\
&\left.+\frac{\omega}{\omega_0}\sin\omega_0 t\cos\omega t\right)+\frac{1}{\omega_0^2-\omega^2}[-\Delta n_y\sin\varphi\cos\varphi+\\
&\qquad+\Delta n_z\cos^2\varphi+r\omega\Delta m_y\cos\varphi(1+\cos^2\varphi)+\\
&+r\omega\Delta m_z\sin\varphi\cos\varphi]\left(1-\cos\omega_0 t\cos\omega t-\frac{\omega}{\omega_0}\sin\omega_0 t\sin\omega t\right)+\\
&+\frac{1}{\omega_0^2}(\Delta n_y\sin\varphi\cos\varphi+\Delta n_z\sin^2\varphi+\\
&+r\omega\Delta m_y\cos\varphi\sin^2\varphi-r\omega\Delta m_z\sin\varphi\cos^2\varphi)(1-\cos\omega_0 t).
\end{aligned}
\tag{6.236}
$$

Formulas (6.235) and formulas (6.236) deriving from them were obtained from the general solution (6.219) of equation (6.201), which are equivalent to equations (6.14). For motion along a parallel, the coefficients of equations (6.14) become constant. The characteristic equation of system (6.14) has the roots

$$p_{1,2}=\pm j\omega_0,\quad p_{3,4,5,6}=\pm j(\omega_0\pm\omega). \tag{6.237}$$

Formulas (6.235) and (6.236) may therefore also be obtained directly as a solution of the system of differential equations (6.14) with constant coefficients.

6.6.3. <u>The relation between the errors of an inertial system and instrument errors and errors in initial conditions</u>. Comparing the case under consideration with preceding cases, we see that, again, only the equations of the second group have changed. These changes effect the relation between system errors and instrument errors and errors in initial conditions.

Unlike the preceding cases, the system is now stable (non-asymptotically) for arbitrary motion on the surface of the earth. This fact determines the character of the relation between the system errors and errors in the initial conditions. In the $O_1\xi_*\eta_*\zeta_*$

coordinate system the errors in the initial conditions $\delta\xi_*^0$, $\delta\eta_*^0$, $\delta\zeta_*^0$, $\delta\dot{\xi}_*^0$, $\delta\dot{\eta}_*^0$, $\delta\dot{\zeta}_*^0$ lead to harmonic oscillations at a Schuler frequency ω_0 for arbitrary motion on the surface of the earth. In the three-newtonometer systems analyzed in the preceding chapters, these errors increased exponentially with time for the special case of motion along a parallel, and in two-newtonometer systems increase in errors with time proved to be possible for variable ω_x, ω_y and ω_z.

The relation between errors in the class of three-newtonometer systems in question and instrument errors is fully determined by solution (6.219) of the error equations. It is in general impossible to compare this relation with the corresponding relations for the preceding cases, since we do not have the solutions of equations (5.1) and (6.27) for arbitrary motion. We must limit ourselves here to those cases of motion for which there exist exact solutions of equations (5.1) and (6.27), i.e., the case of an object which is stationary in inertial space, the case of motion in a fixed plane passing through the center of the earth, and the case of motion along a parallel.

Let us compare the solutions to equations (5.1), (6.27) and (6.14) for the indicated cases of motion at constant instrument errors. For the case of an object which is stationary in inertial space, it is necessary to compare formulas (5.97), (6.184) and (6.224). It is evident that the formulas for δx and δy coincide in all three solutions. Only the formulas for δz in solutions (5.97) and (6.224) differ. The nature of this difference is obvious.

For the case of motion in a fixed plane (for example, motion in the plane of the equator), it is necessary to compare formulas (5.101) and (5.112) with formulas (6.182) and (6.232). In order to facilitate the comparison, we will confine ourselves to small values of ω_y. We will retain in these formulas only terms in the first degree of ω_y. We then obtain in place of formulas (5.101) and (5.112):

$$
\left.
\begin{aligned}
\delta x &= \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t + \\
&\quad + \frac{2\sqrt{2}\,\omega_y}{3\omega_0}\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\left(\sqrt{2}\sin\omega_0 t - \operatorname{sh}\omega_0\sqrt{2}\,t\right) + \\
&\quad + \frac{2\omega_y\,\delta\dot{z}^0}{3\omega_0^2}\left(\cos\omega_0 t - \operatorname{ch}\omega_0\sqrt{2}\,t\right), \\
\delta y &= \frac{\Delta n_y - \omega_y r\,\Delta m_z}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y - \omega_y r\,\Delta m_z}{\omega_0^2}\right)\cos\omega_0 t + \\
&\quad + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t, \\
\delta z &= -\frac{\Delta n_z + 2r\omega_y\,\Delta m_y}{2\omega_0^2} + \left(\delta z^0 + \frac{\Delta n_z + 2r\omega_y\,\Delta m_y}{2\omega_0^2}\right)\operatorname{ch}\omega_0\sqrt{2}\,t + \\
&\quad + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\operatorname{sh}\omega_0\sqrt{2}\,t + \frac{\omega_y\sqrt{2}}{3\omega_0}\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\times \\
&\quad \times\left(\sqrt{2}\sin\omega_0 t - \operatorname{sh}\omega_0\sqrt{2}\,t\right) - \frac{2\omega_y\,\delta\dot{x}^0}{3\omega_0^2}\left(\cos\omega_0 t - \operatorname{ch}\omega_0\sqrt{2}\,t\right).
\end{aligned}
\right\} \qquad (6.238)
$$

In place of formulas (6.182) we will have:

$$
\left.
\begin{aligned}
\delta x &= \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t, \\
\delta y &= \frac{\Delta n_y - \omega_y r\,\Delta m_z}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y - \omega_y r\,\Delta m_z}{\omega_0^2}\right)\cos\omega_0 t + \\
&\quad + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t.
\end{aligned}
\right\} \qquad (6.239)
$$

Finally, in place of formulas (6.232) and the second formula (6.182) we find that

$$
\left.
\begin{aligned}
\delta x &= \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t - \\
&\quad - \left(\delta z^0 - \frac{\Delta n_z}{\omega_0^2}\right)\left(\cos\omega_0 t\sin\omega_y t - \frac{\omega_y}{\omega_0}\sin\omega_0 t\cos\omega_y t\right) - \\
&\quad - \frac{\delta\dot{z}^0}{\omega_0}\sin\omega_y t\sin\omega_0 t, \\
\delta y &= \frac{\Delta n_y - \omega_y r\,\Delta m_z}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y - \omega_y r\,\Delta m_z}{\omega_0^2}\right)\cos\omega_0 t + \\
&\quad + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t, \\
\delta z &= \frac{\Delta n_z + 2r\omega_y\,\Delta m_y}{\omega_0^2} + \left(\delta z^0 - \frac{\Delta n_z + 2r\omega_y\,\Delta m_y}{\omega_0^2}\right)\cos\omega_0 t + \\
&\quad + \frac{\delta\dot{z}^0}{\omega_0}\sin\omega_0 t - \left(\delta x^0 + \frac{\Delta n_x}{\omega_0^2}\right)\left(\cos\omega_0 t\sin\omega_y t - \right. \\
&\quad \left. - \frac{\omega_y}{\omega_0}\sin\omega_0 t\cos\omega_y t\right) + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_y t\sin\omega_0 t.
\end{aligned}
\right\} \qquad (6.240)
$$

Comparison of expressions (6.238), (6.239) and (6.240) for δy in a direction normal to the direction of motion, shows that all of these expressions are identical. Only the expressions for δx and δz differ. For the case of motion on the surface of the earth, the error δz is insignificant, and therefore the difference between the systems appears only from the difference in the error δx in the determination of the coordinate in the direction of motion of the object. Let us compare the first formulas of (6.238), (6.239) and (6.240). The first three terms in formulas (6.238) and (6.240) coincide. The first formula (6.239) consists entirely of these terms. The remaining terms of formula (6.238) and (6.240) differ basically in that, in place of the rapidly increasing hyperbolic functions sinh $\omega_0\sqrt{2}t$ and cosh $\omega_0\sqrt{2}t$ which appear in formulas (6.238), in formulas (6.240) the harmonic functions $\sin\omega_0 t$ and $\cos\omega_0 t$ appear.

The errors δx, δy and δz may be compared in an analogous fashion for the case of motion of an object along a parallel.

In conclusion we note that in the three-newtonometer system in question, as in the two-newtonometer system, constant newtonometer instrument errors lead only to oscillatory and constant errors in the determination of coordinates. Constant gyroscope errors lead, as a result of the second group of equations, to errors which increase with time. The gyroscope errors in these systems are the main errors limiting the independent functioning time of the inertia system.

NOTES

1. As was already noted in the preface, these problems were analyzed in the author's, Teoriya inertsial'noy navigatsü (korrektiruyemyye sistemy) (Theory of Inertial Navigation. Correctable Systems) (in press) directly relating to the current work.

2. Andreyev, V. D. On the general equations of inertial navigation, Prikladnaya matematika i mekhanika, vol. XXVIII, Issue 2, 1964.

3. Compare Note 1.

4. Compare Note of Chapter 5.

5. Ishlinskiy, A. Yu. On the relative equibrium of a compound pendulum with a moving point of support, Prikladnaya matematika i mekhanika, vol. XX, Issue 3, 1956.

6. Grammel', R. Giroskop, ego teoriya i primeneniye (The Gyroscope. Theory and Application), vol. 2, Foreign Literature Publishing House, 1952; Bulgakov, B. V. Prikladnaya teoriya giroskopov (Applied Gyroscope Theory), Gostekhizdat, 1955.

7. Andreyev, V. D. On a case of small oscillations of a compound pendulum with a moving point of support, Prikladnaya matematika i mekhanika, vol. XXII, Issue 6, 1958.

8. Andreyev, V. D. On errors in inertial navigation systems, Izvestiya (Bulletin) of the USSR Academy of Sciences, tekhnicheskaya kibernetika, No. 2, 1964.

9. Klimov, D. M. On nonperturbed conditions for a gyroscope frame, Prikladnaya matematika i mekhanika, vol. XXVII, Issue 3, 1964.

10. Andreyev, V. D. Regarding the theory of a gyroscopic-pendulum system satisfying the Schuler condition, Prikladnaya matematika i mekhanika, vol. XXIX, Issue 6, 1965.

11. Ishlenskiy, A. Yu. Regarding the theory of the pitch control gyro, Prikladnaya matematika i mekhanika, vol. XX, Issue 4, 1956; Theory of the twin-vertical gyro, ibid., vol. XXI, Issue 2, 1957.

12. See Note 10.

13. See, for example, Smirnov, V. I. Kurs vysshey matematiki (Course in Higher Mathematics), vol. III, Part 1, Gostekhizdat, 1953.

14. Mak-Lakhlan [MacLachlan], N. V. Teoriya i prilozheniya funktsiy Mat'ye (Theory and Applications of Mathieu Functions), Foreign Literature Publishing House, 1953.

15. See, for example, Maltsev, A. I. Osnovy lineynoy algebry (Fundamentals of Linear Algebra), Gostekhizdat, 1948.

16. Chetayev, N. G. Ystoychivost'dvizheniya (Stability of Motion), Gostekhizdat, 1955.

17. ibid.

18. Zhbanov, Yu. K. Regarding the investigation of natural oscillations in a system for independent calculation of the coordinates of a moving object, Prikladnaya matematika i mekhanika, vol. XXIV, Issue 6, 1960; Merkin, D. P. On the stability of the motion of a gyroframe, Prikladnaya matematika i mekhanika, vol. XXV, Issue 6, 1961.

19. Let us note that if we are speaking not of motion along a parallel, but of some other motion when all three projections ω_x, ω_y, and ω_z are different from zero, the corresponding conditions of stability are obtained by replacing in inequalities (6.171) and (6.172) the value ω_y^2 by $\omega_y^2 + \omega_x^2$.

20. Chetayev, N. G. op. cit.

21. Koshlyakov, V. N. On the stability of a pitch control gyrocompass in the presence of dissipative forces, Prikladnaya matematika i mekhanika, vol. XXVI, Issue 3, 1962.

Chapter 7

PREPARATION OF AN INERTIAL SYSTEM FOR BEGINNING OPERATION AT A TAKE-OFF POINT WHICH IS STATIONARY RELATIVE TO THE EARTH

§7.1. Initial Considerations

For an inertial system, beginning at some moment of time t^0, to function normally, the initial conditions of its operation must be correctly specified at this moment of time.

The problem consists in guaranteeing, at the initial moment of time, the selected orientation of the axes of sensitivity of the newtonometers and gyroscopes, and also in introducing into the system (or obtaining from the system itself) the values of the coordinates and their rates of change at the moment t^0, which constitute the initial conditions of the solution to the main differential equation of inertial navigation. Of course, in the process of preparing an inertial system for operation, operations associated with the start up and testing of its equipment must also be performed. We will not deal with this aspect of the process here.

In theory, and even more so in practice, the content of the operations by means of which an inertial system is prepared for operation depends to a great extent on the system in question, its kinematics, the orientation of its sensing elements, and also on the selection of the reference grid in which it operates. Much also depends on the external information available during the process of the initial preparation of the system.

It should be noted that the task of putting an inertial system into the correct initial state is in many ways similar to that of correcting it. In both cases the problem consists in moving the system from some "incorrect" state to a correct state. The task of putting the system into the correct initial state cannot be performed without external information, using the term external information in the broad sense. Thus, for example, as will become clear below, it is possible to automatically solve the problem of preparing the system if at the initial moment the object on which it is mounted is stationary relative to the earth. However, simple knowledge of this fact does not constitute additional information.

The required precision in the establishment of the initial operational conditions of an inertial system depends to a great extent on the possibilities for correcting its operation after it has begun. If correction during operation is not provided for, the accuracy of the specification of the initial state of the inertial system must be extremely high, since errors in initial conditions, as was shown in the course of analysis of the error equations, leads to errors in the determination of the navigation parameters, and these errors are retained throughout the further operation of the inertial system.

Below, the initial preparation of an inertial system is analyzed for the case of an object which is stationary relative to the earth.

§7.2. The Case of Arbitrary Initial Orientation of the Inertial System Platform

Let us assume that the functional diagram of an inertial system is a gyrostablized platform. Let the axes of sensitivity of the newtonometers coincide with the x, y, and z axes of the platform. Let the coordinates determined by the system be the earth body-axis coordinate system $O_1\xi\eta\zeta$. Let the origin of this coordinate system coincide with the center of the earth, the ζ axis be directed along its axis of rotation, and the ξ axis lie along the intersection of the planes of the equator and the Greenwich meridian. The coordinates determined by the system might be, for example, the Cartesian coordinates ξ, η, and ζ, the geocentric coordinates r, φ and λ, the geographic coordinates h, φ and λ the geodetic coordinates r, z and S, etc.

For the system, beginning at some moment of time t^0, to operate correctly, the following must be determined for this moment of time: the relative position of the xyz and $\xi\eta\zeta$ coordinate systems, i.e., the direction cosines between the axes of these coordinate systems, the coordinates x^0, y^0 and z^0, and the time derivatives of these coordinates $\dot{x}^0$, $\dot{y}^0$ and $\dot{z}^0$.

We will show that if the longitude λ^0 of the take-off point of the object is known and introduced into the computational apparatus of the inertial system, the further preparation of the system for operation may be performed automatically and autonomously on the basis of the newtonometer readings of the system itself.

The newtonometer readings give the projections n_x, n_y and n_z of the vector

$$n = \frac{d^2 r}{dt^2} - g(r) \tag{7.1}$$

on the x, y, z axes of the platform, which after the starting up of the gyroscopes and stabilization, occupies a fixed position in inertial space.

Since the object on which the inertial system is mounted is stationary relative to the earth, it follows that

$$\frac{dr}{dt} = u \times r, \quad \frac{d^2 r}{dt^2} = u \times (u \times r) = u(u \cdot r) - u^2 r. \tag{7.2}$$

Here differentiation is performed in the xyz body-axis coordinate system of the stabilized platform. This coordinate system may be considered as inertial. Substituting the second equality (7.2) into formula (7.1), we obtain:

$$n = u(u \cdot r) - u^2 r - g(r). \tag{7.3}$$

Time-differentiating the vector $\vec{n}$ in the xyz coordinate system and noting that the vector $\vec{u}$ is constant in this system, and that the vector $\vec{g}$ is constant in the coordinate earth body-axis system we find:

$$\frac{dn}{dt} = u \times n. \tag{7.4}$$

A second differentiation gives:

$$\frac{d^2n}{dt^2} = u(u \cdot n) - u^2 n. \tag{7.5}$$

The vector $\vec{u}$ may be found from equations (7.4) and (7.5). Indeed, performing the vector multiplication of equations (7.4) and $\vec{n}$ and the scaler multiplication of equations (7.5) and $\vec{n}$, we obtain:

$$\left.\begin{aligned} \frac{dn}{dt} \times n &= n(u \cdot n) - u \cdot n^2, \\ \frac{d^2n}{dt^2} \cdot n &= (u \cdot n)^2 - u^2 n^2. \end{aligned}\right\} \tag{7.6}$$

From the second equation we find:

$$u \cdot n = \sqrt{\frac{d^2n}{dt^2} \cdot n + u^2 n^2}. \tag{7.7}$$

From the first equation (7.6) we have:

$$u = \frac{n\sqrt{\frac{d^2n}{dt^2} \cdot n + u^2 n^2} - \frac{dn}{dt} \times n}{n^2}. \tag{7.8}$$

Since the vector $\vec{u}$ coincides with the ζ axis, the unit vector $\vec{\zeta}$ of this axis is

$$\zeta = \frac{n\sqrt{\frac{d^2n}{dt^2} \cdot n + u^2 n^2} - \frac{dn}{dt} \times n}{un^2}. \tag{7.9}$$

Finding the unit vector $\vec{\zeta}$ in the xyz coordinate system implies that the third column of the table of direction cosines

$$\begin{array}{c|ccc} & \xi & \eta & \zeta \\ \hline x & \beta_{11} & \beta_{12} & \beta_{13} \\ y & \beta_{21} & \beta_{22} & \beta_{23} \\ z & \beta_{31} & \beta_{32} & \beta_{33} \end{array} \tag{7.10}$$

between the x, y, z and ξ, η, ζ axes is known, and, consequently, that the angles between the axes of the stabilized platform and the earth's axis are also known.

The first and second columns of table (7.10) remain to be found. To do this we introduce the trihedron $O_1\xi_1\eta_1\zeta_1$, the ζ_1 axis of which coincides with the earth's axis, and the ξ_1 axis of which is situated on the intersection of the plane of the equator and the plane of the meridian of the point at which the object is located. Since the longitude λ^0 of this point is known, the unit vectors $\vec{\xi}$ and $\vec{\eta}$ of the ξ and η axes are related to the unit vectors $\vec{\xi}_1\vec{\eta}_1$ of the ξ_1 and η_1 axes by the obvious equalities:

$$\xi = \xi_1 \cos\lambda^0 - \eta_1 \sin\lambda^0, \quad \eta = \xi_1 \sin\lambda^0 + \eta_1 \cos\lambda^0. \tag{7.11}$$

We now note that the vector $\vec{n}$ is the sum of the centrifugal force due to the earth's rotation and the gravitational force. It lies therefore in the plane of the meridian on which the object is located. This means that

$$\xi_1 = \frac{n \times \zeta}{|n \times \zeta|}, \quad \eta_1 = \zeta \times \xi_1. \tag{7.12}$$

Formulas (7.12), (7.11) and (7.9) fully determine the relative orientation of the xyz and $\xi\eta\zeta$ coordinate systems. We note that only the longitude λ^0 was used in the derivation of these formulas, i.e., only one of the coordinates of the object. The remaining parameters were found from the newtonometer readings. It is assumed, of course, that the computational apparatus of the inertial system is able to perform the computations required by these formulas, including the differentiation of the newtonometer readings.

We note also that these formulas cannot be used if the take-off point of the object is located at a geographic pole of the earth. More precisely, formula (7.9) remains valid and, consequently, the third column of table (7.10) is reinstated. This is understandable, since, as follows from formula (7.9), in this case the direction of the ζ axis coincides with the direction of the vector $\vec{n}$. Formulas (7.12) become invalid, since, because of the collinearity of the vectors $\vec{\zeta}$ and $\vec{n}$, their vector product is equal to zero and does not define the direction of the ξ_1 axis. Therefore, if the take-off point is located near a geographic pole, determination of the orientation of the platform of the initial system requires additional information, for example the bearing to some star or terrestrial landmark at t^0.

If the orientation of the platform relative to the earth has been determined by the means indicated above, there remains the problem of finding the coordinates x^0, y^0, and z^0 and their derivatives. We will show that preliminary knowledge of only the longitude λ^0 of the take-off point is sufficient for this purpose as well.

Indeed, since the table of the direction cosines $\beta_{ij}(t)$ has been found, the time derivatives $\dot{\beta}_{ij}$ of the elements of this table may also be found, and, therefore, to find x^0, y^0, z^0, $\dot{x}^0$, $\dot{y}^0$, $\dot{z}^0$, it is sufficient to know the initial values of the coordinates ξ^0, η^0, and ζ^0. To determine ξ^0, η^0 and ζ^0, it is in turn sufficient to know the geocentric coordinates r^0, and λ^0 of the take-off point We assume that λ^0 is known. The other coordinates φ^0 and r^0 may be found from the solution to the system of equations.

$$|n| = g_e(\varphi), \quad r = r(\varphi). \tag{7.13}$$

where $g_e(\varphi)$ is the acceleration of the gravitational force at the take-off point, and $r = r(\varphi)$ is the equation for the radius vector of the terrestial meridian.

This procedure for determining the orientation of the sensing elements of an inertial system is easily extended to the case in which the basis of the functional liagram of the inertial system is a three-dimensional gyroscopic gauge of absolute angular velocity which is rigidly attached to the object.

If the object is stationary relative to the earth, the gyroscopic gauge will measure the absolute earth rate $\vec{u}$. This permits immediate determination of the unit vector $\vec{\zeta}$ in the xyz coordinate system attached to the platform of the gauge:

$$\zeta = \frac{u}{u}. \tag{7.14}$$

Equalities (7.12) may be used to determine β_{ij} (now constant), since the vector $\vec{n}$ lies, as before, in the plane of the meridian of the take-off point.

The initial values of the coordinates x^0, y^0 and z^0 are found from ξ^0, η^0, ζ^0 and β_{ij}. The initial velocities $\dot{x}^0$, $\dot{y}^0$ and $\dot{z}^0$ are equal to zero, since the vector $\vec{r}$ is constant in the xyz coordinate system attached to the object and stationary relative to the earth.

The quantities ξ^0, η^0 and ζ^0 are found from the known values λ^0 and the quantities r^0 and φ^0, obtained from the solution to the system of equations (7.13).

§7.3. The Case of the Orientation of One of the Platform Axes Along the Geocentric or Geographic Vertical

Let us now consider the practically important case of a maneuverable gyroplatform as the basis of the functional diagram of an internal system.

Let us designate the right orthogonal trihedron attached to the gyroplatform as xyz. Let the coordinates to be determined be the goegraphic coordinates h, φ' and λ. Then, at the moment at which the system begins operating, the z axis of the platform should be superposed on the direction of the gravitational force, and the y axis should be in the plane of the earth's meridian. Thus, the question here is not that of determining the orientation of trihedron xyz relative to the earth, as in preceding cases, but of positioning this trihedron in a particular relation to trihedron $\xi\eta\zeta$, attached to the earth, and of maintaining it in this position until the moment at which operation begins. As in the preceding cases, part of the problem of preparing the system for operation is the introduction (or determination within the system itself) of the initial values of the coordinates φ' and λ^0 (on the surface of the earth $h = 0$).

We will show that, as in the two preceding variants, the problem of preparing the system for operation may be solved autonomously and automatically, if the longitude λ^0 of the take-off meridian is known.

Let us begin with the problem of the orientation of the platform. We will use x_0, y_0, z_0 to designate the position into which trihedron xyz should be placed. In this position the z_0 axis is directed along the vector $\vec{g}_e$, and the y_0 is in the plane of the meridian and directed towards the north. Let the relative positions of the xyz and $x_0y_0z_0$ trihedra be defined by the small angles α, β and γ in accordance with the table of direction cosines

$$\begin{array}{cccc} & x & y & z \\ x_0 & 1 & -\gamma & \beta \\ y_0 & \gamma & 1 & -\alpha \\ z_0 & -\beta & \alpha & 1. \end{array} \tag{7.15}$$

The deviation of the platform from the position which it should occupy may, clearly, be considered as small. This does not diminish the generality of the analysis.

To solve this problem we will use the precession equations of motion of the platform:

$$\left.\begin{array}{c} -H\omega_x = M_y^{\text{у}}, \quad H\omega_y = M_x^{\text{у}}, \\ H\omega_z = M_z^{\text{у}}. \end{array}\right\} \tag{7.16}$$

We form the controlling moments in accordance with the equalities

$$\left.\begin{array}{c} M_y^{\text{у}} = -H\omega_{x_0} - kn_y, \quad M_x^{\text{у}} = H\omega_{y_0} - kn_x, \\ M_z^{\text{у}} = H\omega_{z_0}. \end{array}\right\} \tag{7.17}$$

Here n_x and n_y are the readings of the newtonometers oriented long the x and y axes, and ω_{x_0}, ω_{y_0} and ω_{z_0} are the projections f the absolute angular velocity of trihedron $x_0y_0z_0$ on its axes.

According to table (7.15) the newtonometer readings are

$$n_x = g_e\beta, \quad n_y = -g_e\alpha. \tag{7.18}$$

where g_e is the acceleration of the gravitational force at the take off point. The projections of the absolute angular velocity of trihedron xyz on its axes are determined by the following equalities:

$$\left.\begin{aligned} \omega_x &= \omega_{x_0} + \omega_{y_0}\gamma - \omega_{z_0}\beta + \dot{\alpha}, \\ \omega_y &= \omega_{y_0} - \omega_{x_0}\gamma + \omega_{z_0}\alpha + \dot{\beta}, \\ \omega_z &= \omega_{z_0} + \omega_{x_0}\beta - \omega_{y_0}\alpha + \dot{\gamma}. \end{aligned}\right\} \qquad (7.19)$$

Substituting expressions (7.18) and (7.19) into equality (7.17), we obtain the following equations describing the motion of the platform in the process of bringing it into the prestart position:

$$\left.\begin{aligned} \dot{\alpha} + \frac{kg_e}{H}\alpha + \omega_{y_0}\gamma - \omega_{z_0}\beta &= 0, \\ \dot{\beta} + \frac{kg_e}{H}\beta - \omega_{x_0}\gamma + \omega_{z_0}\alpha &= 0, \\ \dot{\gamma} + \omega_{x_0}\beta - \omega_{y_0}\alpha &= 0. \end{aligned}\right\} \qquad (7.20)$$

The system of equations (7.20) has, clearly, a trivial solution: $\alpha = \beta = \gamma = 0$. The characteristic equation of this system has the form:

$$\begin{aligned} p^3 + 2\frac{kg_e}{H}p^2 + \left[\left(\frac{kg_e}{H}\right)^2 + \omega_{x_0}^2 + \omega_{y_0}^2 + \omega_{z_0}^2\right]p + \\ + \frac{kg_e}{H}\left(\omega_{x_0}^2 + \omega_{y_0}^2\right) = 0. \end{aligned} \qquad (7.21)$$

In the case under consideration here

$$\omega_{x_0} = 0, \quad \omega_{y_0} = u\cos\varphi'^0, \quad \omega_{z_0} = u\sin\varphi'^0 \qquad (7.22)$$

and the characteristic equation takes the form:

$$p^3 + 2\frac{kg_e}{H}p^2 + \left[\left(\frac{kg_e}{H}\right)^2 + u^2\right]p + \frac{kg_e}{H}u^2\cos^2\varphi'^0 = 0. \qquad (7.23)$$

Application of the Hurwitz criterion gives the following conditions for the stability for the process of bringing the platform into the prestart position:

$$\left.\begin{aligned} k > 0, \quad 2\left(\frac{kg_e}{H}\right)^2 + u^2\left(2 - \cos^2\varphi'^0\right) > 0, \\ \cos^2\varphi'^0 > 0. \end{aligned}\right\} \qquad (7.24)$$

Thus, the only significant stability condition is the requirement

$$\varphi \neq \frac{\pi}{2}. \tag{7.25}$$

If this condition is not met a zero root appears in equations (7.23).

The prestart orientation proves to be impossible at the geographic poles of the earth: the closer the take-off point is to a pole the greater the time required for the prestart orientation.

Indeed, at the equator, where $\cos \varphi'^{0} = 1$, the left side of equation (7.23) may be factored, as a result of which the characteristic equation takes the form:

$$\left(p + \frac{kg_e}{H}\right)\left(p^2 + \frac{kg_e}{H} p + u^2\right) = 0. \tag{7.26}$$

Its roots are:

$$\left.\begin{aligned} p_1 &= -\frac{kg_e}{H}, \\ p_{2,3} &= -\frac{kg_e}{H} \pm \sqrt{\frac{1}{4}\left(\frac{kg_e}{H}\right)^2 - u^2}. \end{aligned}\right\} \tag{7.27}$$

The real parts of the roots are negative. The maximum value of the smallest of the moduli of these parts is equal to u and occurs at $k = 2uH/g_e$. If we take this as the working value of k, for values of φ'^{0} sufficiently close to $\pi/2$, two of the roots of equation (7.23) will be close to the values

$$p_{1,2} = -2u \pm ju,$$

and the third will be close to

$$p_3 = -\frac{u \cos^2 \varphi'^{0}}{2}.$$

The magnitudes of the real parts of the roots of the characteristic equation determine the speed at which the platform is brought to the required (prestart) position. It is evident from the expression for p_3 that, as φ'^{0} increases, this speed will decrease proportionally to $\cos^2 \varphi'$.

We note that, in addition to the method described above for placing the y axis in the plane of the meridian, leading to equations (7.20), an equivalent method, based on the working principle of the gyrocompass, may be used.[1]

Let us consider the following. To form the controlling moments (7.17), the quantities ω_{x_0}, ω_{y_0} and ω_{z_0} are required. To compute these from formulas (7.22), we need to know the geographic latitude φ'^0 of the take-off point. The lattitude φ is also required as an initial condition. If φ'^0 is unknown, it may be determined from the equation

$$|n| = g_r(\varphi'). \tag{7.28}$$

For this, of course, a third newtonometer is required.

It is evident that it is fundamental in the method under consideration that the y axis should be located in the plane of the meridian of the take-off point. It would also be acceptable to orient it along some direction in the plane of the geographic horizon, forming a given angle ψ^0 with the meridian. This requires only that other values of ω_{x_0}, ω_{y_0} and ω_{z_0} be substituted, in place of those given by (7.22), into expressions (7.17) for the controlling moments, namely:

$$\left.\begin{array}{c}\omega_{x_0} = u\cos\varphi'^0\sin\psi^0, \quad \omega_{y_0} = u\cos\varphi'^0\cos\psi^0, \\ \omega_{z_0} = u\sin\varphi'^0.\end{array}\right\} \tag{7.29}$$

Equations (7.20) retain their form in this case. The coefficients of the characteristic equation (7.23) likewise do not change, since, once again,

$$\omega_{x_0}^2 + \omega_{y_0}^2 = u^2\cos^2\varphi'^0, \quad \omega_{x_0}^2 + \omega_{y_0}^2 + \omega_{z_0}^2 = u^2. \tag{7.30}$$

In conclusion, let us consider an inertial system operating in spherical, for example geocentric or geodetic coordinates. In this case the xyz trihedron of the inerital system should be oriented as follows at the moment at which operation begins: the z axis should be directed along the radius vector $\vec{r}$, and the y axis should form some angle ψ^0 with the plane of the meridian. For the case of geocentric coordinates $\psi^0 = 0$. For the case of geocentric coordinates, the initial conditions will be r^0, φ^0 and λ^0, and for the case of geodetic coordinates, r^0, z^0 and S^0, respectively.

As before, the longitude λ^0 should be given. The quantities φ^0 and r^0 are found from the equations

$$|n| = g_0(\varphi), \quad r = r(\varphi). \tag{7.31}$$

The quanties z^0 and S^0 may easily be found from λ^0 and φ^0 by using, for example, relations (3.303), regarding them as equations in z and S.

The problem of the prestart orientation of the platform of the inertial system remains to be solved. This problem may be solved in a manner which is completely analogous to that used in the preceding case, in which the z axis of the platform was oriented along the normal to the reference ellipsoid. The only difference is that the controlling moments of the gyroplatform should be formed not in accordance with equalities (7.17), but in accordance with the formulas

$$\left.\begin{aligned} M_y^j &= -H\omega_{x_0} - k(n_y - n_{y_0}), \\ M_x^j &= H\omega_{y_0} - k(n_x - n_{x_0}), \\ M_z^j &= H\omega_{z_0}. \end{aligned}\right\} \tag{7.32}$$

Here n_{x_0} and n_{y_0} designate the projections of the vector $\vec{n}$ on the x_0 and y_0 axes. These projections are equal to the negatives of the known (since r^0, φ^0 and ψ^0 are known) projections of the acceleration of the gravitational force on the x_0 and y_0 axes. The ω_{x_0}, ω_{y_0} and ω_{z_0} of the absolute angular velocity entering into formulas (7.32) are also known if φ'^0 and ψ^0 are known. These projections are given by equality (7.29).

As before, we will define the relative positions of the xyz and $x_0y_0z_0$ trihedra by means of the angles α, β and γ, in accordance with the table of direction cosines (7.15). Formulas (7.19) for the projections ω_x, ω_y and ω_z will remain valid, and the expressions for n_x and n_y will take the form:

$$\left.\begin{aligned} n_x &= n_{x_0} + n_{y_0}\gamma - n_{z_0}\beta, \\ n_y &= -n_{x_0}\gamma + n_{y_0} + n_{z_0}\alpha. \end{aligned}\right\} \tag{7.33}$$

Substitution of expression (7.33) into formula (7.32) yields the following values of the controlling moments:

$$
\begin{aligned}
M_y^1 &= -H\omega_{x_0} - k(n_{z_0}\alpha - n_{x_0}\gamma),\\
M_x^1 &= H\omega_{y_0} - k(n_{y_0}\gamma - n_{z_0}\beta),\\
M_z^0 &= H\omega_{z_0}.
\end{aligned}
$$

Substituting these values into equations (7.16) and simultaneously substituting the expressions (7.19) for the projections ω_x, ω_y and ω_z into these equations, we obtain the system of equations:

$$
\left.
\begin{aligned}
\dot{\alpha} - \frac{kn_{z_0}}{H}\alpha + \left(\frac{kn_{x_0}}{H} + \omega_{y_0}\right)\gamma - \omega_{z_0}\beta = 0,\\
\dot{\beta} - \frac{kn_{z_0}}{H}\beta + \left(\frac{kn_{y_0}}{H} - \omega_{x_0}\right)\gamma + \omega_{z_0}\alpha = 0,\\
\dot{\gamma} + \omega_{x_0}\beta - \omega_{y_0}\alpha = 0.
\end{aligned}
\right|
\tag{7.34}
$$

The characteristic equation of this system is written in the following form:

$$
\begin{aligned}
p^3 - \frac{2kn_{z_0}}{H}p^2 + \Bigg[\left(\frac{kn_{z_0}}{H}\right)^2 + u^2 + \\
+ \frac{k}{H}(n_{x_0}\omega_{y_0} - n_{y_0}\omega_{x_0})\Bigg]p - \frac{kn_{z_0}}{H}u^2\cos^2\varphi - \\
- \frac{k^2n_{z_0}}{H}(n_{x_0}\omega_{y_0} - n_{y_0}\omega_{x_0}) + \frac{k}{H}\omega_{z_0}(n_{x_0}\omega_{x_0} + n_{y_0}\omega_{y_0}) = 0.
\end{aligned}
\tag{7.35}
$$

But

$$
\begin{aligned}
n_{x_0} &\approx -\frac{ge^2}{4}\sin 2\varphi^0 \sin\psi^0,\\
n_{y_0} &\approx -\frac{ge^2}{4}\sin 2\varphi^0 \cos\psi^0,\\
n_{z_0} &\approx g,
\end{aligned}
\tag{7.36}
$$

where g is the acceleration of the gravitational force, and e is the eccentricity of the terrestrial ellipsoid.

Taking into account relations (7.36) and (7.29), we obtain from equations (7.35):

$$
\begin{aligned}
p^3 + 2\frac{kg}{H}p^2 + \left[\left(\frac{kg}{H}\right)^2 + u^2\right]p + \\
+ \frac{kg}{H}u^2\cos^2\varphi^0\left(1 - \frac{e^2}{2}\sin^2\varphi^0\right) = 0.
\end{aligned}
\tag{7.37}
$$

Since

$$\frac{e^2}{2}\sin^2\varphi^0 \ll 1,$$

equation (7.37) is equivalent to equation (7.23), and the stability of the process of bringing the placform of the inertial system into its prestart position is guaranteed for all locations except the geographic poles of the earth.

Note

1. See for example, Roytenberg, Ya.N. on accelarated actuation of a gyroscopic compass on a meridian, Prikladnaya matematika i mekhnika, vol. XXXIII, Issue 5, 1959.

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ERRATA

If the initial conditions are null, then

$$
\left.
\begin{aligned}
\Delta x &= \frac{\Delta a_x}{\omega_0^2}\left[-\frac{3(\omega_0 t)^2}{2} + 4(1-\cos\omega_0 t)\right] + \\
&\quad + \left(\frac{r\,\Delta\omega_y}{\omega_0} + \frac{2\Delta a_z}{\omega_0^2}\right)(\sin\omega_0 t - \omega_0 t),\\
\Delta y &= \frac{1}{\omega_0^2}(\Delta a_y - r\omega_0\,\Delta\omega_x)(1-\cos\omega_0 t),\\
\Delta z &= \frac{2\Delta a_x}{\omega_0^2}(\omega_0 t - \sin\omega_0 t) + \frac{\Delta a_z}{\omega_0^2}(1-\cos\omega_0 t) + \\
&\quad + \frac{2r\,\Delta\omega_y}{\omega_0}(1-\cos\omega_0 t)
\end{aligned}
\right| \qquad (5.119)
$$

5.3.4. <u>The motion of an object along a parallel at constant velocity</u>. Let us now consider the case of the motion of an object along a parallel in which the first group of the coordinate error equations also reduces to equations with constant coefficients.

In this case

$$\omega_x = 0,\quad \dot\omega_y = \dot\omega_z = 0,\quad \dot r = 0 \qquad (5.120)$$

and the error equations (5.19) form the following system of sixth-order differential equations:

$$
\left.
\begin{aligned}
\Delta\ddot x &+ (\omega_0^2 - \omega_y^2 - \omega_z^2)\Delta x - 2\omega_z\Delta\dot y + 2\omega_y\Delta\dot z = \\
&= \Delta a_x - \Delta\dot\omega_y r - \omega_z\Delta\omega_x r,\\
\Delta\ddot y &+ (\omega_0^2 - \omega_z^2)\Delta y + \omega_y\omega_z\Delta z + 2\omega_z\Delta\dot x = \\
&= \Delta a_y + \Delta\dot\omega_x r - \omega_y\Delta\omega_y r - \omega_z\Delta\omega_z r,\\
\Delta\ddot z &- (2\omega_0^2 + \omega_y^2)\Delta z - 2\omega_y\Delta\dot x + \omega_y\omega_z\Delta y = \\
&= \Delta a_z + 2r\omega_y\Delta\omega_y
\end{aligned}
\right| \qquad (5.121)
$$

where

$$\omega_0^2 = \mu/r^3$$

The initial conditions for these equations will be the quantities given by equalities (5.91) and (5.92).

Equations (5.121) are projections of the error equations (5.1) onto the x, y, z axes of a geocentric moving trihedron, the z axis of which coincides with the vector $\vec r$, with its y axis directed